The Relation of Energy to Momentum

Thus far, we've described how to translate between space and time intervals from one frame to another. And beyond a constant speed of light, we've introduced very little actual physics into the mix. Where's momentum? Where's energy?

I want to begin by reminding you of the fundamental definition of energy:

$$\Delta E = v\Delta p \tag{1}$$

Remember, we derived this from the Work-Kinetic Energy theorem. Turning this into a derivative, we get:

$$\frac{dE}{dp} = v$$

Remind you of anything?

How about:

$$\frac{dx}{dt} = v$$

In other words, if we switch frames, momentum and energy chance. In the transforms, momentum plays the same role as time, and energy plays the same role as position. Thus:

$$E = \gamma E' + v\gamma p' \tag{2}$$

$$p = \gamma p' + \gamma\frac{v}{c^2}E' \tag{3}$$

But wait! There's more! Consider a particle to be stationary in the primed frame. There sits a particle with some (unknown) energy,$E' = E_0$, and $p' = 0$. Now, do a Lorentz transform into the unprimed frame with a velocity $v << c$. In this case, $\gamma \simeq 1$, and $p = mv$, since after all, we're in the Newtonian limit.

Thus, according to equation 3:

$$mv = 0 + \frac{v}{c^2}E_0$$

or, rearranging:

$$E_0 = mc^2 \tag{4}$$

Even the most stoic among you must admit that this is freakin' awesome!

But now that we know the rest energy, we can solve for momentum in general. Plugging in the energy to equation 3:

$$\begin{aligned} p &= 0 + \gamma\frac{v}{c^2}mc^2 \\ &= mv\gamma \qquad (5) \\ & \qquad\qquad (6) \end{aligned}$$

Exactly as we asserted in the first place!

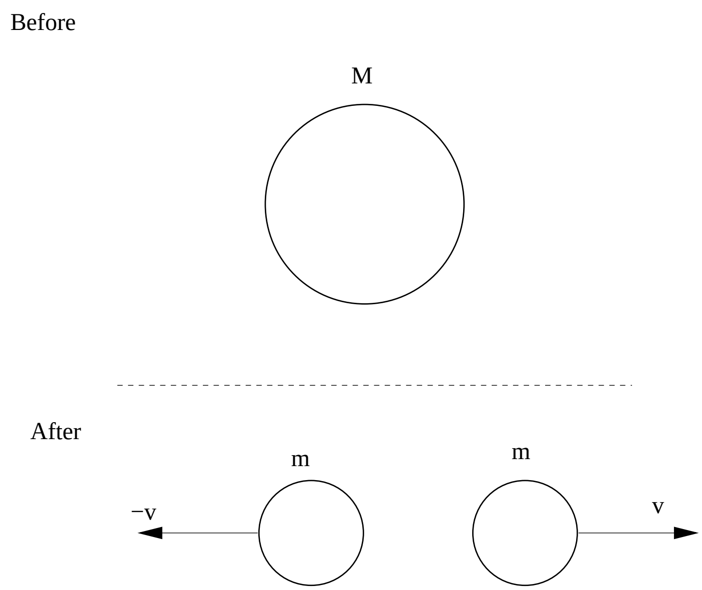
Or, using equation 2, we find:

$$E = \gamma mc^2 \tag{7}$$

An example using an explosion

Okay, so perhaps you have a little trouble buying into my math tricks. Who could blame you? To demonstrate how all of this fits together, we'll do a little example.

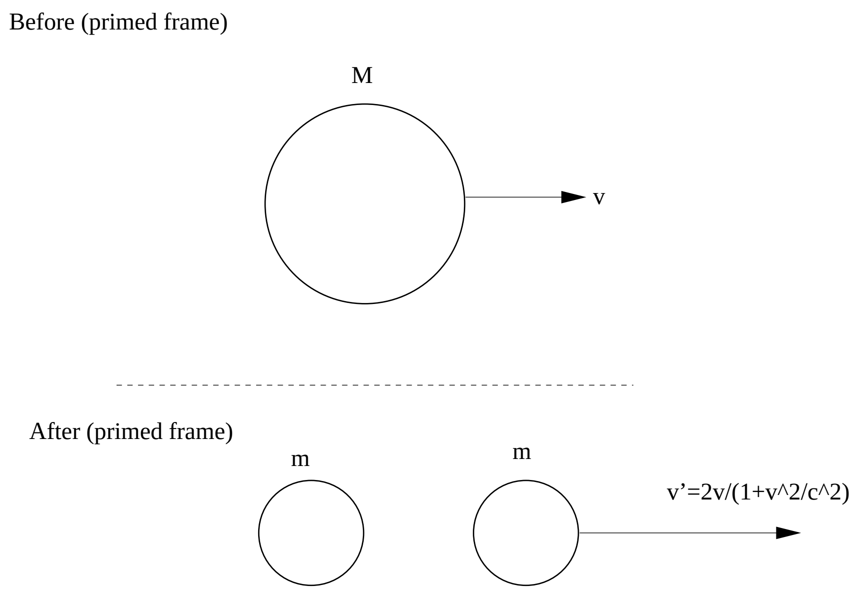
> Now, the important thing to remember here is that we physicists love conserved quantities. Momentum and Energy should be conserved no matter who is looking at the experiment. If they're not conserved, we've done something wrong.



This is what we call a nuclear bomb. We start with a big mass (at rest), and it explodes into two equal parts (each of mass, m).

