Chapter 1

Interactions and Motion

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This course deals with the nature of matter and its interactions. The variety of phenomena that we will be able to explain and understand is very wide, including the orbit of stars around a black hole, nuclear fusion, and the speed of sound in a solid.

> The main goal of this course is to have you engage in a process central to science: the attempt to explain in detail a broad range of phenomena using a small set of powerful fundamental principles.

> The specific focus is on learning how to model the nature of matter and its interactions in terms of a small set of physical laws that govern all mechanical interactions, and in terms of the atomic structure of matter.

This first chapter introduces the notion of interactions and the changes they produce. The major topics are:
- The kinds of matter we will deal with
- How to detect interactions
- Precise description of position and motion in 3D space
- Momentum

1.1 Kinds of matter

In this course we will deal with material objects of many sizes, from subatomic particles to galaxies. All of these objects have certain things in common.

Atoms and nuclei

Ordinary matter is made up of tiny atoms. An atom isn't the smallest type of matter, for it is composed of even smaller objects (electrons, protons, and neutrons), but many of the ordinary everyday properties of ordinary matter can be understood in terms of atomic properties and interactions. As you probably know from studying chemistry, atoms have a very small, very dense core, called the nucleus, around which is found a cloud of electrons. The nucleus contains protons and neutrons, collectively called nucleons. Electrons are kept close to the nucleus by electric attraction to the protons (the neutrons don't interact with the electrons).

> **?** Recall your previous studies of chemistry. How many protons and electrons are there in a hydrogen atom? In helium or carbon atoms?

Throughout this text you will encounter questions like the preceding one, which ask you to stop and think before reading further. An important part of reading and understanding a scientific text is to ask yourself questions and to try to answer them. You will learn more from reading this text if you try to answer these questions before looking at the discussion in the subsequent paragraph.

If you don't remember the properties of these atoms, see the periodic table on the inside front cover of this textbook. Hydrogen is the simplest atom, with just one proton and one electron. A helium atom has two protons and two electrons. A carbon atom has six protons and six electrons. Near the other end of the chemical periodic table, a uranium atom has 92 protons and 92 electrons. Figure 1.1 shows the approximate cloud of electrons for

Hydrogen
1 electron

$1{\times}10^{-10}$ m

Carbon
6 electrons
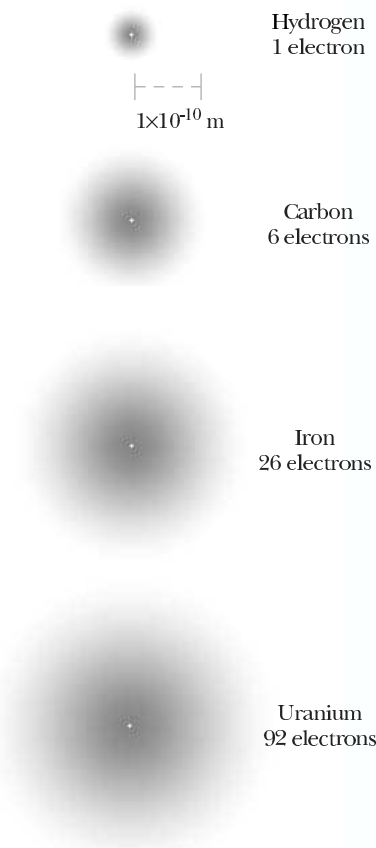
Iron
26 electrons

Uranium
92 electrons

Figure 1.1 Atoms of hydrogen, carbon, iron, and uranium. The white dot shows the location of the nucleus. On this scale, however, the nucleus would be much too small to see.

several elements but cannot show the nucleus to the same scale; the tiny dot marking the nucleus in the figure is much larger than the actual nucleus.

The radius of the electron cloud for a typical atom is about 1×10^{-10} meter. The reason for this size can be understood using the principles of quantum mechanics, a major development in physics in the early 20th century. The radius of a proton is about 1×10^{-15} meter, very much smaller than the radius of the electron cloud.

Nuclei contain neutrons as well as protons (Figure 1.2). The most common form or "isotope" of hydrogen has no neutrons in the nucleus. However, there exist isotopes of hydrogen with one or two neutrons in the nucleus (in addition to the proton). Hydrogen atoms containing one or two neutrons are called deuterium or tritium. The most common isotope of helium has two neutrons (and two protons) in its nucleus, but a rare isotope has only one neutron; this is called helium-3.

The most common isotope of carbon has six neutrons together with the six protons in the nucleus (carbon-12), while carbon-14 with eight neutrons is an isotope that plays an important role in dating archeological objects.

Near the other end of the periodic table, uranium-235, which can undergo a fission chain reaction, has 92 protons and 143 neutrons, while uranium-238, which does not undergo a fission chain reaction, has 92 protons and 146 neutrons.

Molecules and solids

When atoms come in contact with each other, they may stick to each other ("bond" to each other). Several atoms bonded together can form a molecule—a substance whose physical and chemical properties differ from those of the constituent atoms. For example, water molecules (H_2O) have properties quite different from the properties of hydrogen atoms or oxygen atoms.

An ordinary-sized rigid object made of bound-together atoms and big enough to see and handle is called a solid, such as a bar of aluminum. A new kind of microscope, the scanning tunneling microscope (STM), is able to map the locations of atoms on the surface of a solid, which has provided new techniques for investigating matter at the atomic level. Two such images appear in Figure 1.3. You can see that atoms in a crystalline solid are arranged in a regular three-dimensional array. The arrangement of atoms on the surface depends on the direction along which the crystal is cut. The irregularities in the bottom image reflect "defects," such as missing atoms, in the crystal structure.

Liquids and gases

When a solid is heated to a higher temperature, the atoms in the solid vibrate more vigorously about their normal positions. If the temperature is raised high enough, this thermal agitation may destroy the rigid structure of the solid. The atoms may become able to slide over each other, in which case the substance is a liquid.

At even higher temperatures the thermal motion of the atoms or molecules may be so large as to break the interatomic or intermolecular bonds completely, and the liquid turns into a gas. In a gas the atoms or molecules are quite free to move around, only occasionally colliding with each other or the walls of their container.

In this course we will learn how to analyze many aspects of the behavior of solids and gases. We won't have much to say about liquids, because their properties are much harder to analyze. Solids are simpler to analyze than liquids because the atoms stay in one place (though with thermal vibration about their usual positions). Gases are simpler to analyze than liquids be-

Hydrogen nucleus
1 proton

1×10^{-15} m

Deuterium nucleus
1 proton + 1 neutron

Tritium nucleus
1 proton + 2 neutrons

Helium-3 nucleus
2 protons +1neutron

Helium-4 nucleus
2 protons + 2 neutrons
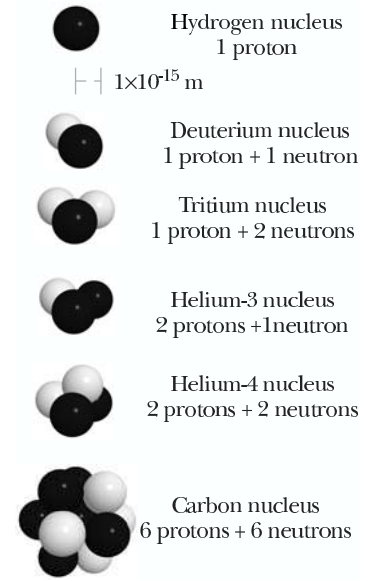
Carbon nucleus
6 protons + 6 neutrons

Figure 1.2 Nuclei of hydrogen, helium, and carbon. Note the *very* much smaller scale than in Figure 1.1!
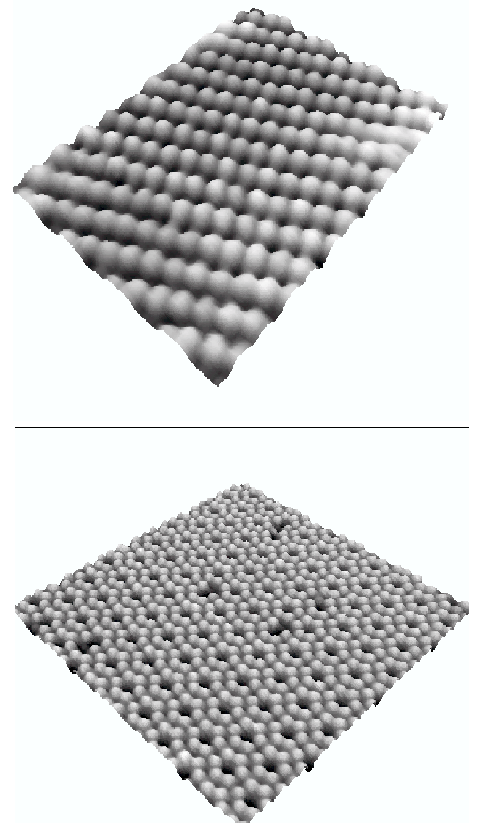

Figure 1.3 Two different surfaces of a crystal of pure silicon. The images were made with a scanning tunneling microscope.
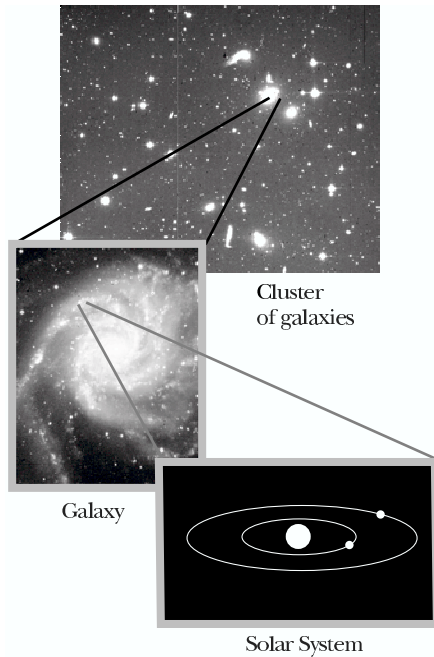
Figure 1.4 Our Solar System exists inside a galaxy, which itself is a member of a cluster of galaxies.

cause between collisions the gas molecules are approximately unaffected by the other molecules. Liquids are the awkward intermediate state, where the atoms move around rather freely, but always in contact with other atoms. This makes the analysis of liquids very complex.

Planets, stars, solar systems, and galaxies

In our brief survey of the kinds of matter that we will study, we make a giant leap in scale from atoms all the way up to planets and stars, such as our Earth and Sun. In this course we will see that many of the same principles that apply to atoms apply to planets and stars. By making this leap we bypass an important physical science, geology, whose domain of interest includes the formation of mountains and continents. We will study objects that are much bigger than mountains, and we will study objects that are much smaller than mountains, but we don't have time in one course to apply the principles of physics to every important kind of matter.

Our Sun and its accompanying planets constitute our Solar System (Figure 1.4). It is located in the Milky Way galaxy, a giant rotating disk-shaped system of stars. On a clear dark night you can see a band of light (the Milky Way) coming from the huge number of stars lying in this disk, which you are looking at from a position in the disk, about two-thirds of the way out from the center of the disk. Our galaxy is a member of a cluster of galaxies that move around each other much as the planets of our Solar System move around the Sun. The Universe contains many such clusters of galaxies.

1.2   Detecting interactions

Objects made of different kinds of matter interact with each other in various ways: gravitationally, electrically, magnetically, and through the strong and weak interactions. How can we detect that an interaction has occurred? In this section we consider various kinds of observations that indicate the presence of interactions.

? Before you read further, take a moment to think about your own ideas of interactions. How can you tell that two objects are interacting with each other?

1.2.1   Change of direction

Suppose you observe a proton moving through a region of outer space, far from almost all other objects. The proton moves along a path like the one shown in Figure 1.5. The arrow indicates the initial direction of the proton's motion, and the "x's" in the diagram indicate the position of the proton at equal time intervals.

? Do you see evidence that the proton is interacting with another object?

Evidently a change in direction is a vivid indicator of interactions. If you observe a change in direction of the motion of a proton, you will find another object somewhere that has interacted with this proton.

? Suppose that the only other object nearby was another proton. What was the approximate initial location of this second proton?

Figure 1.5 A proton moves through space, far from almost all other objects. The initial direction of the proton's motion is upward, as indicated by the arrow. The x's represent the position of the proton at equal time intervals.

Since two protons repel each other electrically, the second proton must have been located to the right of the bend in the first proton's path.

1.2.2   Change of speed

Suppose that you observe an electron traveling in a straight line through outer space far from almost all other objects (Figure 1.6). The path of the electron is shown as though a camera had taken multiple exposures at equal time intervals.

? Where is the electron's speed largest? Where is the electron's speed smallest?

The speed is largest at the top, where the dots are farther apart. It is smallest at the bottom, where the dots are closer together.

? Suppose that the only other object nearby was another electron. What was the approximate initial location of this other electron?

The other electron must have been located directly below the starting point of the path, since electrons repel each other electrically.

Evidently a change in speed is an indicator of interactions. If you observe a change in speed of an electron, you will find another object somewhere that has interacted with the electron.

1.2.3 Change of velocity: change of speed or direction

In physics, the word "velocity" has a special technical meaning which is different from its meaning in everyday speech. In physics, the quantity called "velocity" indicates a combination of speed and direction. (In contrast, in everyday speech, "speed" and "velocity" are often used as synonyms. In physics, however, all words have precise meanings and there are no synonyms.)

For example, consider an airplane that is flying with a speed of 1000 kilometers/hour in a direction that is due east. We say the velocity is 1000 km/hr, east, where we specify both speed and direction. An airplane flying west with a speed of 1000 km/hr would have the same speed, but a different velocity.

We have seen that a change in an object's speed, or a change in the direction of its motion, indicates that the object has interacted with at least one other object. The two indicators of interaction, change of speed and change of direction, can be combined into one compact statement:

> A change of velocity (speed or direction or both) indicates the existence of an interaction.

Diagrams showing changes in velocity

In physics diagrams, the velocity of an object is represented by an arrow: a line with an arrowhead. The tail of the arrow is placed at the location of the object, and the arrow points in the direction of the motion of the object. The length of the arrow is proportional to the speed of the object. Figure 1.7 shows two successive positions of a particle at two different times, with velocity arrows indicating a change in speed of the particle (it's slowing down). Figure 1.8 shows three successive positions of a different particle at three different times, with velocity arrows indicating a change in direction but no change in speed.

We will see a little later in the chapter that velocity is only one example of a physical quantity that has a "magnitude" (an amount or a size) and a direction. Other examples of such quantities are position relative to an origin in 3D space, force, and magnetic field. Quantities having magnitude and direction can be usefully described as "vectors". Vectors are mathematical quantities which have their own special rules of algebra, similar (but not identical) to the rules of ordinary algebra. Arrows are commonly used in di-

Figure 1.6 An electron moves through space, far from almost all other objects. The initial direction of the electron's motion is upward, as indicated by the arrow. The x's represent the position of the electron at equal time intervals.

Figure 1.7 Two successive positions of a particle (indicated by a dot), with arrows indicating the velocity of the particle at each location. The shorter arrow indicates that the speed of the particle at location 2 is less than its speed at location 1.
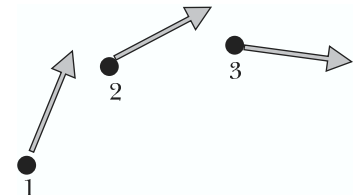

Figure 1.8 Three successive positions of a particle (indicated by a dot), with arrows indicating the velocity of the particle at each location. The arrows are the same length, indicating the same speed, but they point in different directions, indicating a change in direction and therefore a change in velocity.

agrams to denote vector quantities. We will use vectors extensively in this course.

1.2.4 Uniform motion

Suppose you observe a rock moving along in outer space far from all other objects. We don't know what made it start moving in the first place; presumably a long time ago an interaction gave it some velocity and it has been coasting through the vacuum of space ever since.

It is an observational fact that such an isolated object moves at constant, unchanging speed, in a straight line. Its velocity does not change (neither its direction nor its speed changes). We call such motion with unchanging velocity "uniform motion" (Figure 1.9).

An object at rest

A special case of uniform motion is the case in which an object's speed is zero and remains zero—the object remains at rest. In this case the object's speed is constant (zero) and the direction of motion, while undefined, is not changing.

Uniform motion implies no net interaction

When we observe an object in uniform motion, we conclude that since its velocity is not changing, either it is not interacting significantly with any other object, or else it is undergoing multiple interactions that cancel each other out. In either case, we can say that there is no "net" (total) interaction.

Figure 1.9 "Uniform motion"—no change in speed or direction.

1.3 Newton's first law of motion

The basic relationship between change of velocity and interaction is summarized qualitatively by Newton's "first law of motion":

NEWTON'S FIRST LAW OF MOTION

**An object moves in a straight line and at constant speed
except to the extent that it interacts with other objects.**

The words "to the extent" imply that the stronger the interaction, the more change there will be in direction and/or speed. The weaker the interaction, the less change. If there is no net interaction at all, the direction doesn't change and the speed doesn't change (uniform motion). This case can also be called "uniform velocity" or "constant velocity," since velocity refers to both speed and direction. It is important to remember that if an object is not moving at all, its velocity is not changing, so it too may be considered to be in uniform motion.

Newton's first law of motion is only qualitative, because it doesn't give us a way to calculate quantitatively how much change in speed or direction will be produced by a certain amount of interaction, a subject we will take up in the next chapter. Nevertheless, Newton's first law of motion is important in providing a conceptual framework for thinking about the relationship between interaction and motion.

The English physicist Isaac Newton was the first person to state this law clearly. Newton's first law of motion represented a major break with ancient tradition, which assumed that constant pushing was required to keep something moving. This law says something radically different: no interactions at all are needed to keep something moving!

Does Newton's first law apply in everyday life?

Superficially, Newton's first law of motion may at first seem not to apply to many everyday situations. To push a chair across the floor at constant speed, you have to keep pushing all the time.

> **?** Doesn't Newton's first law of motion say that the chair should keep moving at constant speed without anyone pushing it? In fact, shouldn't the speed or direction of motion of the chair change due to the interaction with your hands? Does this everyday situation violate Newton's first law of motion? Try to answer these questions before reading farther.

The complicating factor here is that your hands aren't the only objects that are interacting with the chair. The floor also interacts with the chair, in a way that we call friction. If you push just hard enough to compensate exactly for the floor friction, the sum of all the interactions is zero, and the chair moves at constant speed as predicted by Newton's first law. (If you push harder than the floor does, the chair's speed does increase.)

Motion without friction

It is difficult to observe motion without friction in everyday life, because objects almost always interact with many other objects, including air, flat surfaces, etc. This explains why it took people such a long time (Newton lived in the 1600's) to understand clearly the relationship between interaction and change.

You may be able to think of situations in which you have seen an object keep moving at constant (or nearly constant) velocity, without being pushed or pulled. One example of a nearly friction-free situation is a hockey puck sliding on ice. The puck slides a long way at nearly constant speed in a straight line (constant velocity) because there is little friction with the ice. An even better example is the uniform motion of an object in outer space, far from all other objects.

Exercises

> At the end of a section you will usually find exercises such as those that follow. It is important to work through exercises as you come to them, to make sure that you can apply what you have just read. Simply reading about concepts in physics is not enough—you must be able to use the concepts in answering questions and solving problems.
>
> Make a serious attempt to do an exercise before checking the answer at the end of the chapter. This will help you assess your own understanding.

