1 Writing matrix equations

All battery-resistor-current problems produce systems of linear equations. Linear equations are just a equations where the unknowns (usually the currents $I_i$) are never multiplied together. For example from Recitation 6, Problem 35, we have

$$0 = I_1 - I_2 - I_3$$
$$0 = \epsilon_2 - I_2 R_2 - I_1 R_1$$
$$0 = \epsilon_3 - I_3 R_3 - I_1 R_1 ,$$

from Kirchhoff’s laws. We’re given everything except the currents, and no terms have two currents multiplied by each-other in them. We can rewrite these equations and adjust the spacing a bit to get

$$0 = -I_1 + I_2 + I_3$$
$$\epsilon_2 = R_1 I_1 + R_2 I_2 + 0 \cdot I_3$$
$$\epsilon_3 = R_1 I_1 + 0 \cdot R_2 + R_3 I_3 .$$

We can see that each line has the same format

$$b = a_1 I_1 + a_2 I_2 + a_3 I_3 = \sum_{j=1}^{3} a_j I_j ,$$

with constant $a_i$s and $b$s.

We can rewrite the system of equations as a matrix equation

$$\begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ R_1 & R_2 & 0 \\ R_1 & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

Comparing this to Eqns. 4–6, we can see that all we’ve done is erased a few equals signs in the middle, drawn some big parenthesis, and dragged the unknown currents off by themselves, giving them their own parenthesis. We’ve also rotated order we wrote the currents in

$$\spadesuit I_1 + \blacklozenge I_2 + \blacklozenge I_3 \rightarrow \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}.$$ 

Things look fairly different at first, so go back and compare Eqn. 8 with Eqns. 4–6 until you get comfortable with the changes.

2 Solving matrix equations

All the things we could do when we had regular equations, we can still do with the equations written in matrix form. We’ll go through and solve these side by side so you can see a solution in action and become more familiar with the matrix notation. The only difference in solving strategy is that we keep all the symbols on the same side they started on, and add or subtract equations instead of plugging in.

$$0 = -I_1 + I_2 + I_3$$
$$\epsilon_2 = R_1 I_1 + R_2 I_2 + 0$$
$$\epsilon_3 = R_1 I_1 + 0 + R_3 I_3$$

Let’s solve the middle equation for $I_2$ by dividing by $R_2$.

$$0 = -I_1 + I_2 + I_3$$
$$\frac{\epsilon_2}{R_2} = \frac{R_1 I_1}{R_2} + \frac{I_2}{R_2} + 0$$
$$\epsilon_3 = R_1 I_1 + 0 + R_3 I_3$$

$$\begin{pmatrix} 0 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ \frac{R_1}{R_2} & \frac{R_2}{R_2} & 0 \\ \frac{R_1}{R_2} & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \frac{\epsilon_2}{R_2} \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ \frac{R_1}{R_2} & \frac{R_2}{R_2} & 0 \\ \frac{R_1}{R_2} & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$

$$\frac{\epsilon_2}{R_2} = \frac{R_1 I_1}{R_2} + \frac{I_2}{R_2} + 0$$
$$\epsilon_3 = R_1 I_1 + 0 + R_3 I_3$$

$$\begin{pmatrix} 0 \\ \frac{\epsilon_2}{R_2} \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ \frac{R_1}{R_2} & \frac{R_2}{R_2} & 0 \\ \frac{R_1}{R_2} & 0 & R_3 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix}$$
Now we can subtract (just like plugging in) the second equation to the first, to get rid of the $I_2$ in the first equation in terms of $I_1$.

\[
\frac{\epsilon_2}{R_2} = -\left( \frac{R_1}{R_2} + 1 \right) I_1 + 0 + I_3 \quad \left( \frac{\epsilon_2}{R_2} \right) = \left( \begin{array}{ccc} \frac{R_1}{R_2} + 1 & 0 & 1 \\ \frac{R_1}{R_2} & 1 & 0 \\ R_1 & 0 & R_2 \end{array} \right) \left( \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} \right)
\] (12)

Now lets solve the last equation for $I_3$ by dividing by $R_3$...

\[
\frac{\epsilon_1}{R_3} = -\left( \frac{R_1}{R_3} \right) I_1 + I_2 + 0 \quad \left( \frac{\epsilon_1}{R_3} \right) = \left( \begin{array}{ccc} \frac{R_1}{R_3} & 1 & 0 \\ R_1 & 0 & R_3 \end{array} \right) \left( \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} \right)
\] (13)

...and subtracting the last equation to the first, to get rid of the $I_3$ in the first equation in terms of $I_1$.

