

PHYS 501: Mathematical Physics I

Fall 2022

Solutions to Homework #4

1. (a) (i) Setting $z = 1$, we find

$$\sum_{n=0}^{\infty} P_n(1)h^n = (1 - 2h + h^2)^{-1/2} = (1 - h)^{-1}.$$

Application of the binomial theorem gives

$$(1 - h)^{-1} = \sum_{n=0}^{\infty} h^n,$$

so $P_n(1) = 1$ for all n .

- (ii) Setting $z = 0$, we have

$$\sum_{n=0}^{\infty} P_n(1)h^n = (1 + h^2)^{-1/2}.$$

The binomial expansion of the expression on the right is

$$(1 + h^2)^{-1/2} = 1 - \frac{1}{2}h^2 + \frac{3}{8}h^4 + \dots + \frac{(-1)^m 1.3.5 \dots (2m-1)}{2^m m!} h^{2m} + \dots$$

Thus $P_{2m+1}(0) = 0$ (no odd powers of h) and

$$P_{2m}(0) = \frac{(-1)^m 1.3.5 \dots (2m-1)}{2^m m!} = \frac{(-1)^m (2m-1)!!}{2^m m!} = \frac{(-1)^m (2m)!}{2^{2m} (m!)^2}.$$

- (iii) Differentiating the generating function equation with respect to z gives

$$h(1 - 2zh + h^2)^{-3/2} = \sum_{n=0}^{\infty} P'_n(z)h^n,$$

so

$$\begin{aligned} \sum_{n=0}^{\infty} P'_n(1)h^n &= h(1 - h)^{-3} = h \left(1 + 3h + \dots + \frac{(-3) \cdot (-4) \dots (-n-2)}{n!} (-h)^n + \dots \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(n+1)(n+2)h^{n+1} \\ &= \sum_{m=1}^{\infty} \frac{1}{2}m(m+1)h^m \end{aligned}$$

so

$$P'_m(1) = \frac{1}{2}m(m+1).$$

Since $P_0(z) = 1$, the result is true for $m = 0$ too.

(iv) Similarly,

$$\begin{aligned}\sum_{n=0}^{\infty} P'_n(0)h^n &= h(1+h^2)^{-3/2} \\ &= h - \frac{3}{2}h^3 + \dots + \frac{(-3)(-5)\dots(-2m-1)}{2^m m!} h^{2m+1} + \dots\end{aligned}$$

so $P'_{2m}(0) = 0$ and

$$P'_{2m+1}(0) = \frac{(-1)^m (2m+1)!!}{2^m m!} = \frac{(-1)^m (2m+1)!}{2^{2m} (m!)^2} = (2m+1)P_{2m}(0).$$

(b) Using Stirling's formula, as $m \rightarrow \infty$

$$\begin{aligned}P_{2m}(0) &= \frac{(-1)^m (2m)!}{2^{2m} (m!)^2} \\ &\approx \frac{(-1)^m \sqrt{4\pi m} (2m)^{2m} e^{-2m}}{2^{2m} (\sqrt{2\pi m} m^m e^{-m})^2} \\ &= (-1)^m \frac{\sqrt{4\pi m}}{2\pi m} \\ &= \frac{(-1)^m}{\sqrt{\pi m}}.\end{aligned}$$

(c) Since $P_0(x) = 1$, the integral is $\int_{-1}^1 P_0(x)P_n(x)dx = 2\delta_{n0}$ by the orthogonality property of the P_n .

(d) We can use one of the recurrence relations to write

$$P_n(x) = \frac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1},$$

so

$$\begin{aligned}I_n &\equiv \int_0^1 P_n(x)dx = \frac{1}{2n+1} [P_{n+1}(1) - P_{n+1}(0) - (P_{n-1}(1) - P_{n-1}(0))] \\ &= \frac{1}{2n+1} [P_{n-1}(0) - P_{n+1}(0)].\end{aligned}$$

Thus the integral is zero for even n ; for odd $n = 2m+1$, using the result from part (a)(ii), we find

$$\begin{aligned}I_{2m+1} &= \frac{1}{4m+3} \left[\frac{(-1)^m (2m)!}{2^{2m} (m!)^2} - \frac{(-1)^{m+1} (2m+2)!}{2^{2m+2} [(m+1)!]^2} \right] \\ &= \frac{(-1)^m (2m)!}{(4m+3)2^{2m} (m!)^2} \left[1 + \frac{2m+1}{2(m+1)} \right] \\ &= \frac{(-1)^m (2m)!}{2^{2m+1} (m+1)(m!)^2}.\end{aligned}$$