Ex. 1.1 Which of the following objects are moving with constant velocity?
(a) A ship sailing northeast at a speed of 5 meters per second
(b) The moon orbiting the Earth
(c) A tennis ball traveling across the court after having been hit by a tennis racket
(d) A can of soda sitting on a table
(e) A person riding on a Ferris wheel which is turning at a constant rate

Ex. 1.2 Apply Newton's first law to each of the following situations. In which situations can you conclude that the object must have undergone a net interaction with one or more other objects?
(a) A book slides across the table and comes to a stop
(b) A proton in a particle accelerator moves faster and faster

(c) A car travels at constant speed around a circular race track

(d) A spacecraft travels at a constant speed toward a distant star

(e) A hydrogen atom remains at rest in outer space

Ex. 1.3 A spaceship far from all other objects uses its thrusters to attain a speed of 10^4 m/s. The crew then shuts off the power. According to Newton's first law, what will happen to the motion of the spaceship from then on?

1.4 Other indicators of interaction

Change of identity

Change of velocity (change of speed and/or direction) is not the only indicator of interactions. Another is change of identity, such as the formation of water (H_2O) from the burning of hydrogen in oxygen. A water molecule behaves very differently from the hydrogen and oxygen atoms of which it is made.

Change of shape or configuration

Another indicator of interaction is change of shape or configuration (arrangement of the parts). For example, slowly bend a pen or pencil, then hold it in the bent position. The speed hasn't changed, nor is there a change in the direction of motion (it's not moving!). The pencil has not changed identity. Evidently a change of shape can be evidence for interactions, in this case with your hand.

Other changes in configuration include "phase changes" such as the freezing or boiling of a liquid, brought about by interactions with the surroundings. In different phases (solid, liquid, gas), atoms or molecules are arranged differently. Changes in configuration at the atomic level are another indication of interactions.

Change of temperature

Another indication of interaction is change of temperature. Place a pot of cold water on a hot stove. As time goes by, a thermometer will indicate a change in the water due to interaction with the hot stove.

Other indications of interactions

Is a change of position an indicator of an interaction? That depends. If the change of position occurs simply because a particle is moving at constant speed and direction, then a mere change of position is not an indicator of an interaction, since uniform motion is an indicator of no net interaction.

? If however you observe an object at rest in one location, and later you observe it again at rest but in a different location, did an interaction take place?

Yes. You can infer that there must have been an interaction to give the object some velocity to move the object toward the new position, and another interaction to slow the object to a stop in its new position.

In later chapters we will consider interactions involving change of identity, change of shape, and change of temperature, but for now we'll concentrate on interactions that cause a change of velocity (speed and/or direction).

1.4.1 Indirect evidence for an interaction

Sometimes there is indirect evidence for an interaction. When something doesn't change although you would normally expect a change due to a known interaction, this indicates that another interaction is present. Consider a balloon that hovers motionless in the air despite the downward gravitational pull of the Earth. Evidently there is some other kind of interaction that opposes the gravitational interaction. In this case, interactions with air molecules have the net effect of pushing up on the balloon ("buoyancy"). The lack of change implies that the effect of the air molecules exactly compensates for the gravitational interaction with the Earth.

When you push a chair across the floor and it moves with constant velocity despite your pushing on it (which ought to change its speed), that means that something else must also be interacting with it (the floor).

The stability of the nucleus of an atom is another example of indirect evidence for an interaction. The nucleus contains positively charged protons that repel each other electrically, yet the nucleus remains intact. We conclude that there must be some other kind of interaction present, a nonelectric attractive interaction that overcomes the electric repulsion. This is evidence for the "strong interaction" that acts between protons and neutrons in the nucleus.

1.4.2 Summary: changes as indicators of interactions

Here then are the most common indicators of interactions:
* change of velocity (change of direction and/or change of speed)
* change of identity
* change of shape
* change of temperature
* lack of change when change is expected (indirect evidence)

The important point is this: **Interactions cause change.**

In the absence of interactions, there is no change, which is usually uninteresting. The exception is the surprise when nothing changes despite our expectations that something should change. This is indirect evidence for some interaction that we hadn't recognized was present, that more than one interaction is present and the interactions cancel each others' effects.

For the next few chapters we'll concentrate on change of velocity as evidence for an interaction (or lack of change of velocity, which can give indirect evidence for additional interactions).

1.5 Describing the 3D world: Vectors

Physical phenomena take place in the 3D world around us. In order to be able to make quantitative predictions and give detailed, quantitative explanations, we need tools for describing precisely the positions and velocities of objects in 3D, and the changes in position and velocity due to interactions. These tools are mathematical entities called 3D "vectors."

1.5.1 3D coordinates

We will use a 3D coordinate system to specify positions in space and other vector quantities. Usually we will orient the axes of the coordinate system as shown in Figure 1.10: $+x$ axis to the right, $+y$ axis upward, and $+z$ axis coming out of the page, toward you. This is a "right-handed" coordinate system: if you hold the thumb, first, and second fingers of your right hand perpendicular to each other, and align your thumb with the x axis and your first finger with the y axis, your second finger points along the z axis. (In some math and physics textbook discussions of 3D coordinate systems, the y axis points
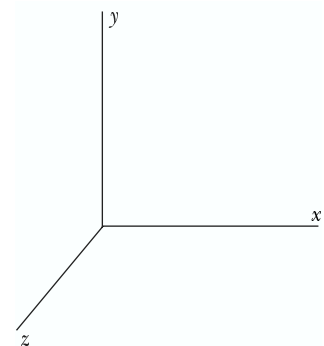

Figure 1.10 Right-handed 3D coordinate system.

out and the z axis points up, but we will also use a 2D coordinate system with y up, so it makes sense always to have the y axis point up.)

1.5.2 Basic properties of vectors: magnitude and direction

A vector is a quantity that has a magnitude and a direction. For example, the velocity of a baseball is a vector quantity. The magnitude of the baseball's velocity is the speed of the baseball, for example 20 meters/second. The direction of the baseball's velocity is the direction of its motion at a particular instant, for example "up" or "to the right" or "west" or "in the $+y$ direction." A symbol denoting a vector is written with an arrow over it:

$$\vec{v} \text{ is a vector.}$$

A position in space can also be considered to be a vector, called a position vector, pointing from an origin to that location. Figure 1.11 shows a position vector that might represent your final position if you started at the origin and walked 4 meters along the x axis, then 2 meters parallel to the z axis, then climbed a ladder so you were 3 meters above the ground. Your new position relative to the origin is a vector that can be written like this:

$$\vec{r} = \langle 4, 3, 2 \rangle \text{ m}$$

x component $r_x = 4$ m
y component $r_y = 3$ m
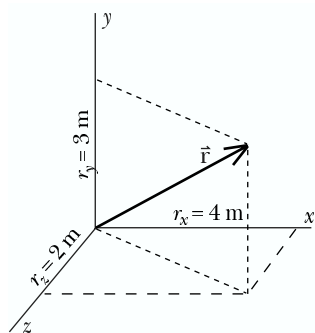z component $r_z = 2$ m

Figure 1.11 A position vector and its x, y, and z components.

In three dimensions a vector is a triple of numbers $\langle x, y, z \rangle$. Quantities like the position of an object and the velocity of an object can be represented as vectors:

$$\vec{r} = \langle x, y, z \rangle \quad \text{(a position vector)}$$

$$\vec{r}_1 = \langle 3.2, -9.2, 66.3 \rangle \text{ m} \quad \text{(a position vector)}$$

$$\vec{v} = \langle v_x, v_y, v_z \rangle \quad \text{(a velocity vector)}$$

$$\vec{v}_1 = \langle -22.3, 0.4, -19.5 \rangle \text{ m/s} \quad \text{(a velocity vector)}$$

Components of a vector

Each of the numbers in the triple is referred to as a "component" of the vector. The x component of the vector $\vec{v}$ is the number v_x. The z component of the vector $\vec{v}_1$ above is -19.5 m/s. A component such as v_x is not a vector, since it is only one number.

It is important to note that the x component of a vector specifies the difference between the x coordinate of the tail of the vector and the x coordinate of the tip of the vector. It does not give any information about the location of the tail of the vector (compare Figure 1.11 and Figure 1.12).
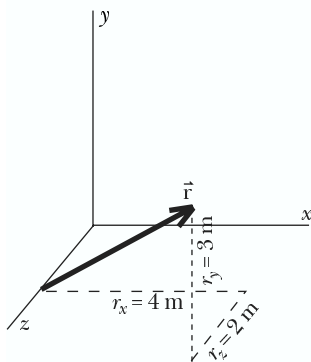
Figure 1.12 The arrow represents the vector $\vec{r} = \langle 4, 3, 2 \rangle$ m, drawn with its tail at location $\langle 0, 0, 2 \rangle$.

1.5.3 Equality of vectors

A vector is equal to another vector if and only if all the components of the vectors are equal. If $\vec{r} = \langle 4, 3, 2 \rangle$ m,

$$\vec{w} = \vec{r} \text{ means that}$$

$$w_x = r_x \text{ and } w_y = r_y \text{ and } w_z = r_z, \text{ so } \vec{w} = \langle 4, 3, 2 \rangle \text{ m}$$

If two vectors are equal, their magnitudes and directions are the same.

1.5.4 Drawing vectors

In Figure 1.11 we represented your position vector relative to the origin graphically by an arrow whose tail is at the origin and whose arrowhead is at your position. The length of the arrow represents the distance from the origin, and the direction of the arrow represents the direction of the vector, which is the direction of a direct path from the initial position to the final position (the "displacement"; by walking and climbing you "displaced" yourself from the origin to your final position).

Since it is difficult to draw a 3D diagram on paper, when working on paper you will usually be asked to draw vectors which all lie in a single plane. Figure 1.13 shows an arrow in the *xy* plane representing the vector $\langle -3, -1, 0 \rangle$.

1.5.5 Vectors and scalars

A quantity which is represented by a single number is called a *scalar*. A scalar quantity does not have a direction. Examples include the mass of an object, such as 5 kg, or the temperature, such as −20 C. Vectors and scalars are very different entities; a vector can never be equal to a scalar, and a scalar cannot be added to a vector. Scalars can be positive or negative:

$$m = 5 \text{ kg}$$
$$T = -20 \text{ C}$$

Ex. 1.4 How many numbers are needed to specify a 3D position vector?

Ex. 1.5 How many numbers are needed to specify a scalar?

Ex. 1.6 Does the symbol $\overset{\circ}{a}$ represent a vector or a scalar?

Ex. 1.7 Which of the following are vectors?
a) 5 m/s, b) $\langle -11, 5.4, -33 \rangle$ m, c) $\overset{\circ}{r}$, d) v_z

Ex. 1.8 $\overset{\circ}{a} = \langle -3, 7, 0.5 \rangle$. If $\vec{b} = \overset{\circ}{a}$, what is the *y* component of $\vec{b}$?

1.5.6 Magnitude of a vector

Consider again the vector in Figure 1.14, showing your displacement from the origin. Using a 3D extension of the Pythagorean theorem for right triangles (Figure 1.15), the net distance you have moved from the starting point is

$$\sqrt{(4 \text{ m})^2 + (3 \text{ m})^2 + (2\text{m})^2} = \sqrt{29} \text{ m} = 5.39 \text{ m}$$

We say that the *magnitude* $|\overset{\circ}{r}|$ of the position vector $\overset{\circ}{r}$ is

$$|\overset{\circ}{r}| = 5.39 \text{ m}$$

The magnitude of a vector is written either with absolute-value bars around the vector as $|\overset{\circ}{r}|$, or simply by writing the symbol for the vector without the little arrow above it, r.

In general, the magnitude of a vector can be calculated by taking the square root of the sum of the squares of its components (see Figure 1.15).

MAGNITUDE OF A VECTOR

If the vector $\overset{\circ}{r} = \langle r_x, r_y, r_z \rangle$ then $|\overset{\circ}{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2}$ (a scalar).
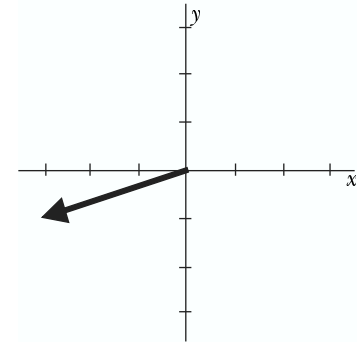

Figure 1.13 The position vector $\langle -3, -1, 0 \rangle$, drawn at the origin, in the *xy* plane. The components of the vector specify the displacement from the tail to the tip. The z axis, which is not shown, comes out of the page, toward you.
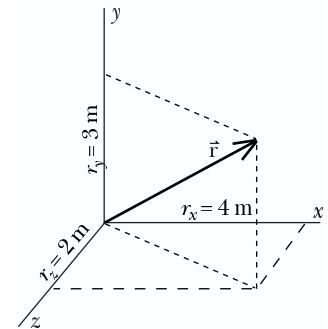

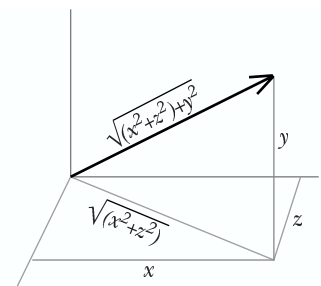Figure 1.14 A vector representing a displacement from the origin.

Figure 1.15 The magnitude of a vector is the square root of the sum of the squares of its components (3D version of the Pythagorean theorem).

Note that the magnitude of a vector is always a positive number. Since the magnitude of a vector is a single number, and not a triple of numbers, it is a scalar, not a vector.

Can a *vector* be positive or negative?

? Consider the vector $\vec{v} = \langle 8\times10^6, 0, -2\times10^7 \rangle$ m/s. Is this vector positive? Negative? Zero?

None of these descriptions is appropriate. The *x* component of this vector is positive, the *y* component is zero, and the *z* component is negative. Vectors aren't positive, or negative, or zero. Their components can be positive or negative or zero, but these words just don't mean anything when used with the vector as a whole.

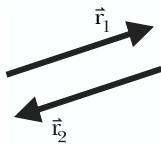On the other hand, the *magnitude* of a vector such as $|\vec{v}|$ is always positive.

Ex. 1.9 Does the symbol $|\vec{v}|$ represent a vector or a scalar?

Ex. 1.10 Consider the vectors $\vec{r}_1$ and $\vec{r}_2$ represented by arrows in Figure 1.16. Are these two vectors equal?

Ex. 1.11 If $\vec{r} = \langle -3, -4, 1 \rangle$ m, find $|\vec{r}|$.

Ex. 1.12 Can the magnitude of a vector be a negative number?

Ex. 1.13 What is the magnitude of the vector $\vec{v}$, where $\vec{v} = \langle 8\times10^6, 0, -2\times10^7 \rangle$ m/s?

Figure 1.16 Two vectors (Exercise 1.10).

1.5.7 Mathematical operations involving vectors

Although the algebra of vectors is similar to the scalar algebra with which you are very familiar, it is not identical. There are some algebraic operations that cannot be performed on vectors.

Algebraic operations that *are* legal for vectors include the following operations, which we will discuss in this chapter:
- adding one vector to another vector: $\vec{a} + \vec{w}$
- subtracting one vector from another vector: $\vec{b} - \vec{d}$
- finding the magnitude of a vector: $|\vec{r}|$
- finding a unit vector (a vector of magnitude 1): $\hat{r}$
- multiplying (or dividing) a vector by a scalar: $3\vec{v}$
- finding the rate of change of a vector: $\dfrac{\Delta\vec{r}}{\Delta t}$ or $\dfrac{d\vec{r}}{dt}$

In later chapters we will also see that there are two more ways of combining two vectors:

the vector dot product, whose result is a scalar

the vector cross product, whose result is a vector

Operations that are *not* legal for vectors

Although vector algebra is similar to the ordinary scalar algebra you have used up to now, there are certain operations that are not legal (and not meaningful) for vectors:

A vector cannot be set equal to a scalar.

A vector cannot be added to or subtracted from a scalar.

A vector cannot occur in the denominator of an expression. (Although you can't divide by a vector, note that you can legally divide by the *magnitude* of a vector, which is a scalar.)

1.5.8 Multiplying a vector by a scalar

A vector can be multiplied (or divided) by a scalar. If a vector is multiplied by a scalar, each of the components of the vector is multiplied by the scalar:

$$\text{If } \vec{r} = \langle x, y, z \rangle \text{ then } a\vec{r} = \langle ax, ay, az \rangle$$

$$\text{If } \vec{v} = \langle v_x, v_y, v_z \rangle \text{ then } \frac{\vec{v}}{b} = \langle \frac{v_x}{b}, \frac{v_y}{b}, \frac{v_z}{b} \rangle$$

$$\left(\frac{1}{2} \right) \langle 6, -20, 9 \rangle = \langle 3, -10, 4.5 \rangle$$

Multiplication by a scalar "scales" a vector, keeping its direction the same but making its magnitude larger or smaller (Figure 1.17). Multiplying by a negative scalar reverses the direction of a vector.

Magnitude of a scalar

You may wonder how to find the magnitude of a quantity like $-3\vec{r}$, which involves the product of a scalar and a vector. This expression can be factored:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}|$$

The magnitude of a scalar is its absolute value, so:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}| = 3\sqrt{r_x^2 + r_y^2 + r_z^2}$$
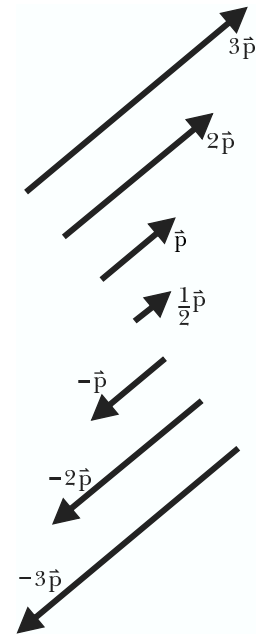

Figure 1.17 Multiplying a vector by a scalar changes the magnitude of the vector. Multiplying by a negative scalar reverses the direction of the vector.

Ex. 1.14 If $\vec{v} = \langle 2, -3, 5 \rangle$ m/s, what is $3\vec{v}$?

Ex. 1.15 If $\vec{r} = \langle 2, -3, 5 \rangle$ m, what is $\frac{\vec{r}}{2}$?

Ex. 1.16 What is the result of multiplying the vector $\vec{a}$ by the scalar f, where $\vec{a} = \langle 0.02, -1.7, 30.0 \rangle$ and $f = 2.0$?

Ex. 1.17 How does the direction of the vector $-\vec{a}$ compare to the direction of the vector $\vec{a}$?

Ex. 1.18 Is $3 + \langle 2, -3, 5 \rangle$ a meaningful expression? If so, what is its value?

Ex. 1.19 Is $\dfrac{4}{\langle 6, -7, 4 \rangle}$ a meaningful expression? If so, what is its value?

1.5.9 Direction of a vector: Unit vectors

One way to describe the direction of a vector is by specifying a *unit vector*. A unit vector is a vector of magnitude 1, pointing in some direction. A unit vector is written with a "hat" (caret) over it instead of an arrow. The unit vector $\hat{a}$ is called "a-hat".

? Is the vector $\langle 1, 1, 1 \rangle$ a unit vector?

The magnitude of $\langle 1, 1, 1 \rangle$ is $\sqrt{1^2 + 1^2 + 1^2} = 1.73$. so this is not a unit vector.