Now, before getting into detail, the nice thing about this explosion is that it is symmetric. No matter whether you use the relativistic or non-relativistic version of momentum, the explosion seems reasonable since as much of the material flies to the left as to the right. Thus, momentum is conserved.

However, this is *not* necessarily true if we look at the same explosion in a frame moving to the left at speed v:



Now, if Newton came along and looked at the explosion from *this* point of view, he'd be baffled. The mass of each of the pieces of debris should just be half that of the original, and thus, the total momentum before the explosion is:

$$p_i^{(Newton)} = Mv$$

while the total momentum afterwards is:

$$p_f^{(Newton)} = \frac{M}{2}\frac{2v}{1+v^2/c^2}$$

which is not the same thing at all! In other words, our traditional interpretation of momentum does not work.

We can get around this if we use the form of the momentum:

$$p = mv\gamma \tag{8}$$

Well, with one more complication. In Einstein's view, the energy of a particle (kinetic+mass energy) is given by:

$$E = mc^2\gamma \tag{9}$$

where γ is based on the velocity that the particle is moving in a particular frame. Unsuprisingly, a particle which is at rest according to one observer, and moving according to another, will appear to have a higher energy according to the moving observer.

Because the mass energy is so big, it is a huge reservoir for kinetic energy, and thus is no longer conserved. How much mass is lost? Looking at it in the unprimed frame, the initial energy is:

$$E_i = Mc^2$$

(since the bomb is initially at rest $\gamma = 1$).

The final energy is:

$$E_f = 2(mc^2\gamma)$$

where the factor of "2" comes from the fact that there are two pieces of debris and the $mc^2\gamma$ includes all of the energy (mass energy+kinetic energy). Thus, conservation of energy gives us:

$$\begin{aligned} 2mc^2\gamma &= Mc^2 \\ \frac{2m}{M} &= \frac{1}{\gamma} \end{aligned} \qquad (10)$$

$$(11)$$

In other words, if the explosion is very mild (small v), then γ will be close to 1, and it looks like mass will be more or less conserved. However, if the pieces fly off at, say, $v = c/2$, then $\gamma = 1.15$, and thus, about 13.4% of the mass will be converted into energy!

Putting some numbers in.

The unprimed frame:

Let's do a concrete example to show that our approach does work. Consider a bomb which has an initial mass of 1kg. Its rest energy is thus:

$$E_i = 1kg \times c^2 = 9 \times 10^{16} J$$

It then explodes into two equal masses, each with a speed of $0.8c$, giving us $\gamma = 1.67$ (plug into the γ equation and verify for yourself!). Thus, according to equation 11, we have:

$$\frac{2m}{1kg} = \frac{1}{1.666} = 0.6$$

In other words, each of the two particles will have a mass of $0.3kg$. However, $\Delta M = 0.4kg$ were converted into energy.

Using this, energy (because that's how we *derived* the mass loss relation) and momentum (because it's zero before and after) are conserved in the rest (unprimed) frame.

The primed frame:

The primed frame is more complicated. Imagine that we're running to the left at a speed 0.8c. Before the explosion, the big mass is moving within that frame at a speed $u'_M = 0.8c$ ($\gamma = 1.67$). One of the pieces of debris is at rest, while the other is moving at a speed:

$$u'_m = \frac{0.8c + 0.8c}{1 + 0.8c \times 0.8c/c^2} = 0.976c$$

which puts $\gamma' = 4.55$.

So, before the explosion, the big mass has a momentum of:

$$\begin{aligned} p'_M &= mu'_M\gamma \\ &= 1kg \times (0.8 \times 3 \times 10^8 m/s) \times 1.67 \\ &= 4 \times 10^8 kg\ m/s \end{aligned}$$

Likewise, the energy is:

$$\begin{aligned} E'_M &= Mc^2\gamma \\ &= 1kg \times (3 \times 10^8 m/s)^2 \times 1.67 \\ &= 1.5 \times 10^{17} J \end{aligned}$$

Now, after the explosion, the stationary mass obviously has zero momentum, while the moving piece has a momentum of:

$$\begin{aligned} p'_m &= 0.3kg \times (0.976 \times 3 \times 10^8 m/s) \times 4.55 \\ &= 4 \times 10^8 kg\ m/s \end{aligned}$$

And momentum is conserved! (As it must be)

Likewise, the energy afterwards from both the stationary and the moving pieces are:

$$\begin{aligned} E'_{fin} &= mc^2 + mc^2\gamma' \\ &= 0.3kg \times (3 \times 10^8 m/s)^2 + 0.3kg \times (3 \times 10^8 m/s)^2 \times 4.55 \\ &= 1.5 \times 10^{17} J \end{aligned}$$

Tada!

<u>A real world example</u>

You'll notice that the number of Joules, above, tend to be enormous, and thus particle physicists tend to work in units of energy called MeV (mega-electronvolts), and units of mass of MeV/c^2. It cuts down on the exponents.

So, consider the following relation (the basic nuclear fusion relation we've discussed earlier):

$$4H \rightarrow +2e^- + {}^4He + 2\nu + \gamma \tag{12}$$

In this case, we find that the total masses on the left hand side of the equation are considerably less than the masses on the right.

$m_p = 938.3 MeV/c^2$, $m_{He} = 3728.1 MeV/c^2$, and $m_e = 0.511 MeV/c^2$.

So, $E_i = 3754.2$, $E_f = 3728.1$, so, $\Delta E = 26.1 MeV$.

Where does it go? It goes into the kinetic energy of the resulting helium atom, and the energy of the neutrinos and photons.