2. (a) The general solution to Laplace's equation is

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\alpha_{lm} r^l + \beta_{lm} r^{-l-1} \right) Y_l^m(\theta, \phi).$$

For $r < b$, the solution must satisfy the boundary condition that $\Phi = \Phi_0$ when $r = a$. Hence

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\alpha_{lm} a^l + \beta_{lm} a^{-l-1} \right) Y_l^m(\theta, \phi) = \Phi_0 = \sqrt{4\pi} Y_0^0(\theta, \phi) \Phi_0$$

(since $\sqrt{4\pi} Y_0^0 = 1$). For equality to hold, it must do so for every (l, m) pair. Hence

$$\begin{aligned} \alpha_{00} + \beta_{00} a^{-1} &= \sqrt{4\pi} \Phi_0 \\ \alpha_{lm} a^l + \beta_{lm} a^{-l-1} &= 0, \end{aligned}$$

so

$$\begin{aligned} \alpha_{00} &= \sqrt{4\pi} \Phi_0 - \beta_{00} a^{-1} \\ \alpha_{lm} &= -\beta_{lm} a^{-2l-1} \quad (l \neq 0, m \neq 0). \end{aligned}$$

(b) For $r > b$ (replacing α and β inside by γ and δ outside) the boundary condition at infinity implies $\gamma_{lm} = 0$.

(c) The solutions then are

$$\Phi(r, \theta, \phi) = \begin{cases} \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(\alpha_{lm} r^l + \beta_{lm} r^{-l-1} \right) Y_l^m(\theta, \phi) & (r < b), \\ \sum_{l=0}^{\infty} \sum_{m=-l}^l \delta_{lm} r^{-l-1} Y_l^m(\theta, \phi) & (r > b). \end{cases}$$

Continuity at $r = b$ implies

$$\alpha_{lm} b^l + \beta_{lm} b^{-l-1} = \delta_{lm} b^{-l-1}.$$

for all l, m .

(d) Due to the surface charge on the shell, the radial component of the electric field $-\partial\Phi/\partial r$ has a discontinuous jump at $r = b$, as described by Gauss's Law:

$$\left(-\frac{\partial\Phi}{\partial r} \right)_{r=b+} - \left(-\frac{\partial\Phi}{\partial r} \right)_{r=b-} = \frac{\sigma}{\epsilon_0}.$$

If σ is expressed as a spherical harmonic expansion

$$\sigma(b, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sigma_{lm} Y_l^m(\theta, \phi),$$

then, applying this condition term by term, we must have

$$(l+1) \delta_{lm} b^{-l-2} - \left[-l \alpha_{lm} b^{l-1} + (l+1) \beta_{lm} b^{-l-2} \right] = \frac{\sigma_{lm}}{\epsilon_0}.$$

Note that the given expression for σ consists of just two modes (using the conventional definitions of Y_l^m given in Riley & Hobson, p.340):

$$\begin{aligned}\sigma_{21} &= -\sqrt{\frac{8\pi}{15}} \sigma_0 \\ \sigma_{2,-1} &= -i\sigma_{21}.\end{aligned}$$

We can now solve for α_{lm} , β_{lm} , and δ_{lm} . Dropping the subscripts to avoid clutter, we have two cases:

(i) For $l = m = 0$, we have

$$\begin{aligned}\alpha + \beta a^{-1} &= \sqrt{4\pi} \Phi_0 \\ \alpha + \beta b^{-1} &= \delta b^{-1} \\ \delta b^{-2} - \beta b^{-2} &= \frac{\sigma}{\epsilon_0} = 0 \text{ here.}\end{aligned}$$

The solutions are easily shown to be

$$\begin{aligned}\alpha &= 0 \\ \beta &= \delta = \sqrt{4\pi} a \Phi_0.\end{aligned}$$