The vector $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$ is a unit vector, since its magnitude is 1:

$$\sqrt{\left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{1}{\sqrt{3}} \right)^2 + \left(\frac{1}{\sqrt{3}} \right)^2} = 1$$

Note that every component of a unit vector must be less than or equal to 1.
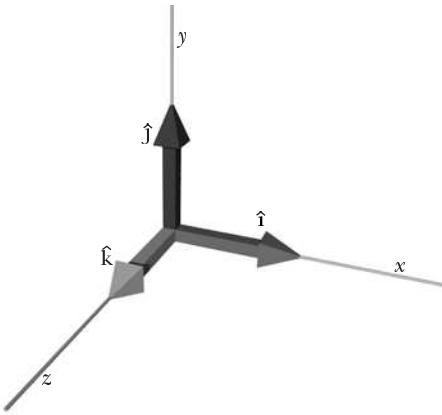
Figure 1.18 The unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.
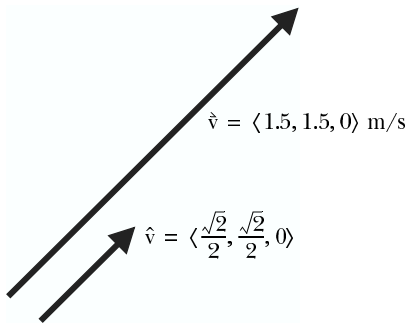
Figure 1.19 The unit vector $\hat{v}$ has the same direction as the vector $\vec{v}$, but its magnitude is 1, and it has no physical units.

In our 3D Cartesian coordinate system, there are three special unit vectors, oriented along the three axes. They are called i-hat, j-hat, and k-hat, and they point along the *x, y,* and *z* axes, respectively (Figure 1.18):

$$\hat{\imath} = \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

One way to express a vector is in terms of these special unit vectors:

$$\langle 0.02, -1.7, 30.0 \rangle = 0.02\hat{\imath} + (-1.7)\hat{\jmath} + 30.0\hat{k}$$

We will usually use the $\langle x, y, z \rangle$ form rather than the ijk form in this book, because $\langle x, y, z \rangle$ is the familiar notation used in many calculus textbooks.

Not all unit vectors point along an axis, as shown in Figure 1.19. For example, the vectors

$$\hat{g} = \langle 0.5774, 0.5774, 0.5774 \rangle \text{ and } \hat{F} = \langle 0.424, 0.566, 0.707 \rangle$$

are both unit vectors, since the magnitude of each is equal to 1. Note that every component of a unit vector is less than or equal to 1.

Calculating unit vectors

Any vector may be factored into the product of a unit vector in the direction of the vector, multiplied by a scalar equal to the magnitude of the vector.

$$\vec{w} = |\vec{w}| \cdot \hat{w}$$

For example, a vector of magnitude 5, aligned with the y axis, could be written as:

$$\langle 0, 5, 0 \rangle = 5\langle 0, 1, 0 \rangle$$

Therefore, to find a unit vector in the direction of a particular vector, we just divide the vector by its magnitude:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{(x^2 + y^2 + z^2)}}$$

$$\hat{r} = \left\langle \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{y}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \right\rangle$$

For example, if $\vec{v} = \langle -22.3, 0.4, -19.5 \rangle\,\text{m/s}$, then

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -22.3, 0.4, -19.5 \rangle \text{ m/s}}{\sqrt{(-22.3)^2 + (0.4)^2 + (-19.5)^2} \text{ m/s}} = \langle -0.753, 0.0135, -0.658 \rangle$$

Remember that to divide a vector by a scalar, you divide each component of the vector by the scalar. The result is a new vector. Note also that a unit vector has no physical units (such as meters per second), because the units in the numerator and denominator cancel.

Ex. 1.20 What is the unit vector in the direction of $\langle 0, 6, 0 \rangle$?

Ex. 1.21 What is the unit vector in the direction of $\langle -300, 0, 0 \rangle$?

Ex. 1.22 What is the unit vector in the direction of $\langle 2, 2, 2 \rangle$? What is the unit vector in the direction of $\langle 3, 3, 3 \rangle$?

Ex. 1.23 What is the unit vector $\hat{a}$ in the direction of $\vec{a}$, where $\vec{a} = \langle 400, 200, -100 \rangle$ m/s^2?

Ex. 1.24 Write the vector $\vec{a} = \langle 400, 200, -100 \rangle$ m/s^2 as the product $|\vec{a}| \cdot \hat{a}$.

1.5.10 Vector addition

The sum of two vectors is another vector, obtained by adding the components of the vectors:

$$\vec{A} = \langle A_x, A_y, A_z \rangle$$

$$\vec{B} = \langle B_x, B_y, B_z \rangle$$

$$\vec{A} + \vec{B} = \langle (A_x + B_x), (A_y + B_y), (A_z + B_z) \rangle$$

For example,

$$\langle 1, 2, 3 \rangle + \langle -4, 5, 6 \rangle = \langle -3, 7, 9 \rangle$$

Don't add magnitudes!

The magnitude of a vector is *not* in general equal to the sum of the magnitudes of the two original vectors! For example, the magnitude of the vector $\langle 3, 0, 0 \rangle$ is 3, and the magnitude of the vector $\langle -2, 0, 0 \rangle$ is 2, but the magnitude of the vector $(\langle 3, 0, 0 \rangle + \langle -2, 0, 0 \rangle)$ is 1, not 5!

Adding vectors graphically: Tip to tail

The sum of two vectors has a geometric interpretation. In Figure 1.20 you first walk along displacement vector $\vec{A}$, followed by walking along displacement vector $\vec{B}$. What is your net displacement vector $\vec{C} = \vec{A} + \vec{B}$? The x component C_x of your net displacement is the sum of A_x and B_x. Similarly, the y component C_y of your net displacement is the sum of A_y and B_y.

To add two vectors $\vec{A}$ and $\vec{B}$ graphically (Figure 1.20):

- Draw the first vector $\vec{A}$
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tip* of the first vector
- Draw a new vector from the tail of vector $\vec{A}$ to the tip of vector $\vec{B}$

1.5.11 Vector subtraction

The difference of two vectors will be very important in this and subsequent chapters. To subtract one vector from another, we subtract the components of the second from the components of the first:

$$\vec{A} - \vec{B} = \langle (A_x - B_x), (A_y - B_y), (A_z - B_z) \rangle$$

$$\langle 1, 2, 3 \rangle - \langle -4, 5, 6 \rangle = \langle 5, -3, -3 \rangle$$

Subtracting vectors graphically: Tail to tail

To subtract one vector $\vec{B}$ from another vector $\vec{A}$ graphically:

- Draw the first vector $\vec{A}$
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tail* of the first vector
- Draw a new vector from the tip of vector $\vec{B}$ to the tip of vector $\vec{A}$

Note that you can check this algebraically and graphically. As shown in Figure 1.21, since the tail of $\vec{A} - \vec{B}$ is located at the tip of $\vec{B}$, then the vector $\vec{A}$ should be the sum of $\vec{B}$ and $\vec{A} - \vec{B}$, as indeed it is:

$$\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$$

To add $\vec{A} + \vec{B}$ graphically,

move $\vec{B}$ so the tail of $\vec{B}$ is at the tip of $\vec{A}$,

then draw a new arrow starting at the tail of $\vec{A}$ and ending at the tip of $\vec{B}$.
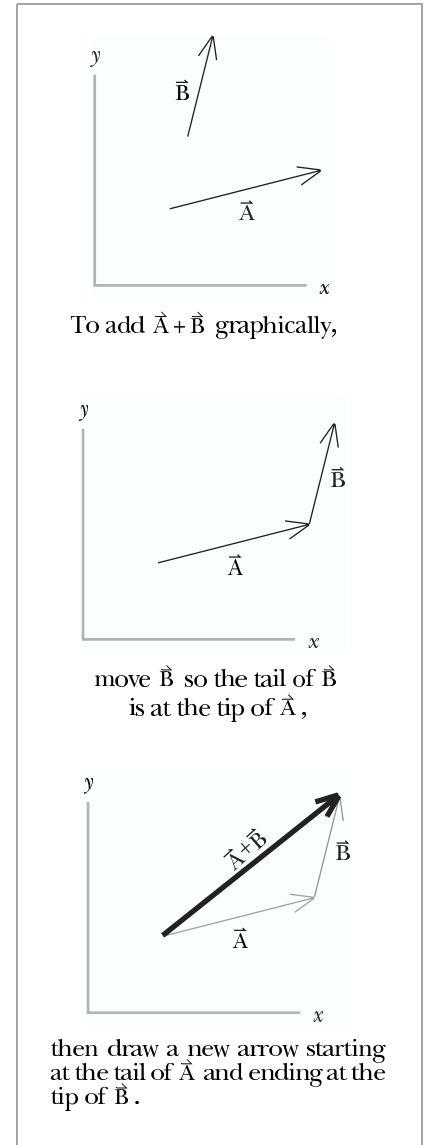
Figure 1.20 The procedure for adding two vectors graphically: draw vectors tip to tail.
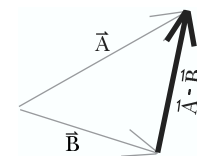

Figure 1.21 The procedure for subtracting vectors graphically: draw vectors tail to tail; draw new vector from tip of second vector to tip of first vector.

1.5.12 The zero vector

It is convenient to have a compact notation for a vector whose components are all zero. We will use the symbol $\vec{0}$ to denote a zero vector, in order to distinguish it from a scalar quantity that has the value 0.

$$\vec{0} = \langle 0, 0, 0 \rangle$$

For example, the sum of two vectors $\vec{B} + (-\vec{B}) = \vec{0}$.

1.5.13 Change in a quantity: The Greek letter Δ

Frequently we will want to calculate the change in a quantity. For example, we may want to know the change in an moving object's position or the change in its velocity during some time interval. The Greek letter Δ (capital delta) is used to denote the change in a quantity (either a scalar or a vector).

We typically use the subscript i to denote an *initial* value of a quantity, and the subscript f to denote the *final* value of a quantity. If a vector $\vec{r}_i$ denotes the initial position of an object relative to the origin (its position at the beginning of the time interval), and $\vec{r}_f$ denotes the final position of the object, then

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$\Delta\vec{r}$ means "change of $\vec{r}$" or $\vec{r}_f - \vec{r}_i$ (displacement)

Δt means "change of t" or $t_f - t_i$ (time interval)

The symbol Δ (delta) always mean "final minus initial", not "initial minus final". For example, when a child's height changes from 1.1 m to 1.2 m, the change is $\Delta y = +0.1$ m, a positive number. If your bank account dropped from \$150 to \$130, what was the change in your balance? Δ(bank account) $= -20$ dollars.

Ex. 1.25 If $\vec{F}_1 = \langle 300, 0, -200 \rangle$ and $\vec{F}_2 = \langle 150, -300, 0 \rangle$, what is the sum $\vec{F}_1 + \vec{F}_2$?

Ex. 1.26 What is the magnitude of $\vec{F}_1$ (see Exercise 1.25)? What is the magnitude of $\vec{F}_2$? What is the magnitude of $\vec{F}_1 + \vec{F}_2$?

Ex. 1.27 What is the magnitude of $\vec{F}_1$ (see Exercise 1.25) plus the magnitude of $\vec{F}_2$? Is $\left|\vec{F}_1 + \vec{F}_2\right| = \left|\vec{F}_1\right| + \left|\vec{F}_2\right|$?

Ex. 1.28 What is the difference $\vec{F}_1 - \vec{F}_2$? What is $\vec{F}_2 - \vec{F}_1$?

Ex. 1.29 A snail is initially at location $\vec{r}_1 = \langle 3, 0, -7 \rangle$ m. At a later time the snail has crawled to location $\vec{r}_2 = \langle 2, 0, -8 \rangle$ m. What is $\Delta\vec{r}$, the change in the snail's position?
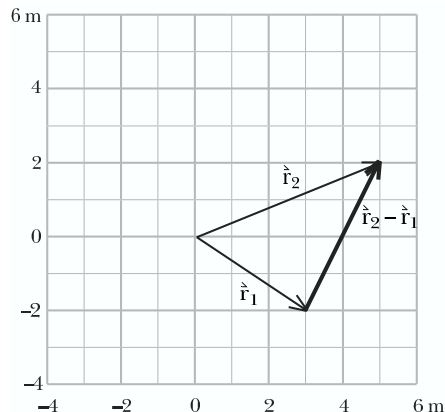

1.5.14 Relative position vectors

Vector subtraction is used to calculate relative position vectors, vectors which represent the position of an object relative to another object. In Figure 1.22 object 1 is at location $\vec{r}_1$ and object 2 is at location $\vec{r}_2$. We want the components of a vector that points from object 1 to object 2. This is the vector obtained by subtraction: $\vec{r}_{2\text{ relative to }1} = \vec{r}_2 - \vec{r}_1$. Note that the form is always "final" minus "initial" in these calculations.

Figure 1.22 Relative position vector.

Ex. 1.30 In Figure 1.22, $\vec{r}_1 = \langle 3, -2, 0 \rangle$ m and $\vec{r}_2 = \langle 5, 2, 0 \rangle$ m. Calculate the position of object 2 relative to object 1, as a relative position vector. Before checking the answer at the back of this

chapter, see whether your answer is consistent with the appearance of the vector $\hat{r}_{2\,\text{relative to }1} = \hat{r}_2 - \hat{r}_1$ shown in Figure 1.22.

Ex. 1.31 What is the position of object 1 relative to object 2, as a vector?

1.6 SI units

In this book we use the SI (Système Internationale) unit system. The SI unit of mass is the kilogram (kg), the unit of distance is the meter (m), and the unit of time is the second (s). In later chapters we will encounter other SI units, such as the newton (N), which is a unit of force. It is essential to use SI units in physics equations; this may require that you convert from some other unit system to SI units. If mass is known in grams, you need to divide by 1000 and use the mass in kilograms. If a distance is given in centimeters, you need to divide by 100 and convert the distance to meters. If the time is measured in minutes, you need to multiply by 60 to use a time in seconds. A convenient way to do such conversions is to multiply by factors which are equal to 1, such as (1 min)/(60 s) or (100 cm)/(1 m). As an example, consider converting 60 miles per hour to SI units, meters per second. Start with the 60 mi/hr and multiply by factors of 1:

$$\left(60\,\frac{\text{mi}}{\text{hr}}\right)\left(\frac{1\,\text{hr}}{60\,\text{min}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right)\left(\frac{5280\,\text{ft}}{1\,\text{mi}}\right)\left(\frac{12\,\text{in}}{1\,\text{ft}}\right)\left(\frac{2.54\,\text{cm}}{1\,\text{in}}\right)\left(\frac{1\,\text{m}}{100\,\text{cm}}\right) = 26.8\,\frac{\text{m}}{\text{s}}$$

Observe how most of the units cancel, leaving final units of m/s.

Ex. 1.32 A snail moved 80 cm (80 centimeters) in 5 minutes. What was its average speed in SI units? Write out the factors as was done above.

1.7 Velocity

We use vectors not only to describe the position of an object but also to describe velocity (speed and direction). If we know a object's present speed in meters per second and the object's direction of motion, we can predict where it will be a short time into the future. As we have seen, *change* of velocity is an indication of interaction. We need to be able to work with velocities of objects in 3D, so we need to learn how to use 3D vectors to represent velocities. After learning how to describe velocity in 3D, we will also learn how to describe *change* of velocity, which is related to interactions.

1.7.1 Average speed

The concept of speed is a familiar one. Speed is a single number, so it is a scalar quantity (speed is the magnitude of velocity). A world class sprinter can run 100 meters in 10 seconds. We say the sprinter's average speed is (100 m)/(10 s) = 10 m/s. In SI units speed is measured in meters per second, abbreviated "m/s".

A car that travels 100 miles straight east in 2 hours has an average speed of (100 miles)/(2 hours) = 50 miles per hour, (or about 22 m/s). In symbols:

$$v_{avg} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t}$$

where v_{avg} is "average speed," Δx is the distance the car has traveled, and Δt is the elapsed time.

There are other useful versions of the basic relationship among average speed, distance, and time. For example,

$$\Delta x = v_{avg}\Delta t$$

expresses the fact that if you run 5 m/s for 7 seconds you go 35 meters. Or you can use

$$\Delta t = \frac{\Delta x}{v_{avg}}$$

to calculate that to go 3000 miles in an airplane that flies at 600 miles per hour will take 5 hours.

Units

While it is easy to make a mistake in one of the formulas relating speed, time interval, and change in position, it is also easy to catch such a mistake by looking at the units. If you had written $\Delta t = v_{avg}/\Delta x$, you would discover that the right hand side has units of (m/s)/m, or 1/s, not s. Always check units!

Instantaneous speed compared to average speed

If a car went 70 miles per hour for the first hour and 30 miles an hour for the second hour, it would still go 100 miles in 2 hours, with an average speed of 50 miles per hour. Note that during this two hour interval, the car was almost never actually traveling at its average speed of 50 miles per hour.

To find the "instantaneous" speed—the speed of the car at a particular instant—we should observe the short distance the car goes in a very short time, such as a hundredth of a second: If the car moves 0.3 meters in 0.01 s, its instantaneous speed is 30 meters per second.

1.7.2 Vector velocity

Earlier we calculated vector differences between two different objects. The vector difference $\vec{r}_2 - \vec{r}_1$ represented a *relative* position vector—the position of object 2 relative to object 1 at a particular time. Now we will be concerned with the change of position of *one* object during a time interval, and $\vec{r}_f - \vec{r}_i$ will represent the "displacement" of this single object during the time interval, where $\vec{r}_i$ is the initial 3D position and $\vec{r}_f$ is the final 3D position (note that as with relative position vectors, we always calculate "final minus initial"). Dividing the (vector) displacement by the (scalar) time interval $t_f - t_i$ (final time minus initial time) gives the average (vector) velocity of the object:

DEFINITION: AVERAGE VELOCITY

$$\vec{v}_{avg} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Another way of writing this expression, using the "Δ" symbol (Greek capital delta) to represent a change in a quantity, is:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t}$$

remembering that this is a compact notation for:

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \langle \frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t} \rangle$$

The magnitude of the average velocity, $|\vec{v}_{avg}|$, is called the average speed.

1.7.3 Determining average velocity from change in position

Consider a bee in flight (Figure 1.23). At time t_i = 15 s after 9:00 AM, the bee's position vector was $\vec{r}_i = \langle 2, 4, 0 \rangle$ m. At time t_f = 15.1 s after 9:00 AM, the bee's position vector was $\vec{r}_f = \langle 3, 3.5, 0 \rangle$ m. On the diagram, we draw and label the vectors $\vec{r}_i$ and $\vec{r}_f$.

Next, on the diagram, we draw and label the vector $\vec{r}_f - \vec{r}_i$, with the tail of the vector at the bee's initial position. One useful way to think about this graphically is to ask yourself what vector needs to be added to the initial vector $\vec{r}_1$ to make the final vector $\vec{r}_f$, since $\vec{r}_i$ can be written in the form $\vec{r}_f = \vec{r}_i + (\vec{r}_f - \vec{r}_i)$.

The vector we just drew, the change in position $\vec{r}_f - \vec{r}_i$, is called the "displacement" of the bee during this time interval. This displacement vector points from the initial position to the final position, and we always calculate displacement as "final minus initial".

Note that the displacement $\vec{r}_f - \vec{r}_i$ refers to the positions of one object (the bee) at two different times, not the position of one object relative to a second object at one particular time ("relative position vector"). However, the vector subtraction is the same kind of operation for either kind of situation.

We calculate the bee's displacement vector numerically by taking the difference of the two vectors, final minus initial:

$$\vec{r}_f - \vec{r}_i = \langle 3, 3.5, 0 \rangle \text{ m} - \langle 2, 4, 0 \rangle \text{ m} = \langle 1, -0.5, 0 \rangle \text{ m}$$

This numerical result should be consistent with our graphical construction. Look at the components of $\vec{r}_f - \vec{r}_i$ in Figure 1.23. Do you see that this vector has an x component of +1 and a y component of −0.5 m? Note that the (vector) displacement $\vec{r}_f - \vec{r}_i$ is in the direction of the bee's motion.