\[
\frac{\epsilon_3}{R_3} - \frac{\epsilon_2}{R_2} = -\left( \frac{R_1}{R_3} + \frac{R_1}{R_2} + 1 \right) I_1 + 0 + 0 \quad \left( \frac{\epsilon_3}{R_3} - \frac{\epsilon_2}{R_2} \right) = \left( \begin{array}{ccc} \frac{R_1}{R_3} + \frac{R_1}{R_2} + 1 & 0 & 0 \\ \frac{R_1}{R_3} & 1 & 0 \\ R_1 & 0 & R_3 \end{array} \right) \left( \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} \right)
\] (14)

Now we have an equation with just $I_1$ and the $R_i$ and $\epsilon_i$ that were given. Dividing through by the junk in front of $I_1$ we have

\[
\frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_2}{R_2} + 1}{\frac{R_1}{R_3} + \frac{R_1}{R_2}} = I_1 + 0 + 0 \\
\frac{\epsilon_1}{R_3} = \frac{R_1}{R_3} I_1 + I_2 + 0 \\
\frac{\epsilon_3}{R_3} = \frac{R_1}{R_3} I_1 + 0 + I_3
\]

A solution! We know have an explicit expression for $I_1$. Going back and subtracting $R_1/R_2$ time the first equation from the middle lets us solve for $I_2$.

\[
\frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_2}{R_2} + 1}{\frac{R_1}{R_3} + \frac{R_1}{R_2}} - \frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_2}{R_2} + 1}{\frac{R_1}{R_3} + \frac{R_1}{R_2}} = 0 + I_2 + 0 \\
\frac{\epsilon_3}{R_3} - \frac{\epsilon_2}{R_2} - \frac{\epsilon_3}{R_3} - \frac{\epsilon_2}{R_2} = 0 + I_3
\]

And finally subtracting $R_1/R_3$ time the first equation from the last lets us solve for $I_3$.

\[
\frac{\frac{\epsilon_3}{R_3} + \frac{\epsilon_2}{R_2} + 1}{\frac{R_1}{R_3} + \frac{R_1}{R_2}} = I_1 + 0 + 0 \\
\frac{\epsilon_1}{R_3} = \frac{R_1}{R_3} I_1 + 0 + I_3
\]

Yuck! Why would anyone want to solve equations like that? You have to do all the same work in the matrix version, but you’re rewriting everything at each step! What gives? The power of linear algebra is that, while it is long and tedious for us (see above), it is very easy for a computer (see below). Understanding why means taking a step back and getting an abstract view of the solution we just worked through.

### 3 Solving matrix equations with a computer

When we started, we had an equation like this

\[
[\epsilon] = [R] \cdot [I]
\] (18)

Wait a second, the $\epsilon$s were voltages, so that looks a lot like plain old $V = IR$! We know how to get the current then, it’s just

\[
R^{-1} V = I.
\] (19)

Maybe we can think about our matrix equation like this, we just need to figure out what $[R]^{-1}$ means. What is $R^{-1}$ anyway? It’s the inverse of $R$; the thing that, when multiplied by $R$, gives one. $[R]^{-1}$ is also just the inverse of $[R]$, so $[R]^{-1} \cdot [R] = 1$. 

Inverting a matrix by hand is basically what we were doing in our solution above, but if we can get our computer (or calculator) to find \([R]^{-1}\) for us, we don't have to do any of the messy algebra. We can just get our solution via

\[
[R]^{-1} [c] = [I] \tag{20}
\]

On the TI-83+, that's pretty much all there is to it. You can enter your \([R]\) and \([c]\) matrices in the \([\text{2nd}] \ \text{MATRX} \rightarrow \text{EDIT}\) menu (calling them \([A]\) and \([B]\)). Then just type out

\[
[A]^{-1} \ast B \tag{21}
\]

using the \([\text{2nd}] \ \text{MATRX} \rightarrow \text{NAMES}\) menu to generate the \([A]\) and \([B]\) symbols.

On the TI-89, you can enter the matrices straight from the command line, using commas to separate the columns, and semi-colons to separate the rows. Using the numbers from Problem 35 that gives

\[
\begin{pmatrix}-1, 1, 1; 8, 6, 0; 8, 0, 4 \end{pmatrix} \rightarrow A \tag{22}
\]
\[
\begin{pmatrix}0; 4; 12 \end{pmatrix} \rightarrow I \tag{23}
\]
\[
A^{-1} \ast I \tag{24}
\]
\[
\begin{pmatrix}0.8462 \\ -0.4615 \\ 1.3077\end{pmatrix} \tag{25}
\]

(I don't have a TI-89, so if this is wrong, let me know...).