(ii) For other l and m , we have

$$\begin{aligned}\alpha + \beta a^{-2l-1} &= 0 \\ \alpha b^l + \beta b^{-l-1} &= \delta b^{-l-1} \\ (l+1)\delta b^{-l-2} + l\alpha b^{l-1} - (l+1)\beta b^{-l-2} &= \frac{\sigma}{\epsilon_0}.\end{aligned}$$

The solutions are

$$\begin{aligned}\alpha &= \frac{\sigma b^{-l+1}}{(2l+1)\epsilon_0} \\ \beta &= -\frac{\sigma b^{l+2}}{(2l+1)\epsilon_0} \left(\frac{a}{b}\right)^{2l+1} \\ \delta &= \frac{\sigma b^{l+2}}{(2l+1)\epsilon_0} \left[1 - \left(\frac{a}{b}\right)^{2l+1}\right].\end{aligned}$$

Putting together the pieces, the complete solution for $r < a$ is

$$\begin{aligned}\Phi(r, \theta, \phi) &= \beta_{00} r^{-1} Y_0^0 + \alpha_{21} r^2 Y_{21} + \beta_{21} r^{-3} Y_{21} + \alpha_{2,-1} r^2 Y_{2,-1} + \beta_{2,-1} r^{-3} Y_{2,-1} \\ &= \Phi_0 \left(\frac{a}{r}\right) + \frac{\sigma_0 b}{5\epsilon_0} \left[\left(\frac{r}{b}\right)^2 - \left(\frac{a}{b}\right)^5 \left(\frac{b}{r}\right)^3 \right] \sin 2\theta \cos \phi.\end{aligned}$$

For $r > b$,

$$\begin{aligned}\Phi(r, \theta, \phi) &= \delta_{00} r^{-1} Y_0^0 + \delta_{21} r^{-3} Y_{21} + \delta_{2,-1} r^{-3} Y_{2,-1} \\ &= \Phi_0 \left(\frac{a}{r}\right) + \frac{\sigma_0 b}{5\epsilon_0} \left[1 - \left(\frac{a}{b}\right)^5\right] \left(\frac{b}{r}\right)^3 \sin 2\theta \cos \phi.\end{aligned}$$

3. (a) Inside the spherical cavity formed by the two hemispheres, the general solution of Laplace's equation is

$$\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(b_{lm} r^{-l-1} + c_{lm} r^l \right) P_l^m(\cos \theta) e^{im\phi},$$

where r, θ, ϕ are spherical polar coordinates and the line of contact between the hemispheres is at $\theta = \frac{\pi}{2}$. For ϕ to be regular at $r = 0$ we must have $b_{lm} = 0$; axial symmetry implies that $c_{lm} = 0$ for $m \neq 0$. Thus the solution is of the form

$$\phi(r, \theta) = \sum_{l=0}^{\infty} c_l r^l P_l(\cos \theta).$$

The boundary condition at $r = a$ is

$$\phi(a, \theta) = \sum_{l=0}^{\infty} c_l a^l P_l(\cos \theta) = \begin{cases} +V_0, & 0 \leq \theta < \frac{\pi}{2}, \\ -V_0, & \frac{\pi}{2} < \theta \leq \pi. \end{cases}$$

Inverting this Legendre series, we find

$$\begin{aligned} c_l a^l \left(\frac{2}{2l+1} \right) &= \int_0^\pi \phi(a, \theta) P_l(\cos \theta) d(\cos \theta) \\ &= V_0 \left[\int_{-1}^0 -P_l(\mu) d\mu + \int_0^1 P_l(\mu) d\mu \right] \\ &= \begin{cases} 0 & (l \text{ even}) \\ 2V_0 \int_0^1 P_l(\mu) d\mu & (l \text{ odd}) \end{cases} \end{aligned}$$

Hence, for $l = 2m + 1$, we can evaluate the integral using the solution to problem 1(d), to find

$$\phi(r, \theta) = V_0 \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)(2m)!}{2^{2m+1}(m+1)(m!)^2} \left(\frac{r}{a} \right)^{2m+1} P_{2m+1}(\cos \theta).$$

- (b) Now we are seeking a solution to the wave equation

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = 0,$$

with boundary conditions specified on the same hemispheres as in part (a), except that now the boundary values are variable,

$$\Phi(t, r = a) = \pm V_0 e^{-i\omega t},$$

and we want the solution for $r > a$.