The average velocity of the bee, a vector quantity, is the (vector) displacement $\vec{r}_f - \vec{r}_i$ divided by the (scalar) time interval, $t_f - t_i$. Calculate the bee's average velocity:

$$\vec{v}_{avg} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} = \frac{\langle 1, -0.5, 0 \rangle \text{ m}}{(15.1 - 15) \text{ s}} = \frac{\langle 1, -0.5, 0 \rangle \text{ m}}{0.1 \text{ s}} = \langle 10, -5, 0 \rangle \text{ m/s}$$

Since we divided $\vec{r}_f - \vec{r}_i$ by a scalar ($t_f - t_i$), the average velocity $\vec{v}_{avg}$ points in the direction of the bee's motion, if the bee flew in a straight line.

What is the speed of the bee?

$$\text{speed of bee} = |\vec{v}_{avg}| = \sqrt{10^2 + (-5)^2 + 0^2} \text{ m/s} = 11.18 \text{ m/s}$$

What is the direction of the bee's motion, expressed as a unit vector?

$$\text{direction of bee: } \hat{v}_{avg} = \frac{\vec{v}_{avg}}{|\vec{v}_{avg}|} = \frac{\langle 10, -5, 0 \rangle \text{ m/s}}{11.18 \text{ m/s}} = \langle 0.894, -0.447, 0 \rangle$$

Note that the "m/s" units cancel; the result is dimensionless. We can check that this really is a unit vector:

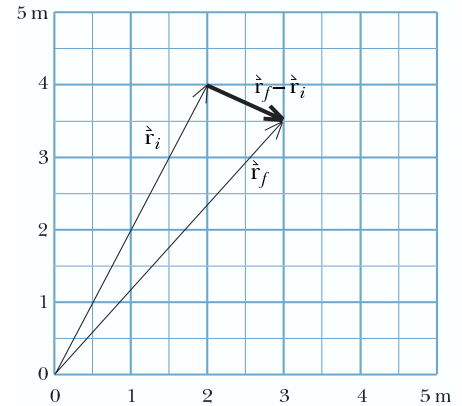$$\sqrt{0.894^2 + (-0.447)^2 + 0^2} = 0.9995$$

Figure 1.23 The displacement vector points from initial position to final position.

This is not quite 1.0 due to rounding the velocity coordinates and speed to three significant figures.

Put the pieces back together and see what we get. The original vector factors into the product of the magnitude times the unit vector:

$$|\vec{v}|\hat{v} = (11.18 \text{ m/s})\langle 0.894, -0.447, 0\rangle = \langle 10, -5, 0\rangle \text{ m/s}$$

This is the same as the original vector $\vec{v}$.

Ex. 1.33 At a time 0.2 seconds after it has been hit by a tennis racket, a tennis ball is located at $\langle 5, 7, 2\rangle$ m, relative to an origin in one corner of a tennis court. At a time 0.7 seconds after being hit, the ball is located at $\langle 9, 2, 8\rangle$ m.
(a) What is the average velocity of the tennis ball?
(b) What is the average speed of the tennis ball?
(c) What is the unit vector in the direction of the ball's velocity?

Ex. 1.34 A spacecraft is observed to be at a location $\langle 200, 300, -400\rangle$ m relative to an origin located on a nearby asteroid, and 5 seconds later is observed at location $\langle 325, 25, -550\rangle$ m.
(a) What is the average velocity of the spacecraft?
(b) What is the average speed of the spacecraft?
(c) What is the unit vector in the direction of the spacecraft's velocity?
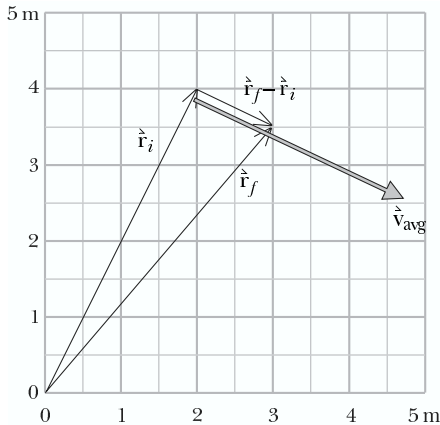

1.7.4 Scaling a vector to fit on a graph

We can plot the average velocity vector on the same graph that we use for showing the vector positions of the bee (Figure 1.24). However, note that velocity has units of m/s while positions have units of m, so in a way we're mixing apples and oranges.

Moreover, the magnitude of the vector, 11.18 m/s, doesn't fit on a graph that is only 5 units wide (in meters). It is standard practice in such situations to scale the arrow representing the vector down to fit on the graph, preserving the correct direction. In Figure 1.24 we've scaled the velocity vector down by about a factor of 3 to make the arrow fit on the graph. Of course if there is more than one velocity vector we use the same scale factor for all the velocity vectors. The same kind of scaling is used with other physical quantities that are vectors, such as force and momentum, which we will encounter later.

Figure 1.24 Average velocity vector: displacement divided by time interval.

1.7.5 Predicting a new position

We can rewrite the velocity relationship in the form

$$(\vec{r}_f - \vec{r}_i) = \vec{v}_{avg}(t_f - t_i)$$

That is, the (vector) displacement of an object is its average (vector) velocity times the time interval. This is just the vector version of the simple notion that if you run at a speed of 7 m/s for 5 s you move a distance of (7 m/s)(5 s) = 35 m, or that a car going 50 miles per hour for 2 hours goes (50 mi/hr)(2 hr) = 100 miles.

Is $(\vec{r}_f - \vec{r}_i) = \vec{v}_{avg}(t_f - t_i)$ a valid vector relation? Yes, multiplying a vector $\vec{v}_{avg}$ times a scalar $t_f - t_i$ yields a vector. We make a further rearrangement to obtain a relation for updating the position when we know the velocity:

THE POSITION UPDATE FORMULA

$$\vec{r}_f = \vec{r}_i + \vec{v}_{avg}(t_f - t_i)$$

This equation says that if we know the starting position, the average velocity, and the time interval, we can predict the final position. This equation will be important throughout this course.

Using the position update formula

The position update formula $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}(t_f - t_i)$ is a vector equation, so we can write out its full component form:

$$\langle x_f, y_f, z_f \rangle = \langle x_i, y_i, z_i \rangle + \langle v_{avg,x}, v_{avg,y}, v_{avg,z} \rangle (t_f - t_i)$$

Because the x component on the left of the equation must equal the x component on the right (and similarly for the y and z components), this compact vector equation represents three separate component equations:

$$x_f = x_i + v_{avg,x}(t_f - t_i)$$

$$y_f = y_i + v_{avg,y}(t_f - t_i)$$

$$z_f = z_i + v_{avg,z}(t_f - t_i)$$

Example

At time $t_i = 12.18$ s after 1:30 PM a ball's position vector is $\vec{r}_i = \langle 20, 8, -12 \rangle$ m. The ball's velocity at that moment is $\hat{v} = \langle 9, -4, 6 \rangle$ m/s. At time $t_f = 12.21$ s after 1:30 PM, where is the ball, assuming that its velocity hardly changes during this short time interval?

$$\vec{r}_f = \vec{r}_i + \vec{v}(t_f - t_i) = \langle 20, 8, -12 \rangle \text{m} + (\langle 9, -4, 6 \rangle \text{m/s})(12.21 - 12.18)\text{s}$$

$$\vec{r}_f = \langle 20, 8, -12 \rangle \text{ m} + \langle 0.27, -0.12, 0.18 \rangle \text{ m}$$

$$\vec{r}_f = \langle 20.27, 7.88, -11.82 \rangle \text{ m}$$

Note that if the velocity changes significantly during the time interval, in either magnitude or direction, our prediction for the new position may not be very accurate.

Ex. 1.35 A proton traveling with a velocity of $\langle 3 \times 10^5, 2 \times 10^5, -4 \times 10^5 \rangle$ m/s passes the origin at a time 9.0 seconds after a proton detector is turned on. Assuming the velocity of the proton does not change, what will its position be at time 9.7 seconds?

Ex. 1.36 How long does it take a baseball with velocity $\langle 30, 20, 25 \rangle$ m/s to travel from location $\vec{r}_1 = \langle 3, 7, -9 \rangle$ m to location $\vec{r}_2 = \langle 18, 17, 3.5 \rangle$ m?

Ex. 1.37 A "slow" neutron produced in a nuclear reactor travels from location $\langle 0.2, -0.05, 0.1 \rangle$ m to location $\langle -0.202, 0.054, 0.098 \rangle$ m in 2 microseconds ($1\mu s = 1 \times 10^{-6}$ s).
(a) What is the average velocity of the neutron?
(b) What is the average speed of the neutron?

1.7.6 Instantaneous velocity

The curved gray line in Figure 1.25 shows the path of a ball through the air. The gray dots mark the ball's position at time intervals of one second. While the ball is in the air, its velocity is constantly changing, due to interactions with the Earth (gravity) and with the air (air resistance).

Suppose we ask: What is the velocity of the ball at the precise instant that it reaches location *B*? This quantity would be called the "instantaneous velocity" of the ball. We can start by approximating the instantaneous velocity of the ball by finding its average velocity over some larger time interval.
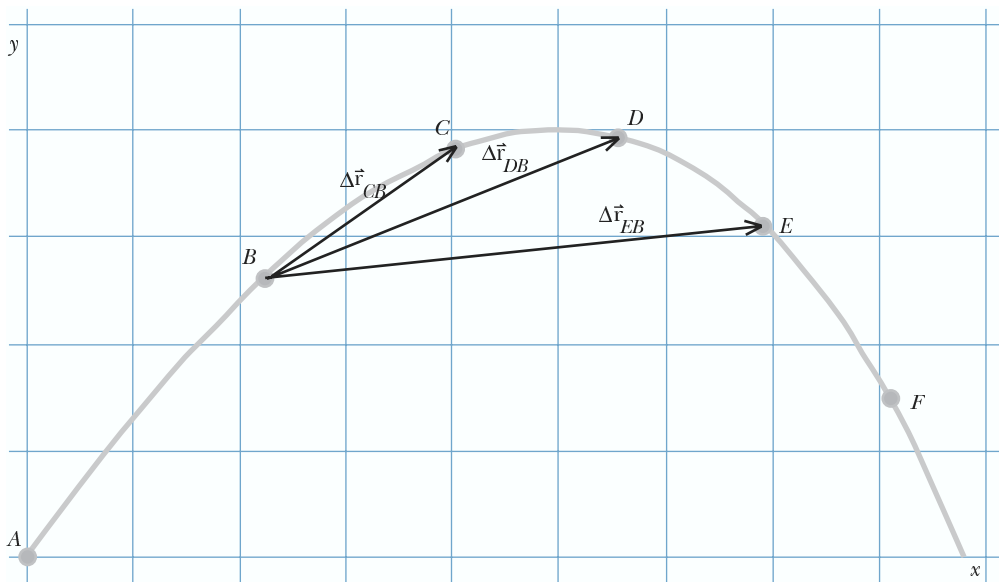

Figure 1.25 The trajectory of a ball through air. The axes represent *x* and *y* distance from the ball's initial location; each square on the grid corresponds to 10 meters. Three different displacements, corresponding to three different time intervals, are indicated by arrows on the diagram.

The table in Figure 1.26 shows the time and the position of the ball for each location marked by a gray dot in Figure 1.25. We can use these data to calculate the average velocity of the ball over three different intervals, by finding the ball's displacement during each interval, and dividing by the appropriate Δt for that interval:

loc.	t (s)	position (m)
A	0.0	$\langle 0, 0, 0 \rangle$
B	1.0	$\langle 22.3, 26.1, 0 \rangle$
C	2.0	$\langle 40.1, 38.1, 0 \rangle$
D	3.0	$\langle 55.5, 39.2, 0 \rangle$
E	4.0	$\langle 69.1, 31.0, 0 \rangle$
F	5.0	$\langle 80.8, 14.8, 0 \rangle$

Figure 1.26 Table showing elapsed time and position of the ball at each location marked by a dot in Figure 1.25.

$$\vec{v}_{EB} = \frac{\Delta \vec{r}_{EB}}{\Delta t} = \frac{\vec{r}_E - \vec{r}_B}{t_E - t_B} = \frac{(\langle 69.1, 31.0, 0 \rangle - \langle 22.3, 26.1, 0 \rangle)\text{m}}{(4.0 - 1.0)\text{s}}$$

$$= \langle 15.6, 1.6, 0 \rangle \frac{\text{m}}{\text{s}}$$

$$\vec{v}_{DB} = \frac{\Delta \vec{r}_{DB}}{\Delta t} = \frac{\vec{r}_D - \vec{r}_B}{t_D - t_B} = \frac{(\langle 55.5, 39.2, 0 \rangle - \langle 22.3, 26.1, 0 \rangle)\text{m}}{(3.0 - 1.0)\text{s}}$$

$$= \langle 16.6, 6.55, 0 \rangle \frac{\text{m}}{\text{s}}$$

$$\vec{v}_{CB} = \frac{\Delta \vec{r}_{CB}}{\Delta t} = \frac{\vec{r}_C - \vec{r}_B}{t_C - t_B} = \frac{(\langle 40.1, 38.1, 0 \rangle - \langle 22.3, 26.1, 0 \rangle)\text{m}}{(2.0 - 1.0)\text{s}}$$

$$= \langle 17.8, 12.0, 0 \rangle \frac{\text{m}}{\text{s}}$$

Not surprisingly, the average velocities we calculate over these different time intervals are not the same, because both the direction of the ball's motion

and the speed of the ball were changing continuously during its flight. The three average velocity vectors that we calculated are shown in Figure 1.27.

Figure 1.27 The three different average velocity vectors calculated above are shown by three arrows, each with its tail at location *B*. Note that since the units of velocity are m/s, these arrows use a different scale from the distance scale used for the path of the ball. The three arrows representing average velocities are drawn with their tails at the location of interest. The dashed arrow represents the actual instantaneous velocity of the ball at location *B*.
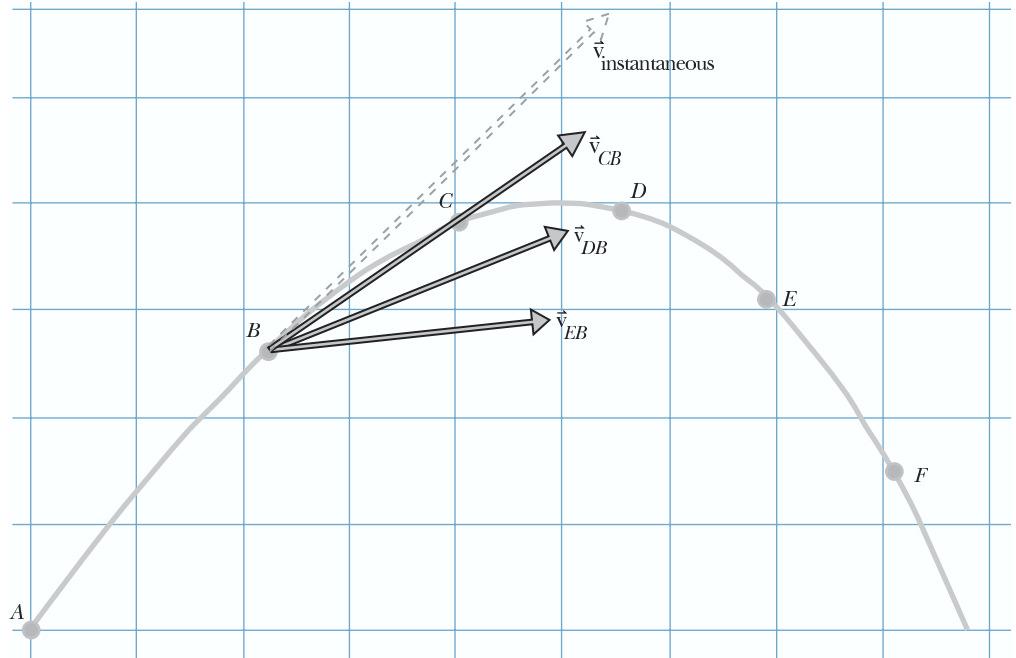
? Which of the three average velocity vectors depicted in Figure 1.27 best approximates the instantaneous velocity of the ball at location *B*?

Simply by looking at the diagram, we can tell that $\vec{v}_{CB}$ is closest to the actual instantaneous velocity of the ball at location *B*, because its direction is closest to the direction in which the ball is actually traveling. Because the direction of the instantaneous velocity is the direction the ball is moving at a particular instant, the instantaneous velocity is tangent to the ball's path. Of the three average velocity vectors we calculated, $\vec{v}_{CB}$ best approximates a tangent to the path of the ball. Evidently $\vec{v}_{CB}$, the velocity calculated with the shortest time interval, $t_C - t_B$, is the best approximation to the instantaneous velocity at location *B*. If we used even smaller values of Δt in our calculation of average velocity, such as 0.1 second, or 0.01 second, or 0.001 second, we would presumably have better and better estimates of the actual instantaneous velocity of the object at the instant when it passes location *B*.

Two important ideas have emerged from this discussion:
- The direction of the instantaneous velocity of an object is tangent to the path of the object's motion.
- Smaller time intervals yield more accurate estimates of instantaneous velocity.

1.7.7 Connection to calculus

You may already have learned about derivatives in calculus. The instantaneous velocity is a derivative, the limit of $\Delta \vec{r} / \Delta t$ as the time interval Δt used in the calculation gets closer and closer to zero:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}, \text{ which is written as } \vec{v} = \frac{d\vec{r}}{dt}$$

In Figure 1.27, the process of taking the limit is illustrated graphically. As smaller values of Δt are used in the calculation, the average velocity vectors approach the limiting value: the actual instantaneous velocity.

A useful way to see the meaning of the derivative of a vector is to consider the components:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\langle x, y, z \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle v_x, v_y, v_z \rangle$$

The derivative of the position vector $\vec{r}$ gives components that are the components of the velocity, as we should expect.

Informally, you can think of $d\vec{r}$ as a very small ("infinitesimal") displacement, and dt as a very small ("infinitesimal") time interval. It is as though we had continued the process illustrated in Figure 1.25 to smaller and smaller time intervals, down to an extremely tiny time interval dt with a correspondingly tiny displacement $d\vec{r}$. The ratio of these tiny quantities is the instantaneous velocity.

The ratio of these two tiny quantities need not be small. For example, suppose an object moves in the x direction a tiny distance of 1×10^{-15} m, the radius of a proton, in a very short time interval of 1×10^{-23} s:

$$\vec{v} = \frac{\langle 1 \times 10^{-15}, 0, 0 \rangle \text{ m}}{1 \times 10^{-23} \text{ s}} = \langle 1 \times 10^8, 0, 0 \rangle \text{ m/s},$$

which is one-third the speed of light (3×10^8 m/s)!

Change of magnitude and/or change in direction

Note that the time rate of change of a vector $\vec{r} = |\vec{r}|\hat{r}$ has two parts:

rate of change of the magnitude of the vector $\dfrac{d|\vec{r}|}{dt}$

rate of change of the direction of the vector $\dfrac{d\hat{r}}{dt}$

We will discuss this further in later sections.

1.7.8   Summary of velocity

DEFINITION OF AVERAGE VELOCITY $\vec{v}_{avg} = \dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$

POSITION UPDATE FORMULA $\vec{r}_f = \vec{r}_i + \vec{v}_{avg}\Delta t$

DEFINITION OF INSTANTANEOUS VELOCITY $\vec{v} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{d\vec{r}}{dt}$

The symbol Δ (delta) means "change of": $\Delta t = t_f - t_i, \Delta \vec{r} = \vec{r}_f - \vec{r}_i$

The instantaneous velocity of an object is tangent to the path of the object.

To approximate the instantaneous velocity of an object, calculate its average velocity over a very short time interval.

1.8 Momentum

Newton's first law of motion:

**An object moves in a straight line and at constant speed
except to the extent that it interacts with other objects.**

gives us a conceptual connection between interactions and their effects on the motion of objects. However, this law does not allow us to make quantitative (numerical) predictions or explanations—we could not have used this law to predict the exact trajectory of the ball shown in Figure 1.25, and we could not use this law alone to figure out how to send a rocket to the Moon. In order to make quantitative predictions or explanations of physical phenomena, we need a quantitative measure of interactions and a quantitative measure of effects of those interactions.

Newton's first law of motion does contain the important idea that if there is no interaction, a moving object will continue to move in a straight line, with no change of direction or speed, and an object that is not moving will remain at rest. A quantitative version of this law would provide a means of predicting the motion of an object, or of deducing how it must have moved in the past, if we could list all of its interactions with other objects.

1.8.1 Changes in velocity

? What factors make it difficult or easy to change an object's velocity?

You have probably noticed that if two objects have the same velocity but one is much more massive than the other, it is more difficult to change the heavy object's speed or direction. It is easier to stop a baseball traveling at a hundred miles per hour than to stop a car traveling at a hundred miles per hour! It is easier to change the direction of a canoe than to change the direction of a large, massive ship such as the Titanic (which couldn't change course quickly enough after the iceberg was spotted).

Momentum involves both mass and velocity

To take into account both an object's mass and its velocity, we can define a vector quantity called "momentum" that involves the product of mass (a scalar) and velocity (a vector). Instead of saying "the stronger the interaction, the bigger the change in the velocity," we now say "the stronger the interaction, the bigger the change in the momentum."

Momentum, a vector quantity, is usually represented by the symbol $\vec{p}$. We might expect that the mathematical expression for momentum would be simply $\vec{p} = m\vec{v}$, and indeed this is almost, but not quite, correct.

Experiments on particles moving at very high speeds, close to the speed of light $c = 3 \times 10^8$ m/s, show that changes in $m\vec{v}$ are not really proportional to the strength of the interactions. As you keep applying a force to a particle near the speed of light, the speed of the particle barely increases, and it is not possible to increase a particle's speed beyond the speed of light.

Through experiments it has been found that changes in the following quantity are proportional to the amount of interaction:

DEFINITION OF MOMENTUM

$$\vec{p} = \gamma m \vec{v}$$

where the proportionality factor γ (lower-case Greek gamma) is defined as

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{|\vec{v}|}{c}\right)^2}}$$

> In these equations $\vec{p}$ represents momentum, $\vec{v}$ is the velocity of the object, m is the mass of the object, $|\vec{v}|$ is the magnitude of the object's velocity (the speed), and c is the speed of light. Momentum has units of $\text{kg} \cdot \text{m/s}$. To calculate momentum in these units, you must specify mass in kg and velocity in meters per second.

This is the "relativistic" definition of momentum. Albert Einstein in 1905 in his special theory of relativity predicted that this would be the appropriate definition for momentum at high speeds, a prediction that has been abundantly verified in a wide range of experiments.

Example

Suppose that a proton (mass 1.7×10^{-27} kg) is traveling with a velocity of $\langle 2 \times 10^7, 1 \times 10^7, -3 \times 10^7 \rangle$ m/s. What is the momentum of the proton?

$$|\vec{v}| = \sqrt{(2 \times 10^7)^2 + (1 \times 10^7)^2 + (-3 \times 10^7)^2} = 3.7 \times 10^7 \frac{m}{s}$$

$$\frac{|\vec{v}|}{c} = \frac{3.7 \times 10^7 \frac{m}{s}}{3 \times 10^8 \frac{m}{s}} = 0.12$$

$$\gamma = \frac{1}{\sqrt{1 - (0.12)^2}} = 1.007$$

$$\vec{p} = (1.007)(1.7 \times 10^{-27} kg)\langle 2 \times 10^7, 1 \times 10^7, -3 \times 10^7 \rangle \frac{m}{s}$$

$$\vec{p} = \langle 3.4 \times 10^{-20}, 1.7 \times 10^{-20}, -5.1 \times 10^{-20} \rangle \frac{kg \cdot m}{s}$$

$\lvert\vec{v}\rvert$ m/s	$\lvert\vec{v}\rvert / c$	γ
0	0	1.0000
3	1×10^{-8}	1.0000
300	1×10^{-6}	1.0000
3×10^6	1×10^{-2}	1.0001
3×10^7	0.1	1.0050
1.5×10^8	0.5	1.1547
2.997×10^8	0.999	22.3663
2.9997×10^8	0.9999	70.7124
3×10^8	1	infinite! (and impossible)

Figure 1.28 Values of γ calculated for some speeds. γ is shown to four decimal places, which is more accuracy than we will usually need in this course.

1.8.2 Approximate formula for momentum

In the example above, we found that $\gamma = 1.007$. Since in that calculation we used only two significant figures, we could have used the approximation that $\gamma \approx 1.0$ without affecting our answer. Let's examine the expression for γ to see if we can come up with a guideline for when it is reasonable to use the approximate expression

$$\vec{p} \approx 1 \cdot m\vec{v} \text{ approximate expression for momentum}$$

Looking at the expression for γ

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{|\vec{v}|}{c}\right)^2}}$$

we see that it depends only on the ratio of the speed of the object to the speed of light (the object's mass doesn't appear in this expression).

If $\frac{|\vec{v}|}{c}$ is a very small number, then $1 - \left(\frac{|\vec{v}|}{c}\right)^2 \approx 1 - 0 \approx 1$, so $\gamma \approx 1$.

Some values of $(|\vec{v}|/c)$ and γ are displayed in Figure 1.28. From this table you can see that even at the very high speed where $|\vec{v}|/c = 0.1$, which means that $|\vec{v}| = 3 \times 10^7$ m/s, the relativistic factor γ is only slightly different from 1. For large-scale objects such as a space rocket, whose speed is typically only about 1×10^4 m/s, we can ignore the factor γ, and momentum is to a good approximation $\vec{p} \approx m\vec{v}$. It is only for high-speed cosmic rays or particles produced in high-speed particle accelerators that we need to use the full relativistic definition for momentum, $\vec{p} = \gamma m\vec{v}$.

From this table you can also get a sense of why it is not possible to exceed the speed of light. As you make a particle go faster and faster, approaching the speed of light, additional increases in the speed become increasingly dif-

ficult, because a tiny increase in speed means a huge increase in momentum, requiring huge amounts of interaction. In fact, for the speed to equal the speed of light, the momentum would have to increase to be infinite! There is a speed limit in the Universe, 3×10^8 m/s.

In some texts the quantity γm is referred to as the "effective mass" of a fast moving particle, since this quantity gets larger as the particle goes faster. To keep terminology clear, the quantity m is then called the "rest mass" of the particle—the mass of the particle when its speed is zero.

We will repeatedly emphasize the role of momentum throughout this course because of its fundamental importance not only in classical (prequantum) mechanics but also in relativity and quantum mechanics. The use of momentum clarifies the physics analysis of certain complex processes such as collisions, including collisions at speeds approaching the speed of light.

1.8.3 Direction of momentum

Like velocity, momentum is a vector quantity, so it has a magnitude and a direction.

? A leaf is blown by a gust of wind, and at a particular instant is traveling straight upward, in the +y direction. What is the direction of the leaf's momentum?

The mathematical expression for momentum can be looked at as the product of a scalar part times a vector part. Since the mass m must be a positive number, and the factor gamma (γ) must be a positive number, this scalar factor cannot change the direction of the vector (Figure 1.29). Therefore the direction of the leaf's momentum is the same as the direction of its velocity: straight up (the +y direction).

Figure 1.29 The expression for momentum is the product of a scalar times a vector. The scalar factor is positive, so the direction of an object's momentum is the same as the direction of its velocity.

Ex. 1.38 A good sprinter can run 100 meters in 10 seconds. What is the magnitude of the momentum of a sprinter whose mass is 65 kg and who is running at a speed of 10 m/s?

Ex. 1.39 A baseball has a mass of about 155 g. What is the magnitude of the momentum of a baseball thrown at a speed of 100 miles per hour? (Note that you need to convert mass to kilograms and speed to meters/second. See the inside back cover of the textbook for conversion factors.)

Ex. 1.40 What is the magnitude (in kg·m/s) of the momentum of a 1000 kg airplane traveling at a speed of 500 miles per hour? (Note that you need to convert speed to meters per second.)

Ex. 1.41 What is the magnitude of the momentum of an electron traveling at a speed of 2×10^8 meters per second? (Masses of particles are given on the inside back cover of this textbook.)

Ex. 1.42 Show that when the speed is within one percent of the speed of light ($|\vec{v}| = 0.99c$), the ratio of the correct relativistic momentum to the approximate nonrelativistic momentum $m|\vec{v}|$ is quite large. Such speeds are attained in particle accelerators.

Ex. 1.43 If a particle has momentum $\vec{p} = \langle 4, -5, 2 \rangle$ kg·m/s, what is the magnitude $|\vec{p}|$ of its momentum?

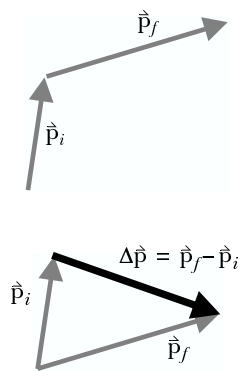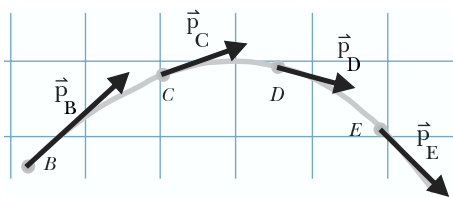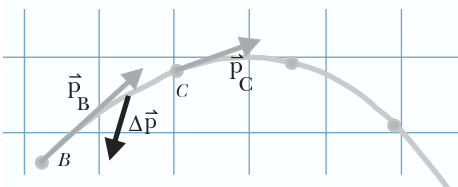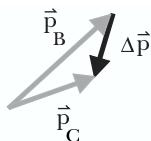Figure 1.30 Calculation of $\Delta\vec{p}$.

1.8.4 Change of momentum

In the next chapter we will introduce "the momentum principle" which quantitatively relates change in momentum $\Delta\vec{p}$ to the strength and duration of an interaction. In order to be able to use the momentum principle we need to know how to calculate changes in momentum.

Momentum is a vector quantity, so just as was the case with velocity, there are two aspects of momentum that can change: magnitude and direction. A mathematical description of change of momentum must include either a change in the magnitude of the momentum, or a change in the direction of the momentum, or both.

Change of the vector momentum

The change in the momentum $\Delta\vec{p}$ during a time interval is a vector: $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$. This vector expression captures both changes in magnitude and changes in direction. Figure 1.30 is a graphical illustration of a change from an initial momentum $\vec{p}_i$ to a final momentum $\vec{p}_f$. Place the initial and final momentum vectors tail to tail, then draw a vector from initial to final. This is the vector representing $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$. This is the same procedure you used to calculate relative position vectors by subtraction, or displacement vectors by subtraction. The rule for subtracting vectors is always the same: Place the vectors tail to tail, then draw from the tip of the initial vector to the tip of the final vector. This resultant vector is "final minus initial".

Figure 1.31 A portion of the trajectory of a ball moving through air, subject to gravity and air resistance. The arrows represent the momentum of the ball at the locations indicated by letters.

Example

Figure 1.31 shows a portion of the trajectory of a ball in air, subject to gravity and air resistance. When the ball is at location B, its momentum is $\vec{p}_B = \langle 3.03, 2.83, 0 \rangle$ kg · m ∕ s. When it is at location C, its momentum is $\vec{p}_C = \langle 2.55, 0.97, 0 \rangle$ kg · m ∕ s. Find the change in the ball's momentum between these locations, and show it on the diagram.

$$\Delta\vec{p} = \vec{p}_C - \vec{p}_B = \langle 2.55, 0.97, 0 \rangle \text{ kg · m ∕ s} - \langle 3.03, 2.83, 0 \rangle \text{ kg · m ∕ s}$$
$$= \langle -0.48, -1.86, 0 \rangle \text{ kg · m ∕ s}$$

Both the x and y components of the ball's momentum decreased, so $\Delta\vec{p}$ has negative x and y components. This is consistent with the graphical subtraction shown in Figure 1.32.

Figure 1.32 Graphical calculation of $\Delta\vec{p}$. The result is also shown superimposed on the ball's path, midway between the initial and final locations.

Change in magnitude of momentum

If an object's speed changes (that is, the magnitude of its velocity changes), the magnitude of the object's momentum also changes. However, note that the change of the magnitude of momentum is not in general equal to the magnitude of change of momentum.

Example

Suppose you are driving a 1000 kilogram car at 20 m/s in the $+x$ direction. After making a 180 degree turn, you drive the car at 20 m/s in the $-x$ direction. (20 m/s is about 45 miles per hour or 72 km per hour.)

(a) What is the change of magnitude of the momentum of the car $\Delta|\vec{p}|$?

These speeds are very small compared to the speed of light, so we can use the approximate nonrelativistic formula for momentum.

$$\Delta|\vec{p}| = |\vec{p}_2| - |\vec{p}_1| \approx |m\vec{v}_2| - |m\vec{v}_1| = |m||\vec{v}_2| - |m||\vec{v}_1|$$
$$\Delta|\vec{p}| = (1000 \text{ kg})(20 \text{ m/s}) - (1000 \text{ kg})(20 \text{ m/s})$$
$$\Delta|\vec{p}| = 0 \text{ kg·m/s}$$

(b) What is the magnitude of the change of momentum of the car $|\Delta\vec{p}|$?

$$\Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = 1000 \text{ kg} \langle 20, 0, 0 \rangle \text{ m/s} - 1000 \text{ kg} \langle -20, 0, 0 \rangle \text{ m/s}$$

$$= \langle 4 \times 10^4, 0, 0 \rangle \text{ kg} \cdot \text{m/s}$$

$$|\Delta\vec{p}| = 4 \times 10^4 \text{ kg} \cdot \text{m/s}$$

So $|\Delta\vec{p}| \neq \Delta|\vec{p}|$.

Change of direction of momentum

There are various ways to specify a change in the direction of motion. For example, if you use compass directions, you could say that an airplane changed its direction from 30° east of North to 45° east of North: a 15° clockwise change. One can imagine various other schemes, involving other kinds of coordinate systems. The standard way to deal with this is to use vectors.

Ex. 1.44 A tennis ball of mass 57 g travels with velocity $\langle 50, 0, 0 \rangle$ m/s toward a wall. After bouncing off the wall, the tennis ball is observed to be traveling with velocity $\langle -48, 0, 0 \rangle$ m/s.
(a) Draw a diagram showing the initial and final momentum of the tennis ball.
(b) What is the change in the momentum of the tennis ball?
(c) What is the change in the magnitude of the tennis ball's momentum?

Ex. 1.45 The planet Mars has a mass of 6.4×10^{23} kg, and travels in a nearly circular orbit around the Sun, as shown in Figure 1.33. When it is at location A, the velocity of Mars is $\langle 0, 0, -2.5 \times 10^4 \rangle$ m/s. When it reaches location B, the planet's velocity is $\langle -2.5 \times 10^4, 0, 0 \rangle$ m/s.
(a) What is $\Delta\vec{p}$, the change in the momentum of Mars between locations A and B?
(b) On a copy of the diagram in Figure 1.33, draw two arrows representing the momentum of Mars at locations *C* and D, paying attention to both the length and direction of each arrow.
(c) What is the direction of the change in the momentum of Mars between locations *C* and D? Draw the vector $|\Delta\vec{p}|$ on your diagram.

Ex. 1.46 A 50 kg child is riding on a carousel (merry-go-round) at a constant speed of 5 m/s. What is the magnitude of the change in the child's momentum $|\Delta\vec{p}|$ in going all the way around (360°)? In going halfway around (180°)? Draw a diagram showing the initial vector momentum and the final vector momentum, then subtract, then find the magnitude.
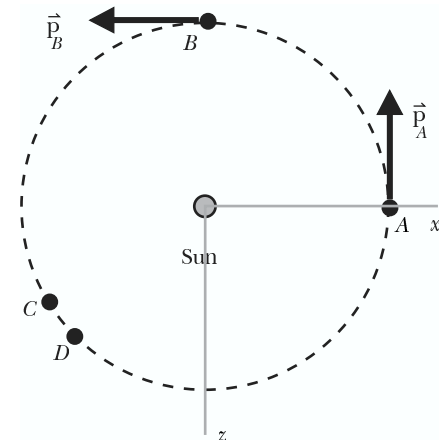

Figure 1.33 The nearly circular orbit of Mars around the Sun, viewed from above the orbital plane. Not to scale: the sizes of the Sun and Mars are exaggerated.

1.8.5 Average rate of change of momentum

The rate of change of the vector position is such an important quantity that it has a special name: "velocity". In Section 1.7 we discussed how to find both the average rate of change of position (average velocity) and the instantaneous rate of change of position (instantaneous velocity) of an object.

The average rate of change of momentum and the instantaneous rate of change of momentum are also extremely important quantities. In some situations, we will only be able to find an average rate of change of momentum:

AVERAGE RATE OF CHANGE OF MOMENTUM

$$\frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{t_f - t_i}$$

This quantity is a vector, and points in the direction of $\Delta \vec{p}$.

Example

If the momentum of a ball changes from $\langle 1, 2, 0 \rangle$ kg · m/s to $\langle 0.5, 0, 0.5 \rangle$ kg · m/s in half a second, the average rate of change of momentum of the ball is

$$\frac{(\langle 0.5, 0, 0.5 \rangle - \langle 1, 2, 0 \rangle) \text{ kg} \cdot \text{m/s}}{0.5 \text{ s}} = \langle -1, -4, 1 \rangle \frac{\text{kg} \cdot \text{m}}{\text{s}^2}$$

1.8.6 Rate of change of momentum along a curving path

When a particle moves along a curving path, such as that shown in Figure 1.34, the direction (and perhaps the magnitude) of its momentum is continuously changing. Using geometry and algebra, we can derive an algebraic expression for the rate of change of momentum of a particle at any location on a curving trajectory, by doing the following:
- Pick a short interval bracketing location A
- Graphically find $\Delta \vec{p}$ over this interval
- Inscribe a "kissing" circle inside the curve
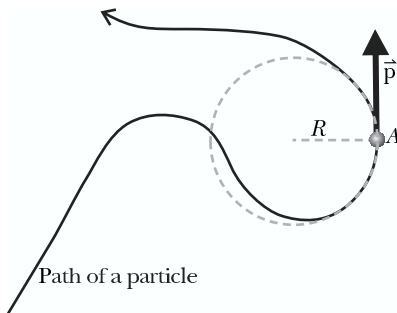- Use similar triangles to relate $\Delta \vec{p}$ to the radius of the circle

We will find that the rate of change of momentum at a location depends on:
- the momentum of the particle when it is at that location
- the particle's speed
- the radius of a circle "kissing" the inside of the curve at that location

We will show that when the particle is (momentarily) at a particular location (such as location A in Figure 1.34),

Figure 1.34 Curving path of a particle. The particle's momentum is shown at location A. At location A the particle's path has a radius of curvature R. The "kissing circle" matches the tangent and radius of curvature of the path.

$$\left| \frac{\Delta \vec{p}}{\Delta t} \right| = \frac{|\vec{v}|}{R} |\vec{p}|$$

where $|\vec{v}|$ is the speed of the particle at the moment that it passes location A, and R is the radius of the kissing circle. Since R is in the denominator, the larger the radius of curvature, the smaller the rate of change of the momentum. This is reasonable, because when the radius of curvature gets extremely large, the path of the particle is essentially a straight line, and the direction of the momentum isn't changing. On the other hand, a tight turn (small radius of curvature R) means that the direction of the momentum is changing very rapidly, and $\Delta \vec{p} / \Delta t$ is large.