As usual, we seek $e^{-i\omega t}$ time dependence, so the spatial part χ of the solution satisfies the Helmholtz equation

$$\nabla^2 \chi + k^2 \chi = 0,$$

with $k = \omega/c$. We require axisymmetry, so the general solution is

$$\Phi(r, \theta, t) = e^{-i\omega t} \sum_l [A_l j_l(kr) + B_l n_l(kr)] P_l(\cos \theta),$$

where we must retain both the j_l and the n_l solutions for $r > a$. Since the asymptotic forms are

$$j_l(x) \sim \frac{1}{x} \cos \left[x - \frac{\pi}{2}(l+1) \right], \quad n_l(x) \sim \frac{1}{x} \sin \left[x - \frac{\pi}{2}(l+1) \right],$$

as $x \rightarrow \infty$, the combination $h_l^{(1)} = j_l + in_l$ clearly satisfies the “outgoing wave” condition as $r \rightarrow \infty$ and the solution takes the form

$$\Phi(r, \theta, t) = e^{-i\omega t} \sum_l C_l h_l^{(1)}(kr) P_l(\cos \theta).$$

At $r = a$,

$$\Phi(a, \theta, t) = e^{-i\omega t} \sum_l C_l h_l^{(1)}(ka) P_l(\cos \theta) = \pm V_0 e^{-i\omega t},$$

and the solution is essentially the same as in part (a), with $l = 2m + 1$ and a^l replaced with $h_l^{(1)}(ka)$:

$$c_{2m+1} h_{2m+1}^{(1)}(ka) = V_0 \frac{4m+3}{I_{2m+1}}.$$

(again using the terminology of Problem 1d). The complete solution therefore is

$$\phi(r, \theta, t) = V_0 e^{-i\omega t} \sum_{m=0}^{\infty} \frac{(-1)^m (4m+3)(2m)!}{2^{2m+1}(m+1)(m!)^2} \frac{h_{2m+1}^{(1)}(kr)}{h_{2m+1}^{(1)}(ka)} P_{2m+1}(\cos \theta).$$

4. We wish to evaluate

$$\phi = \int d^3 r_1 \int d^3 r_2 \psi^*(\mathbf{r}_1) \psi^*(\mathbf{r}_2) \frac{e^2}{r_{12}} \psi(\mathbf{r}_1) \psi(\mathbf{r}_2),$$

where $\psi(\mathbf{r}) = \left(\frac{Z^3}{\pi a_0^3} \right)^{1/2} e^{-Zr/a_0}$, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, and $Z = 2$ here. Expand the r_{12}^{-1} term in spherical harmonics:

$$\frac{1}{r_{12}} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} Y_l^m(\theta_1, \phi_1)^* Y_l^m(\theta_2, \phi_2) \frac{r_{<}^l}{r_{>}^{l+1}},$$

where $r_{<} = \min(r_1, r_2)$ and $r_{>} = \max(r_1, r_2)$, with $r = |\mathbf{r}|$. Writing $d^3 r = r^2 dr d\Omega$, the angular (Ω_1 and Ω_2) integrals give zero except for $l = m = 0$, and

$$\int d\Omega_1 Y_0^0(\theta_1, \phi_1)^* = \int d\Omega_2 Y_0^0(\theta_2, \phi_2) = \sqrt{4\pi},$$

so

$$\phi = 16\pi^2 e^2 \left(\frac{Z^3}{\pi a_0^3} \right)^2 \int r_1^2 dr_1 \int r_2^2 dr_2 e^{-2Zr_1/a_0} e^{-2Zr_2/a_0} \frac{1}{\max(r_1, r_2)}.$$

Splitting the r_2 integral into two parts ($0 < r_2 < r_1$ and $r_1 < r_2 < \infty$), we have

$$\phi = \frac{16Z^6 e^2}{a_0^6} \int_0^{\infty} dr_1 r_1 e^{-2Zr_1/a_0} \left[\int_0^{r_1} dr_2 r_2^2 e^{-2Zr_2/a_0} + r_1 \int_{r_1}^{\infty} dr_2 r_2 e^{-2Zr_2/a_0} \right].$$

After some algebra (or application of Maple), this yields the desired result

$$\phi = \frac{5Ze^2}{8a_0} = \frac{5e^2}{4a_0}.$$