Simplest case: magnitude of momentum constant

To begin, let's assume that the magnitude of the momentum isn't changing, only the direction is changing. Consider motion along the curving path shown in Figure 1.34. We want to find the rate of change of momentum of a particle traveling along this path, at the instant when the particle is at location A.

In Figure 1.34 a dashed circle has been drawn inside the path, tangent to the path at location A. This dashed circle of radius R is called the "osculatory" or "kissing" circle, because it just "kisses" the curving path at ocation A, fitting into the trajectory as smoothly as possible, with the circle and the trajectory sharing the same tangent and same radius of curvature R at location A. We pick a short interval centered on location A, and draw arrows repre-

senting the momentum of the particle at the beginning and end of that interval, as shown in Figure 1.35.

In Figure 1.36 we find $\Delta\vec{p}$ by putting the initial and final momentum vectors tail to tail and drawing the vector $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$ (final minus initial). The important thing to notice is that the *change* in momentum is not in the direction of the momentum but is perpendicular to it. $\Delta\vec{p}$ points to the left, toward the center of the kissing circle. In order to make such a change in the momentum, you would have to push toward the left.

We can use geometry to calculate the rate at which the momentum changes due to the changing direction. In Figure 1.37 the *p*-triangle and the *R*-triangle are similar to each other, with the same acute angles, because the radius vectors in the *R*-triangle and the momentum vectors in the *p*-triangle are at right angles to each other. To say it another way, you can rotate one of the triangles through 90 degrees and the small acute angles will clearly be the same. We're assuming that the magnitude of the momentum isn't changing, only the direction, so $|\vec{p}_f| = |\vec{p}_i|$.

For short times (small angles), the length of the short side of the *R*-triangle is approximately equal to the arc length, which is the distance the particle goes in the short time, which is equal to $|\vec{v}|\Delta t$. Since the two triangles are similar, the ratio of the side opposite the acute angle to the side adjacent to the acute angle must be the same for both triangles:

$$\frac{|\Delta\vec{p}|}{|\vec{p}|} = \frac{|\vec{v}|\Delta t}{R}$$

We have used the fact that we're assuming that the magnitude of the momentum isn't changing, only the direction, so that we can write $|\vec{p}|$ for the magnitude of either the initial or final momentum near location *A*. We are interested in the rate of change of the momentum, so rearrange the formula like this:

$$\frac{|\Delta\vec{p}|}{\Delta t} = \frac{|\vec{v}|}{R}|\vec{p}|$$

Since the time is increasing, Δt is always positive and therefore the same as $|\Delta t|$, so we can express the magnitude of the rate of change of the momentum like this:

$$\left|\frac{\Delta\vec{p}}{\Delta t}\right| = \frac{|\vec{v}|}{R}|\vec{p}|$$

The direction of $\Delta\vec{p}/\Delta t$ is the same as the direction of $\Delta\vec{p}$: perpendicular to the particle's momentum (Figure 1.38).

Checking units

A simple check on the results of a calculation or derivation is a units check.

? Do the units on both sides of the preceding equation match?

$$\text{Left hand side: } \frac{\left(\dfrac{\text{kg}\cdot\text{m}}{\text{s}}\right)}{\text{s}} = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$$

$$\text{Right hand side: } \frac{\left(\dfrac{\text{m}}{\text{s}}\right)}{\text{m}}\left(\frac{\text{kg}\cdot\text{m}}{\text{s}}\right) = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$$

The units match. (If they had not matched, the equation could not be correct, and we would have to look for errors in our derivation.)
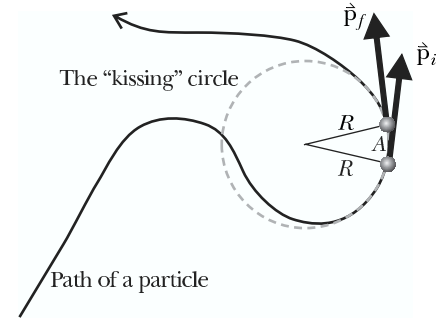

Figure 1.35 The particle's momentum shown at two locations before and after location *A*.
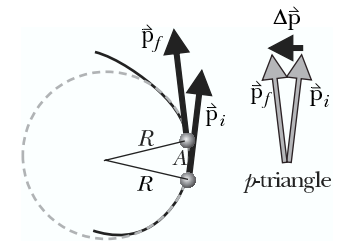

Figure 1.36 We find $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$ graphically, by putting the vectors tail to tail.
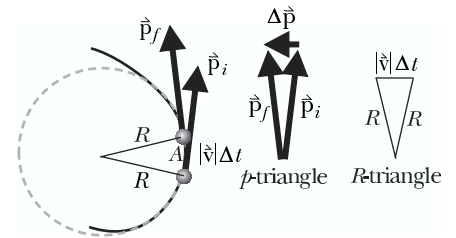

Figure 1.37 The two triangles are similar, with the same acute angle, because the sides of one triangle were originally perpendicular to the side of the other.
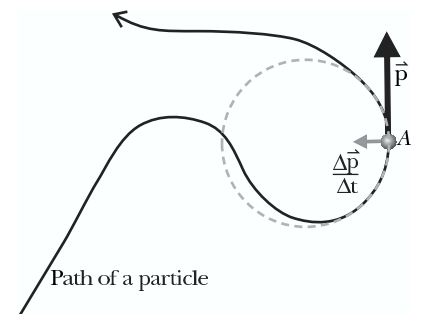

Figure 1.38 For a particle moving at constant speed along a curving path, $\Delta\vec{p}/\Delta t$ is perpendicular to the particle's momentum, and points to the inside of the curve.
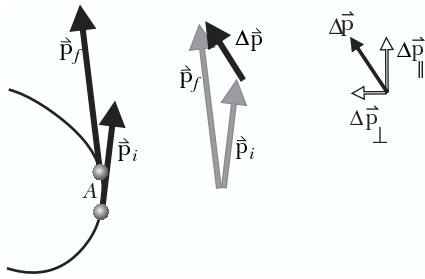
Figure 1.39 If both the magnitude and the direction of the momentum change, then $\Delta\vec{p}$ has a component parallel to the motion as well as a component perpendicular to the motion.

General case: magnitude of momentum changing

Despite the fact that the preceding derivation hinged on the assumption that the magnitude of momentum was constant, we can extend this result to the general case in which both direction and magnitude of momentum are changing. In Figure 1.39 we see what happens if both the speed and direction of motion are changing. Now the initial and final momentum vectors have different lengths (magnitudes). As a result, the momentum change $\Delta\vec{p}$ has a component parallel to the motion as well as a component perpendicular to the motion.

In this general case, $\Delta\vec{p}$ can be separated into two components: one component perpendicular to the momentum $\vec{p}$, along the radius of the kissing circle (this component indicates change of direction), and another component parallel to the momentum $\vec{p}$, along the tangent to the curving trajectory (this component indicates change of speed). If we consider only the perpendicular component, then the result obtained above still applies.

Instantaneous rate of change of momentum

In the limit as Δt becomes infinitesimally small we have a derivative, and we obtain this important result:

PERPENDICULAR COMPONENT OF $d\vec{p}\,/\,dt$
FOR MOTION ALONG A CURVING PATH

Perpendicular component of $\dfrac{d\vec{p}}{dt}$ is given by $\left|\dfrac{d\vec{p}}{dt}\right| = \dfrac{|\vec{v}|}{R}|\vec{p}|$

The direction is toward the center of the kissing circle of radius R.

? Is this result valid for high speeds, when $|\vec{v}| \approx c$?

We did not use the approximate formula for momentum, so our result should be valid even if a particle is traveling at a high speed. Given the full definition of momentum, we can expand the result like this:

$$\left|\frac{d\vec{p}}{dt}\right| = \left(\frac{|\vec{v}|}{R}\right)\frac{1}{\sqrt{1-(|\vec{v}|\,/\,c)^2}}(m|\vec{v}|) = \frac{1}{\sqrt{1-(|\vec{v}|\,/\,c)^2}}\left(\frac{m|\vec{v}|^2}{R}\right)$$

If the particle's speed is small compared to the speed of light, we have the following approximate result for the perpendicular component of $d\vec{p}\,/\,dt$:

APPROXIMATE RESULT

Perpendicular component: $\left|\dfrac{d\vec{p}}{dt}\right| \approx \dfrac{m|\vec{v}|^2}{R}$ if $v \ll c$

It makes sense to remember the more general result, $\left|d\vec{p}\,/\,dt\right| = (|\vec{v}|\,/\,R)|\vec{p}|$, since it is not really any more complicated to use, and it is valid at all speeds, not just low speeds.

Example

The Moon, which has a mass of $7\times10^{22}\ \text{kg}$, orbits the Earth once every 28 days (a lunar month), following a path which is nearly circular. The distance from the Earth to the Moon is $4\times10^8\ \text{m}$. What is the magnitude of the rate of change of the Moon's momentum?

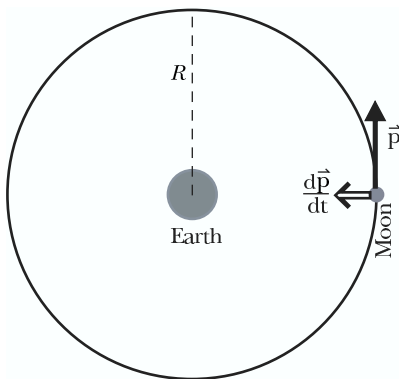Figure 1.40 The Moon's orbit around the Earth is nearly circular. Not to scale: the sizes of the Earth and Moon are exaggerated.

$$|\vec{v}| = \frac{2\pi(4\times10^8 \text{ m})}{(28 \text{ days})(24 \text{ hr/day})(60 \text{ min/hr})(60 \text{ s/min})} = 1\times10^3 \text{m/s}$$

$$|\vec{p}| \approx m|\vec{v}| = (7\times10^{22} \text{ kg})(1\times10^3\text{m/s}) = 7.3\times10^{25} \text{ kg}\cdot\text{m/s}$$

$$\left|\frac{d\vec{p}}{dt}\right| = \frac{|\vec{v}|}{R}|\vec{p}| = \frac{(1\times10^3\text{m/s})}{(4\times10^8 \text{ m})}(7.3\times10^{25} \text{ kg}\cdot\text{m/s}) = 1.8\times10^{20} \frac{\text{kg}\cdot\text{m}}{\text{s}^2}$$

The Moon's speed is not changing, so $d\vec{p}/dt$ is perpendicular to $\vec{p}$, as shown in Figure 1.40.

Ex. 1.47 Which of the dashed circles in Figure 1.41 best represents the "kissing circle" tangent to the path of the particle, with the same radius of curvature as that of the path at the location marked by "x"?

Ex. 1.48 Assume that the particle whose path is shown in Figure 1.41 is traveling at constant speed. At the location marked "x", what is the direction of $d\vec{p}/dt$ for the particle?

Ex. 1.49 A child of mass 40 kg sits on a wooden horse on a carousel. The wooden horse is 5 meters from the center of the carousel, which completes one revolution every 90 seconds. What is the rate of change of the momentum of the child, both magnitude and direction?

Ex. 1.50 The orbit of the Earth around the Sun is approximately circular, and takes one year to complete. The Earth's mass is 6×10^{24} kg, and the distance from the Earth to the Sun is 1.5×10^{11} m. What is the magnitude of the rate of change of the Earth's momentum? What is the direction of the rate of change of the Earth's momentum?
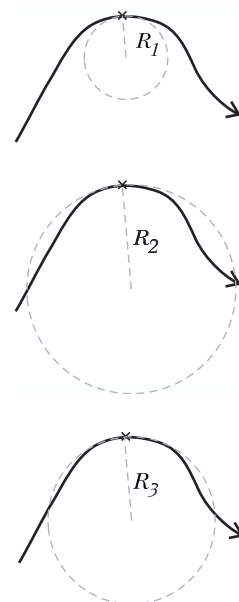

Figure 1.41 Which circle best represents the kissing circle? (Exercise 1.47)

1.9 *The principle of relativity

Sections marked with a "*" are optional. They provide additional information and context, but later sections of the textbook don't depend critically on them. This optional section deals with some deep issues about the "reference frame" from which you observe motion. Newton's first law of motion only applies in an "inertial reference frame," which we will discuss here in the context of the principle of relativity.

A great variety of experimental observations has led to the establishment of the following principle:

THE PRINCIPLE OF RELATIVITY

**Physical laws work in the same way for observers
in uniform motion as for observers at rest.**

This principle is called "the principle of relativity." (Einstein's extensions of this principle are known as "special relativity" and "general relativity.") Phenomena observed in a room in uniform motion (for example, on a train moving with constant speed on a smooth straight track) obey the same physical laws in the same way as experiments done in a room that is not moving. According to this principle, Newton's first law of motion should be true both for an observer moving at constant velocity and for an observer at rest.

For example, suppose you're riding in a car moving with constant velocity, and you're looking at a map lying on the dashboard. As far as you're concerned, the map isn't moving, and no interactions are required to hold it still on the dashboard. Someone standing at the side of the road sees the car go by, sees the map moving at a high speed in a straight line, and can see that no interactions are required to hold the map still on the dashboard. Both you and the bystander agree that Newton's first law of motion is obeyed: the bystander sees the map moving with constant velocity in the absence of interactions, and you see the map not moving at all (a zero constant velocity) in the absence of interactions.

On the other hand, if the car suddenly speeds up, it moves out from under the map, which ends up in your lap. To you it looks like "the map sped up in the backwards direction" without any interactions to cause this to happen, which looks like a violation of Newton's first law of motion. The problem is that you're strapped to the car, which is an accelerated reference frame, and Newton's first law of motion applies only to nonaccelerated reference frames, called "inertial" reference frames. Similarly, if the car suddenly turns to the right, moving out from under the map, the map tends to keep going in its original direction, and to you it looks like "the map moved to the left" without any interactions. So a change of speed or a change of direction of the car (your reference frame) leads you to see the map behave in a strange way.

The bystander, who is in an inertial (non-accelerating) reference frame, doesn't see any violation of Newton's first law of motion. The bystander's reference frame is an inertial frame, and the map behaves in an understandable way, tending to keep moving with the same speed and direction when the car changes speed or direction.

The cosmic microwave background

The principle of relativity, and Newton's first law of motion, apply only to observers who have a constant speed and direction (or zero speed) relative to the "cosmic microwave background," which provides the only backdrop and frame of reference with an absolute, universal character. It used to be that the basic reference frame was loosely called "the fixed stars," but stars and galaxies have their own individual motions within the Universe and do not constitute an adequate reference frame with respect to which to measure motion.

The cosmic microwave background is low-intensity electromagnetic radiation with wavelengths in the microwave region, which pervades the Universe, radiating in all directions. Measurements show that our galaxy is moving through this microwave radiation with a large, essentially constant velocity, toward a cluster of a large number of other galaxies. The way we detect our motion relative to the microwave background is through the "Doppler shift" of the frequencies of the microwave radiation, toward higher frequencies in front of us and lower frequencies behind. This is essentially the same phenomenon as that responsible for a fire engine siren sounding at a higher frequency when it is approaching us and a lower frequency when it is moving away from us.

The discovery of the cosmic microwave background provided major support for the "Big Bang" theory of the formation of the Universe. According to the Big Bang theory, the early Universe must have been an extremely hot mixture of charged particles and high-energy, short-wavelength electromagnetic radiation (visible light, x-rays, gamma rays, etc.). Electromagnetic radiation interacts strongly with charged particles, so light could not travel very far without interacting, making the Universe essentially opaque. Also, the Universe was so hot that electrically neutral atoms could not form with-

out the electrons immediately being stripped away again by collisions with other fast-moving particles.

As the Universe expanded, the temperature dropped. Eventually the temperature was low enough for neutral atoms to form. The interaction of electromagnetic radiation with neutral atoms is much weaker than with individual charged particles, so the radiation was now essentially free, dissociated from the matter, and the Universe became transparent. As the Universe continued to expand (the actual space between clumps of matter got bigger!), the wavelengths of the electromagnetic radiation got longer, until today this fossil radiation has wavelengths in the relatively low-energy, long-wavelength microwave portion of the electromagnetic spectrum.

Inertial frames of reference

It is an observational fact that in reference frames that are in uniform motion with respect to the cosmic microwave background, far from other objects (so that interactions are negligible), an object maintains uniform motion. Such frames are called "inertial frames" and are reference frames in which Newton's first law of motion is valid.

? Is the surface of the Earth an inertial frame?

Figure 1.42 Axes tied to the Earth rotate through 90° in a quarter of a day (6 hours).

No! The Earth is rotating on its axis, so the velocity of an object sitting on the surface of the Earth is constantly changing direction, as is a coordinate frame tied to the Earth (Figure 1.42). Moreover, the Earth is orbiting the Sun, and the Solar System itself is orbiting the center of our Milky Way galaxy, and our galaxy is moving toward other galaxies. So the motion of an object sitting on the Earth is actually quite complicated and definitely not uniform with respect to the cosmic microwave background.

However, for many purposes the surface of the Earth can be considered to be (approximately) an inertial frame. For example, it takes 6 hours for the rotation of the Earth on its axis to make a 90° change in the direction of the velocity of a "fixed" point. If a process of interest takes only a few minutes, during these few minutes a "fixed" point moves in nearly a straight line at constant speed due to the Earth's rotation, and velocity changes in the process of interest are typically much larger than the very small velocity change of the approximate inertial frame of the Earth's surface.

Similarly, although the Earth is in orbit around the Sun, it takes 365 days to go around once, so for a period of a few days or even weeks the Earth's orbital motion is nearly in a straight line at constant speed. Hence for many purposes the Earth represents an approximately inertial frame despite its motion around the Sun.

The special theory of relativity

Einstein's special theory of relativity (published in 1905) built on the basic principle of relativity but added the conjecture that the speed of a beam of light must be the same as measured by observers in different frames of reference in uniform motion with respect to each other. In Figure 1.43, observers on each spaceship measure the speed of the light c emitted by the ship at the top to be the same ($c = 3 \times 10^8$ m/s), despite the fact that they are moving at different velocities.

This additional condition seems peculiar and has far-reaching consequences. After all, the map on the dashboard of your car has different speeds relative to different observers, depending on the motion of the observer. Yet a wide range of experiments has confirmed Einstein's conjecture: all observers measure the same speed for the same beam of light, $c = 3 \times 10^8$ m/s. (The color of the light is different for the different observers, but the speed is the same.)
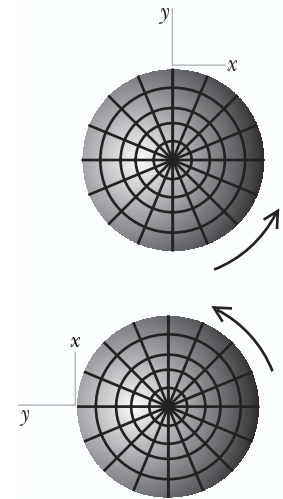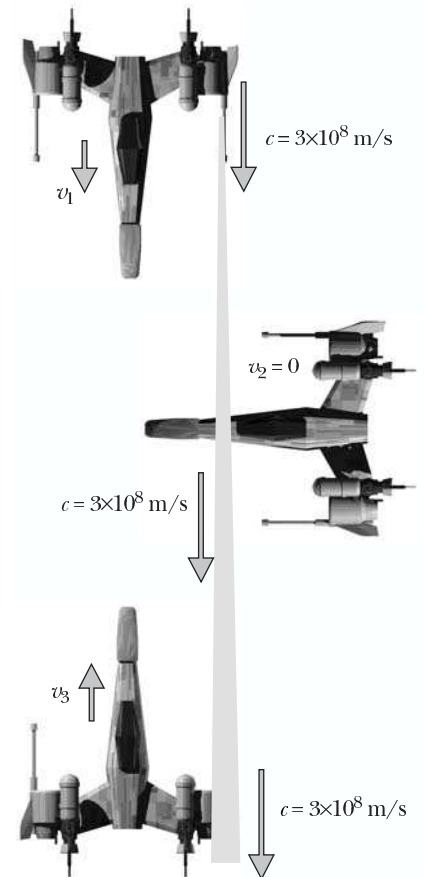

Figure 1.43 Light emitted by the top spaceship is measured to have the same speed by observers in all three ships.

On the other hand, if someone on the ship at the top throws a ball or a proton or some other piece of matter, the speed of the object will be different for observers on the three ships; it is only light whose speed is independent of the observer.

Einstein's theory has interesting consequences. For example, it predicts that time will run at different rates in different frames of reference. These predictions have been confirmed by many experiments. These unusual effects are large only at very high speeds (a sizable fraction of the speed of light), which is why we don't normally observe these effects in everyday life, and why we can use nonrelativistic calculations for low-speed phenomena.

1.10 Summary

Interactions (Section 1.2 and Section 1.4)

Interactions are indicated by
- change of velocity (change of direction and/or change of speed)
- change of identity
- change of shape of multiparticle system
- change of temperature of multiparticle system
- lack of change when change is expected

Newton's first law of motion (Section 1.3)

An object moves in a straight line and at constant speed
except to the extent that it interacts with other objects.

Vectors (Section 1.5)

A 3D *vector* is a quantity with magnitude and a direction, which can
be expressed as a triple $\langle x, y, z \rangle$. A vector is indicated by an arrow: $\vec{r}$

A *scalar* is a single number.

Legal mathematical operations involving vectors include:

- adding one vector to another vector
- subtracting one vector from another vector
- multiplying or dividing a vector by a scalar
- finding the magnitude of a vector
- taking the derivative of a vector

Operations that are *not* legal with vectors include:

- A vector cannot be added to a scalar
- A vector cannot be set equal to a scalar
- A vector cannot appear in the denominator (you can't divide by a vector)

The symbol Δ

The symbol Δ (delta) means "change of": $\Delta t = t_f - t_i$, $\Delta \vec{r} = \vec{r}_f - \vec{r}_i$

Δ always means "final minus initial".

Velocity and change of position (Section 1.7)

DEFINITION: AVERAGE VELOCITY

$$\vec{v}_{avg} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Velocity is a vector. $\vec{r}$ is the position of an object (a vector). t is the time.
Average velocity is equal to the change in position divided by the time
elapsed. SI units of velocity are meters per second (m/s).

THE POSITION UPDATE FORMULA

$$\vec{r}_f = \vec{r}_i + \vec{v}_{avg} \Delta t$$

The final position (vector) is the vector sum of the initial position plus the
product of the average velocity and the elapsed time.

DEFINITION: INSTANTANEOUS VELOCITY

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity is the limiting value of the average velocity as the time elapsed becomes very small.

Momentum (Section 1.8)

DEFINITION: MOMENTUM

$$\vec{p} = \gamma m \vec{v}$$

where $\gamma = \dfrac{1}{\sqrt{1 - (|\vec{v}|/c)^2}}$ (lower-case Greek gamma)

Momentum (a vector) is the product of the relativistic factor "gamma" (a scalar), mass, and velocity.

Combined into one equation: $\vec{p} = \dfrac{1}{\sqrt{1 - (|\vec{v}|/c)^2}} m\vec{v}$

APPROXIMATION: MOMENTUM AT LOW SPEEDS

$\vec{p} \approx m\vec{v}$ at speeds such that $|\vec{v}| \ll c$.

Result: Rate of change of momentum along a curving path (Section 1.8.5)

PERPENDICULAR COMPONENT OF $d\vec{p}/dt$
FOR MOTION ALONG A CURVING PATH

Perpendicular component of $\dfrac{d\vec{p}}{dt}$ is given by $\left| \dfrac{d\vec{p}}{dt} \right| = \dfrac{|\vec{v}|}{R} |\vec{p}|$

The direction is toward the center of the kissing circle of radius R.

Useful numbers:

Radius of a typical atom: about 1×10^{-10} meter.
Radius of a typical atomic nucleus: about 1×10^{-15} meter
Speed of light: 3×10^{8} m/s

These and other useful data and conversion factors are given on the inside back cover of the textbook.

1.11 Review questions

The purpose of review questions is to help you reflect on the most important concepts in the chapter. Try to answer the questions without flipping through the chapter looking for an answer, but think about what you know already from having done the exercises throughout the chapter. If you are stumped, look at the "Summary" on the preceding page.

If you are still stumped after looking at the chapter summary, make a note in your notebook, as a reminder that you may need to spend some extra time studying this particular concept.

Detecting interactions

RQ 1.1 Give two examples (other than those discussed in the text) of interactions that may be detected by observing:
    (a) change in velocity
    (b) change in temperature
    (c) change in shape
    (d) change in identity
    (e) lack of change when change is expected

RQ 1.2 In which of the following situations is there observational evidence for significant interaction between two objects? How can you tell?
    (a) a book rests on a table
    (b) a baseball that was hit by a batter flies toward the outfield
    (c) water freezes in an ice cube tray in the freezer
    (d) a communications satellite orbits the earth
    (e) a space probe leaves the solar system traveling at constant speed toward a distant star
    (f) a charged particle leaves a curving track in a particle detector

RQ 1.3 Which of the following observations give conclusive evidence of an interaction? (Choose all that apply.)
    (a) Change of velocity, either change of direction or change of speed.
    (b) Change of shape or configuration without change of velocity.
    (c) Change of position without change of velocity.
    (d) Change of identity without change of velocity.
    (e) Change of temperature without change of velocity.
    Explain your choice.

RQ 1.4 Moving objects left the traces labelled *A - F* in Figure 1.44. The dots were deposited at equal time intervals (for example, one dot each second). In each case the object starts from the square. Which trajectories show evidence that the moving object was interacting with another object somewhere? If there is evidence for an interaction, what is the evidence?

RQ 1.5 A spaceship far from all other objects uses its rockets to attain a speed of 1×10^4 m/s. The crew then shuts off the power. According to Newton's first law, which of the following statements about the motion of the spaceship after the power is shut off are correct? (Choose all statements that are correct.)
    (a) The spaceship will move in a straight line.
    (b) The spaceship will travel on a curving path.
    (c) The spaceship will enter a circular orbit.
    (d) The speed of the spaceship will not change.
    (e) The spaceship will gradually slow down.
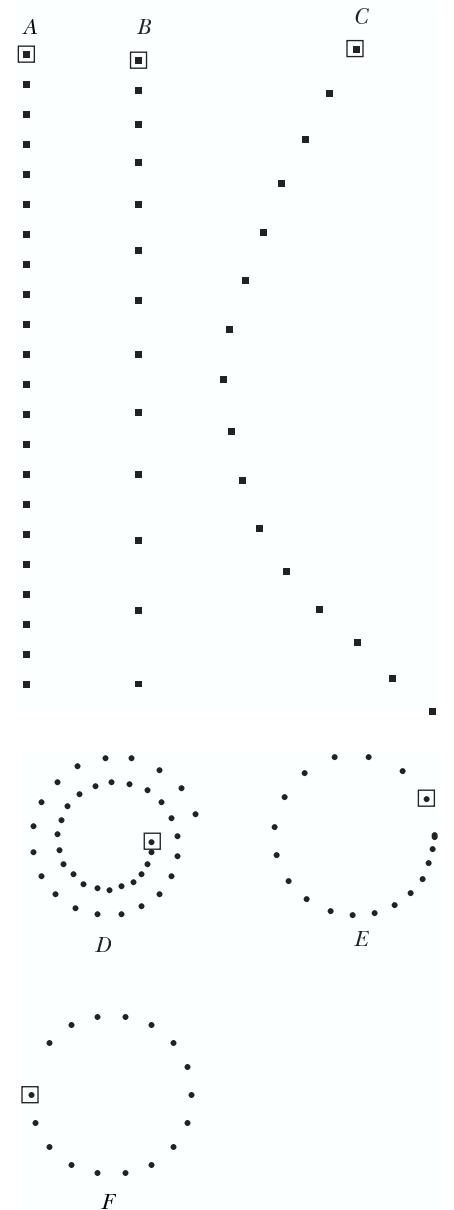   (f) The spaceship will stop suddenly.

Figure 1.44 Traces left by moving objects. The dots mark the objects' positions at equal time intervals (RQ 1.4).

RQ 1.6 Why do we use a spaceship in outer space, far from other objects, to illustrate Newton's first law? Why not a car or a train? (More than one of the following statements may be correct.)

A car or train touches other objects, and interacts with them.

A car or train can't travel fast enough.

The spaceship has negligible interactions with other objects.

A car or train interacts gravitationally with the Earth.

A spaceship can never experience a gravitational force.

RQ 1.7 You slide a coin across the floor, and observe that it travels in a straight line, slowing down and eventually stopping. A sensitive thermometer shows that the coin's temperature increased. What can we conclude? (Choose all statements that are correct.)

(a) Because the coin traveled in a straight line, we conclude that it did not interact with anything.

(b) Because the coin did not change shape, we conclude that it did not interact with anything.

(c) Because the coin slowed down, we conclude that Newton's first law does not apply to objects in everyday life, such as coins.

(d) Because the coin's speed changed, we conclude that it interacted with one or more other objects.

(e) Because the coin got hot, we conclude that it interacted with one or more other objects.

RQ 1.8 Some science museums have an exhibit called a Bernoulli blower, in which a volleyball hangs suspended in a column of air blown upward by a strong fan. If you saw a ball suspended in the air but didn't know the blower was there, why would Newton's first law suggest that something must be holding the ball up?

RQ 1.9 Place a ball on a book and walk with the book in uniform motion. Note that you don't really have to do anything to the ball to keep the ball moving with constant velocity (relative to the ground) or to keep the ball at rest (relative to you). Then stop suddenly, or abruptly change your direction or speed. What does Newton's first law of motion predict for the motion of the ball (assuming the interaction between the ball and the book is small)? Does the ball behave as predicted? It may help to take the point of view of a friend who is standing still, watching you.

Velocity

RQ 1.10 How does average velocity differ from instantaneous velocity?

RQ 1.11 Start with the definition of average velocity and derive the position update formula from it. Show all steps in the derivation.

RQ 1.12 In the expression $\Delta \vec{r}/\Delta t$, what is the meaning of $\Delta \vec{r}$? What is the meaning of Δt?

Momentum

RQ 1.13 Which of the following statements about the velocity and momentum of an object are correct?

(a) The momentum of an object is always in the same direction as its velocity

(b) The momentum of an object can either be in the same direction as its velocity or in the opposite direction

(c) The momentum of an object is perpendicular to its velocity

(d) The direction of an object's momentum is not related to the direction of its velocity

(e) The direction of an object's momentum is tangent to its path

RQ 1.14 In which of these situations is it reasonable to use the approximate formula for the momentum of an object, instead of the full relativistically correct formula?

(a) A car traveling on an interstate highway

(b) A commercial jet airliner flying between New York and Seattle

(c) A neutron traveling at 2700 meters per second

(d) A proton in outer space traveling at 2×10^8 m/s

(e) An electron in a television tube traveling 3×10^6 m/s

RQ 1.15 Answer the following questions about the factor γ (gamma) in the full relativistic formula for momentum.

(a) Is γ a scalar or a vector quantity?

(b) What is the minimum possible value of γ?

(c) Does γ reach its minimum value when an object's speed is high or low?

(d) Is there a maximum possible value for γ?

(e) Does γ become large when an object's speed is high or low?

(f) Does the approximation $\gamma \approx 1$ apply when an object's speed is low or when it is high?

Change of momentum

RQ 1.16 A tennis ball of mass m traveling with velocity $\langle v_x, 0, 0 \rangle$ hits a wall and rebounds with velocity $\langle -v_x, 0, 0 \rangle$.

(a) What was the change in momentum of the tennis ball?

(b) What was the change in the magnitude of the momentum of the tennis ball?

RQ 1.17 The radius of a merry-go-round is 7 meters, and it takes 12 seconds to make a complete revolution.

(a) What is the speed of an atom on the outer rim?

(b) Which statement below (i, ii, or iii) best describes the direction of the momentum of this atom?

(c) Which statement below (i, ii, or iii) best describes the direction of the rate of change of the momentum of this atom?

(i) Inward, toward the center

(ii) Outward, away from the center

(iii) Tangential

RQ 1.18 Figure 1.45 shows the path of a particle. Assuming the particle's speed does not change as it travels along this path, at each location labeled "x" draw an arrow showing the direction of $d\vec{p} / dt$ for the particle as it passes that location.
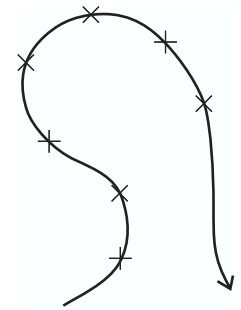

Figure 1.45 Path of a particle traveling at constant speed (RQ 1.18).

Relativity

RQ 1.19 Which of the following observers might observe something that appears to violate Newton's first law of motion? Explain why.

(a) a person standing still on a street corner

(b) a person riding on a roller coaster

(c) a passenger on a starship travelling at $0.75c$ toward the nearby star Alpha Centauri

(d) an airplane pilot doing aerobatic loops

(e) a hockey player coasting across the ice

RQ 1.20 A spaceship at rest with respect to the cosmic microwave background emits a beam of red light. A different spaceship, moving at a speed of 2.5×10^8 m/s toward the first ship, detects the light. Which of the follow-

ing statements are true for observers on the second ship? (More than one statement may be correct.)

(a) They observe that the light travels at 3×10^8 m/s.

(b) They see light that is not red.

(c) They observe that the light travels at 5.5×10^8 m/s

(d) They observe that the light travels at 2.5×10^8 m/s.

1.12 Problems

Vectors

Problem 1.1 On a piece of graph paper, draw arrows representing the following vectors. Make sure the tip and tail of each arrow you draw are clearly distinguishable.

(a) Placing the tail of the vector at $\langle 5, 2, 0 \rangle$ draw an arrow representing the vector $\vec{p} = \langle -7, 3, 0 \rangle$. Label it $\vec{p}$.

(b) Placing the tail of the vector at $\langle -5, 8, 0 \rangle$ draw an arrow representing the vector $-\vec{p}$. Label it $-\vec{p}$.

Problem 1.2 The following questions refer to the vectors depicted by arrows in Figure 1.46.

(a) What are the components of the vector $\vec{a}$? (Note that since the vector lies in the xy plane, its z component is zero.)

(b) What are the components of the vector $\vec{b}$?

(c) Is this statement true or false? $\vec{a} = \vec{b}$

(d) What are the components of the vector $\vec{c}$?

(e) Is this statement true or false: $\vec{c} = -\vec{a}$?

(f) What are the components of the vector $\vec{d}$?
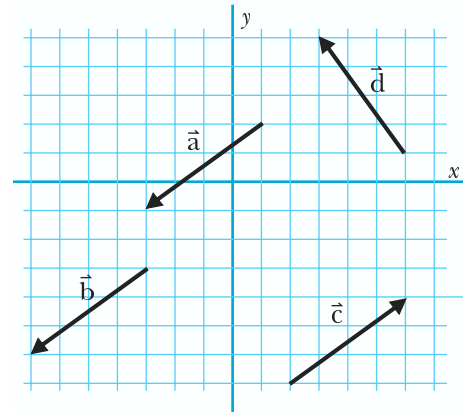
(g) Is this statement true or false: $\vec{d} = -\vec{c}$?

Problem 1.3

(a) What are the components of the vector $\vec{d}$, in Figure 1.47?

(b) If $\vec{e} = -\vec{d}$, what are the components of $\vec{e}$?

(c) If the tail of vector $\vec{d}$ were moved to location $\langle -5, -2, 4 \rangle$ m, where would the tip of the vector be located?

(d) If the tail of vector $-\vec{d}$ were placed at location $\langle -1, -1, -1 \rangle$, where would the tip of the vector be located?
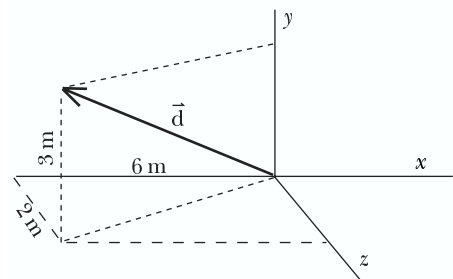
Problem 1.4 Figure 1.48 shows several arrows representing vectors in the *xy* plane.

(a) Which vectors have magnitudes equal to the magnitude of $\vec{a}$?

(b) Which vectors are equal to $\vec{a}$?

Problem 1.5 Consider a vector $\vec{u} = \langle u_x, u_y, u_z \rangle$, and another vector $\vec{p} = \langle p_x, p_y, p_z \rangle$. If $\vec{u} = \vec{p}$, then which of the following statements must be true? Some, all, or none of the following may be true:

(i) $u_x = p_x$ (ii) $u_y = p_y$ (iii) $u_z = p_z$ (iv) The direction of $\vec{u}$ is the same as the direction of $\vec{p}$.

Vector operations

Problem 1.6 In the diagram in Figure 1.49 three vectors are represented by arrows in the *xy* plane. Each square in the grid represents one meter. For each vector:

(a) Write out the components of the vector.

(b) Calculate the magnitude of the vector.

Problem 1.7 Imagine that you have a baseball and a tennis ball at different locations. The center of the baseball is at $\langle 3, 5, 0 \rangle$ m, and the center of the tennis ball is at $\langle -3, -1, 0 \rangle$ m. On a piece of graph paper, do the following:

(a) Draw dots at the locations of the center of the baseball and the center of the tennis ball.

(b) Draw the position vector of the baseball, which is an arrow whose tail is at the origin and whose tip is at thc location of the baseball. Label this position vector $\vec{B}$. Clearly show the tip and tail of each arrow.

(c) Complete this equation: $\vec{B} = \langle$ ____, ____, ____ $\rangle$ m.

Figure 1.46 These arrows represent the vectors referred to in Problem 1.2.

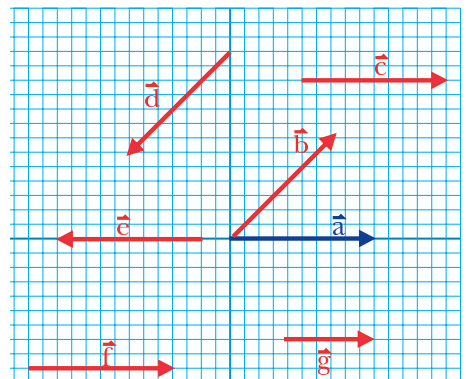Figure 1.47 An arrow representing a position vector (Problem 1.3)

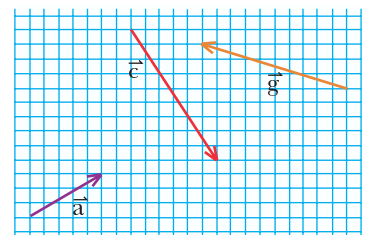Figure 1.48 Arrows representing vectors in the *xy* plane (Problem 1.4).

Figure 1.49 Arrows representing vectors in the *xy* plane (Problem 1.6).

(d) Draw the position vector of the tennis ball. Label it $\vec{T}$.

(e) Complete this equation: $\vec{T} = \langle$ ____ , ____ , ____ $\rangle$ m.

(f) Draw the relative position vector for the tennis ball relative to the baseball. The tail of this vector is at the center of the baseball, and the tip of the vector is at the center of the tennis ball. Label this relative position vector $\vec{r}$.

(g) Complete the following equation by reading the coordinates of $\vec{r}$ from your graph: $\vec{r} = \langle$ ____ , ____ , ____ $\rangle$ m

(h) Calculate the following difference: $\vec{T} - \vec{B} = \langle$ ____ , ____ , ____ $\rangle$ m.

(i) Is the following statement true? $\vec{r} = \vec{T} - \vec{B}$ _____

(j) Write two other equations relating the vectors $\vec{B}$, $\vec{T}$, and $\vec{r}$.

(k) Calculate the magnitudes of the vectors $\vec{B}$, $\vec{T}$, and $\vec{r}$.

(l) Calculate the difference of the magnitudes $|\vec{T}| - |\vec{B}|$.

(m) Does $|\vec{T}| - |\vec{B}| = |\vec{T} - \vec{B}|$?

Problem 1.8 Which of the following are vectors?

(a)   3.5   (b)   0   (c) $\langle 0.7, 0.7, 0.7 \rangle$ (d) $\langle 0, 2.3, -1 \rangle$ (e) -3×10^6

(f) $3 \cdot \langle 14, 0, -22 \rangle$

Problem 1.9 Which of the following are vectors?

(a) $\vec{r}/2$ (b) $|\vec{r}|/2$ (c) $\langle r_x, r_y, r_z \rangle$ (d) $5 \cdot \vec{r}$

Problem 1.10 (a) What is the vector whose tail is at $\langle 9.5, 7, 0 \rangle$ m and whose head is at $\langle 4, -13, 0 \rangle$ m? (b) What is the magnitude of this vector?

Problem 1.11 A man is standing on the roof of a building with his head at the position $\langle 12, 30, 13 \rangle$ m. He sees the top of a tree, which is at the position $\langle -25, 35, 43 \rangle$ m.

(a) What is the relative position vector that points from the man's head to the top of the tree?

(b) What is the distance from the man's head to the top of the tree?

Problem 1.12 (a) On a piece of graph paper, draw the vector $\vec{f} = \langle -2, 4, 0 \rangle$, putting the tail of the vector at $\langle -3, 0, 0 \rangle$. Label the vector $\vec{f}$.

(b) Calculate the vector $2\vec{f}$, and draw this vector on the graph, putting its tail at $\langle -3, -3, 0 \rangle$, so you can compare it to the original vector. Label the vector $2\vec{f}$.

(c) How does the magnitude of $2\vec{f}$ compare to the magnitude of $\vec{f}$?

(d) How does the direction of $2\vec{f}$ compare to the direction of $\vec{f}$?

(e) Calculate the vector $\vec{f}/2$, and draw this vector on the graph, putting its tail at $\langle -3, -6, 0 \rangle$, so you can compare it to the other vectors. Label the vector $\vec{f}/2$.

(f) How does the magnitude of $\vec{f}/2$ compare to the magnitude of $\vec{f}$?

(g) How does the direction of $\vec{f}/2$ compare to the direction of $\vec{f}$?

(h) Does multiplying a vector by a scalar change the magnitude of the vector?

(i) The vector $a(\vec{f})$ has a magnitude three times as great as that of $\vec{f}$, and its direction is opposite to the direction of $\vec{f}$. What is the value of the scalar factor a?

Problem 1.13 (a) On a piece of graph paper, draw the vector $\vec{g} = \langle 4, 7, 0 \rangle$ m. Put the tail of the vector at the origin.

(b) Calculate the magnitude of $\vec{g}$

(c) Calculate $\hat{g}$, the unit vector pointing in the direction of $\vec{g}$.

(d) On the graph draw $\hat{g}$. Put the tail of the vector at $\langle 1, 0, 0 \rangle$ so you can compare $\hat{g}$ and $\vec{g}$.

(e) Calculate the product of the magnitude $|\vec{g}|$ times the unit vector $\hat{g}$: $(|\vec{g}|)(\hat{g})$

Problem 1.14 A proton is located at $\langle 3\times10^{-10}, -3\times10^{-10}, 8\times10^{-10} \rangle$ m .
(a) What is $\vec{r}$, the vector from the origin to the location of the proton?
(b) What is $|\vec{r}|$?
(c) What is $\hat{r}$, the unit vector in the direction of $\vec{r}$?

Problem 1.15 Which of the following statements about the vectors depicted by arrows in Figure 1.50 are correct?
(a) $\vec{s} = \vec{t} - \vec{r}$ (b) $\vec{r} = \vec{t} - \vec{s}$ (c) $\vec{r} + \vec{t} = \vec{s}$ (d) $\vec{s} + \vec{t} = \vec{r}$ (e) $\vec{r} + \vec{s} = \vec{t}$
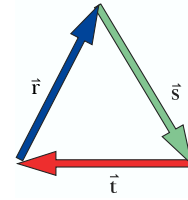
Problem 1.16 Which of the following are unit vectors? (Numerical values are given only to 3 significant figures.)
(a) $\langle 0, 0, -1 \rangle$ (b) $\langle 0.5, 0.5, 0 \rangle$ (c) $\langle 0.333, 0.333, 0.333 \rangle$ (d) $\langle 0.9, 0, 0.1 \rangle$
(e) $\langle 0, 3, 0 \rangle$ (f) $\langle 1, -1, 1 \rangle$ (g) $\langle 0.577, 0.577, 0.577 \rangle$ (h) $\langle 0.949, 0, -0.316 \rangle$

Problem 1.17 Two vectors, $\vec{f}$ and $\vec{g}$, are equal: $\vec{f} = \vec{g}$. Which of the following statements are true?
$\hat{f} = \hat{g}$ (b) $g_x = f_x$ (c) $f_z = g_z$ (d) the directions of $\vec{f}$ and $\vec{g}$ may be different (e) the magnitudes of $\vec{f}$ and $\vec{g}$ may be different

Problem 1.18 A proton is located at $\vec{r}_p = \langle 2, 6, -3 \rangle$ m . An electron is located at $\vec{r}_e = \langle 4, 12, -6 \rangle$ m . Which of the following statements are true?
(a) $2\vec{r}_p = \vec{r}_e$ (b) $2\hat{r}_p = \hat{r}_e$ (c) $|2\vec{r}_p| = |\vec{r}_e|$

Problem 1.19 A proton is located at $\langle x_p, y_p, z_p \rangle$. An electron is located at $\langle x_e, y_e, z_e \rangle$. What is the vector pointing from the electron to the proton? What is the vector pointing from the proton to the electron?

Problem 1.20 The vector $\vec{a} = \langle 0.03, -1.4, 26.0 \rangle$ and the scalar $f = -3.0$. What is $f\vec{a}$?

Problem 1.21 The vector $\vec{g} = \langle 2, -7, 3 \rangle$ and the scalar $h = -2$. What is $h + \vec{g}$?
(a) $\langle 0, -9, 1 \rangle$ (b) $\langle 4, -5, 5 \rangle$ (c) $\langle 4, 9, 5 \rangle$ (d) This is a meaningless expression.

Problem 1.22 Write each of these vectors as the product of the magnitude of the vector and the appropriate unit vector:
(a) $\langle 0, 0, 9.5 \rangle$ (b) $\langle 0, -679, 0 \rangle$ (c) $\langle 3.5\times10^{-3}, 0, -3.5\times10^{-3} \rangle$ (d) $\langle 4\times10^{6}, -6\times10^{6}, 3\times10^{6} \rangle$

Problem 1.23 $\vec{A} = \langle 3\times10^{3}, -4\times10^{3}, -5\times10^{3} \rangle$ and $\vec{B} = \langle -3\times10^{3}, 4\times10^{3}, 5\times10^{3} \rangle$. Calculate the following: (a) $\vec{A} + \vec{B}$ (b) $|\vec{A} + \vec{B}|$ (c) $|\vec{A}|$ (d) $|\vec{B}|$ (e) $|\vec{A}| + |\vec{B}|$

Figure 1.50 Arrows representing three vectors (Problem 1.15).

Problems on velocity and momentum

Problem 1.24 A baseball has a mass of 0.155 kg. A professional pitcher throws a baseball 90 miles per hour, which is 40 m/s. What is the magnitude of the momentum of the pitched baseball?

Problem 1.25 The position of a golf ball relative to the tee changes from <50, 20, 30> m to <53, 18, 31> m in 0.1 second. As a vector, write the velocity of the golf ball during this short time interval.

Problem 1.26 A hockey puck with a mass of 0.4 kg has a velocity of $\langle 38, 0, -27 \rangle$ m/s. What is the magnitude of its momentum, $|\vec{p}|$?

Problem 1.27 A proton in an accelerator attains a speed of $0.88c$. What is the magnitude of the momentum of the proton?

Problem 1.28 The crew of a stationary spacecraft observe an asteroid whose mass is 4×10^{17} kg. Taking the location of the spacecraft as the origin, the asteroid is observed to be at location $\langle -3 \times 10^3, -4 \times 10^3, 8 \times 10^3 \rangle$ m at a time 18.4 seconds after lunchtime. At a time 21.4 seconds after lunchtime, the asteroid is observed to be at location $\langle -1.4 \times 10^3, -6.2 \times 10^3, 9.7 \times 10^3 \rangle$ m. Assuming the velocity of the asteroid does not change during this time interval, calculate the vector velocity $\vec{v}$ of the asteroid.

Problem 1.29 An electron with a speed of $0.95c$ is emitted by a supernova, where c is the speed of light. What is the magnitude of the momentum of this electron?

Problem 1.30 A "cosmic-ray" proton hits the upper atmosphere with a speed $0.9999c$, where c is the speed of light. What is the magnitude of the momentum of this proton?

Problem 1.31 The position of a baseball relative to home plate changes from <15, 8, −3> m to <20, 6, −1> m in 0.1 second. As a vector, write the average velocity of the baseball during this time interval.

Problem 1.32 Figure 1.51 shows the trajectory of a ball traveling through the air, affected by both gravity and air resistance. The table in Figure 1.52 gives the position of the ball at several successive times.
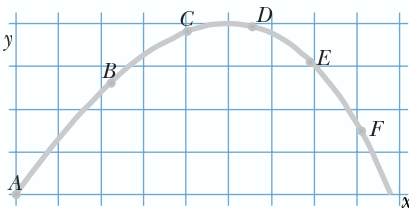
(a) What is the average velocity of the ball as it travels between location A and location B?

(b) If the ball continued to travel at the same average velocity during the next second, where would it be at the end of that second? (That is, where would it be at time $t = 2$ seconds?)

(c) How does your prediction from part (b) compare to the actual position of the ball at $t = 2$ seconds (location C)? If the predicted and observed locations of the ball are different, explain why.

Problem 1.33 In a laboratory experiment, an electron passes location $\langle 0.02, 0.04, -0.06 \rangle$ m, and 2 μs (1 microsecond = 1×10^{-6} s) later is detected at location $\langle 0.02, 1.84, -0.86 \rangle$ m.

(a) What is the average velocity of the electron?

(b) If the electron continues to travel at this average velocity, where will it be in another 5 μs?

Problem 1.34 At 6 seconds after 3:00, a butterfly is observed leaving a flower whose location is $\langle 6, -3, 10 \rangle$ m relative to an origin on top of a nearby tree. The butterfly flies until 10 seconds after 3:00, when it alights on a different flower whose location is $\langle 6.8, -4.2, 11.2 \rangle$ m relative to the same origin.

Figure 1.51 Trajectory of a ball traveling through air, affected by gravity and air resistance (Problem 1.32).

loc.	t (s)	position (m)
A	0.0	$\langle 0, 0, 0 \rangle$
B	1.0	$\langle 22.3, 26.1, 0 \rangle$
C	2.0	$\langle 40.1, 38.1, 0 \rangle$

Figure 1.52 Table showing position of ball in Figure 1.51, at various times.

What was the location of the butterfly at a time 8.5 seconds after 3:00? What assumption did you have to make in calculating this location?

Problem 1.35 The gray line in Figure 1.53 shows a portion of the trajectory of a ball traveling through the air. At various locations, the ball's momentum is:

$\vec{p}_B = \langle 3.03, 2.83, 0 \rangle$ kg·m/s
$\vec{p}_C = \langle 2.55, 0.97, 0 \rangle$ kg·m/s
$\vec{p}_D = \langle 2.24, -0.57, 0 \rangle$ kg·m/s
$\vec{p}_E = \langle 1.97, -1.93, 0 \rangle$ kg·m/s
$\vec{p}_F = \langle 1.68, -3.04, 0 \rangle$ kg·m/s

(a) Calculate the change in the ball's momentum between each pair of adjacent locations.

(b) On a copy of Figure 1.53, draw arrows representing each $\Delta\vec{p}$ you calculated in part (a)

(c) Between which two locations is the magnitude of the change in momentum greatest?

Problem 1.36 A spacecraft traveling at a velocity of $\langle -20, -90, 40 \rangle$ m/s is observed to be at a location $\langle 200, 300, -500 \rangle$ m relative to an origin located on a nearby asteroid. At a later time the spacecraft is observed to be at location $\langle -380, -2310, 660 \rangle$ m.

a) How long did it take the spacecraft to travel between these locations?

b) How far did the spacecraft travel?

c) What is the speed of the spacecraft?

d) What is the unit vector in the direction of the spacecraft's velocity?

Problem 1.37 A person of mass 70 kg rides on a Ferris wheel whose radius is 4 m. The person's speed is constant at 0.3 m/s. The person's location is shown by a dot in the diagram in Figure 1.54.

(a) What is the magnitude of the rate of change of the momentum of the person at the instant shown?.

(b) What is the direction of the rate of change of momentum of the person at the instant shown?

Computational problems

These problems are intended to introduce you to using a computer to model matter, interactions, and motion. You will build on these small calculations to build models of physical systems in later chapters.

Some parts of these problems can be done with almost any tool (spreadsheet, math package, etc.). Other parts are most easily done with a programming language. We recommend the free 3D programming language VPython (http://vpython.org). Your instructor will introduce you to an available computational tool and assign problems, or parts of problems, that can be addressed using the chosen tool.

Problem 1.38 Move an object across a computer screen

(a) Write a program that makes an object move from left to right across the screen at speed v. Make v a variable, so you can change it later. Let the time interval for each step of the computation be a variable dt, so that the position x increases by an amount $v*dt$ each time.

(b) Modify a copy of your program to make the object run into a wall and reverse its direction.

(c) Make a modification so that the object's speed v is no longer a constant but changes smoothly with time. Is the speed change clearly visible to
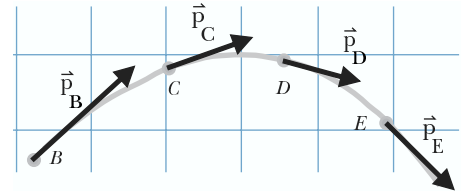

Figure 1.53 The gray line shows a portion of the path of a ball traveling through air, affected by gravity and air resistance. Arrows indicate its momentum at several locations (Problem 1.35).
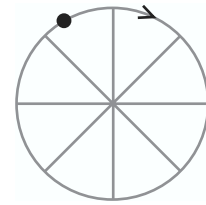

Figure 1.54 A person (represented by a dot) riding on a Ferris wheel (Problem 1.37).

an observer? Try to make one version in which the speed change is clearly noticeable, and another in which it is not noticeable.

(d) Corresponding to part (c), make a computer graph of x vs. t, where t is the time.

(e) Corresponding to part (c), make a computer graph of v vs. t, where t is the time.

Turn in your programs for parts (c), (d), and (e).

Problem 1.39 Move an object at an angle

(a) Write a program that makes an object move at an angle.

(b) Change the component of velocity of the object in the x direction but not in the y direction, or vice versa. What do you observe?

(c) Start the object moving at an angle and make it bounce off at an appropriate angle when it hits a wall.

Turn in your answer to part (b), and the final version of your computation, part (c).

Problem 1.40 Move an object, leave a trail

Write a program that makes an object move smoothly from left to right across the screen at speed v, leaving a trail of dots on the screen at equal time intervals. If the dots are too close together, leave a dot every N steps, and adjust N to give a nice display.

1.13 Answers to exercises

1.1 (page 7)   a, d

1.2 (page 7)   a, b, c

1.3 (page 8)   Continues to move in same direction at 1×10^4 m/s.

1.4 (page 11)   3

1.5 (page 11)   1

1.6 (page 11)   vector

1.7 (page 11)   b, c

1.8 (page 11)   7

1.9 (page 12)   scalar

1.10 (page 12)   no

1.11 (page 12)   5.10 m

1.12 (page 12)   no

1.13 (page 12) 2.15×10^7 **m/s**

1.14 (page 13) $\langle 6, -9, 15 \rangle$ **m/s**

1.15 (page 13) $\langle 1, -1.5, 2.5 \rangle$ **m/s**

1.16 (page 13) $\langle 0.04, -3.4, 60.0 \rangle$

1.17 (page 13)   in the opposite direction

1.18 (page 13)   no

1.19 (page 13)   no

1.20 (page 14) $\langle 0, 1, 0 \rangle$

1.21 (page 14) $\langle -1, 0, 0 \rangle$

1.22 (page 14) $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle , \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$

1.23 (page 14) $\langle 0.873, 0.436, -0.218 \rangle$

1.24 (page 14) $458 \langle 0.873, 0.436, -0.218 \rangle \frac{m}{s^2}$

1.25 (page 16) $\langle 450, -300, -200 \rangle$

1.26 (page 16) $361, 335, 577$

1.27 (page 16) 696, no

1.28 (page 16) $\langle 150, 300, -200 \rangle , \langle -150, -300, 200 \rangle$

1.29 (page 16) $\langle -1, 0, -1 \rangle$ **m**

1.30 (page 16) $\langle 2, 4, 0 \rangle$ **m**

1.31 (page 17) $\langle -2, -4, 0 \rangle$ **m**

1.32 (page 17) 2.67×10^{-3} m/s

1.33 (page 20) $\langle 8, -10, 12 \rangle$ m/s, 17.55 m/s, $\langle 0.456, -0.570, 0.684 \rangle$

1.34 (page 20) $\langle 25, -55, -30 \rangle$ m/s, 67.45 m/s, $\langle 0.371, -0.815, -0.445 \rangle$

1.35 (page 21) $\langle 2.1 \times 10^5, 1.4 \times 10^5, -2.8 \times 10^5 \rangle$ m

1.36 (page 21) 0.5 s

1.37 (page 21) $\langle -2.01 \times 10^5, 5.20 \times 10^4, -1.00 \times 10^3 \rangle$ m/s, 2.08×10^5 m/s

1.38 (page 27) 650 kg·m/s

1.39 (page 27) 6.9 kg·m/s

1.40 (page 27) 2.22×10^5 kg·m/s

1.41 (page 27) 2.415×10^{-22} kg·m/s

1.42 (page 27) $\gamma = 7.09$

1.43 (page 27) 6.71 kg·m/s

1.44 (page 29) $\vec{p}_i$ to the right ($+x$), $\vec{p}_f$ to the left ($-x$),

$\langle -5.59, 0, 0 \rangle$ kg·m/s, 0.114 kg·m/s

1.45 (page 29) $\langle -1.6 \times 10^{28}, 0, 1.6 \times 10^{28} \rangle$ kg·m/s, downward to the left

1.46 (page 29) 0, 500 kg·m/s

1.47 (page 33) The third (bottom) one (R_3)

1.48 (page 33) downward on the page

1.49 (page 33) 0.975 kg·m/s^2, toward the center

1.50 (page 33) 3.57×10^{22} kg·m/s^2, toward Sun