PHYS 231 Lecture Notes – Week 4

Reading from Maoz (2$^{\text{nd}}$ edition):

- **Chapter 3**

A lot of the material presented in class this week is well covered in Maoz, and we simply reference the book, with additional comments and derivations as needed. References to slides on the web page are in the format "slidesx.y/nn," where x is week, y is lecture, and nn is slide number. Note: references to class slides and figures in the new Maoz edition have not yet been checked.

4.1 Virial Theorem

See Maoz §3.1.

Here's an alternative derivation of the virial theorem. Starting from the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}.$$

Multiplying by $4\pi r^3$ and integrating, we have

$$\begin{aligned}
\int_0^R 4\pi r^3 \frac{dP}{dr} dr &= -\int_0^R \frac{GM(r)\, 4\pi r^3 dr}{r^2} \\
&= -\int \frac{GM(r)}{r} dm \\
&= E_{gr},
\end{aligned}$$

where the right side of the equation is the total gravitational potential energy of the star. The left side, as discussed in the text and in class, is

$$-3\int_0^R 4\pi r^2 P(r) dr.$$

Now let's write down some thermodynamic expressions for the pressure P and the internal energy u of the (assumed ideal and nonrelativistic) gas

$$\begin{aligned}
P &= nkT \\
u &= \tfrac{3}{2} nkT,
\end{aligned}$$

so $P = \frac{2}{3}u$ and

$$\begin{aligned}
3\int_0^R 4\pi r^2 P(r) dr &= 2\int_0^R u(r)\, dV \\
&= 2T_{th},
\end{aligned}$$

where E_{th} is the total thermal energy. Thus, we recover the *virial theorem*

$$E_{th} = -\tfrac{1}{2} E_{gr}.$$

This equation is important because it relates the total thermal energy of a star to the total gravitational energy — two quantities determined by very different physical processes.

The total energy of the star is

$$E_{tot} = E_{th} + E_{gr}.$$

Using the above relations, the virial theorem implies that

$$E_{tot} = \tfrac{1}{2} E_{gr}.$$

One obvious takeaway from this is that the total energy is negative. As a result, since the star is radiating energy into space, the total energy is *decreasing*, so the gravitational energy E_{gr} becomes more negative and the thermal energy E_{th} increases. Thus, as the star loses energy, it *gets hotter*. In a sense, it has negative heat capacity, and this spells long-term disaster as the loss of energy drives long-term evolution.

4.2 Mass Continuity Equation

See Maoz §3.3.

4.3 Equation of State

See Maoz §3.6.

Bottom line: the equation of state in the fully ionized regime, spanning most of the interior of the star, is quite simple:

$$P(\rho, T, X, Y) = \frac{\rho k T}{2 m_H} (1 + 3X + Y/2) + \tfrac{1}{3} a T^4.$$

4.4 Radiative Energy Transport in the Stellar Interior

Atomic lines are important near the stellar surface, where the temperature is low enough that atoms or ions can exist, but in the deep interior, where temperatures are high enough that the ionization fraction is 1, a much simpler process dominates — photons scattering off free electrons in the gas. With all atoms fully ionized, the only matter particles around are electrons and nuclei, and the much lighter electrons dominate the photon interactions.

The cross-section for a photon to scatter off a free electron (see Maoz Fig. 3.3), called the *Thomson cross section*, is easily calculated. It is

$$\sigma_T \;=\; \frac{8\pi}{3} \left(\frac{e^2}{4\pi\varepsilon_0 m_e c^2} \right)^2 \;=\; 6.65 \times 10^{-29}\ \mathrm{m}^2.$$

(Note the scaling with mass, which is the reason why electron scattering dominates.) Interestingly, this is the only electromagnetic interaction that is *independent* of frequency.

The mean free path for a photon moving through a gas with electron density n is

$$\ell \;=\; \frac{1}{n\sigma_T} \;=\; 1.5 \times 10^{-2}\ \mathrm{m} \left(\frac{n}{10^{30}\ \mathrm{m}^{-3}} \right)^{-1}.$$

The scaling is appropriate for the average density of the Sun, which is typical of conditions in the solar interior. Thus (see also Maoz §3.3), electromagnetic energy doesn't stream directly from the core to the surface. Instead, it bounces around on its way out, and diffuses outward in a random walk. Because the density decreases with radius, the mean free path in the outward direction is longer than in the inward direction, so the net direction of the diffusion is outward. However, it is well known that the rms distance traveled after N steps of a 3-D random walk of step length ℓ is

$$\langle D^2 \rangle^{1/2} \;\sim\; N^{1/2}\ell,$$

each step taking $\ell/c = 5 \times 10^{-11}$ s. Thus the energy needs $N \sim (R_\odot/\ell)^2 \sim 10^{21}$ hops to escape, taking tens of thousands of years to do so.

4.5 The Radiative Energy Transfer Equation

Here's an alternative derivation of the radiative energy transfer equation. From the discussion in the previous section, the matter in the solar interior is very opaque to radiation. Given that the temperature gradient in the Sun is on the order of $T_{vir}/R_\odot$, it follows that the variation in temperature over radial distance ℓ is

$$\Delta T \;\sim\; \frac{\ell\, T_{vir}}{R_\odot} \;\approx\; 10^{-4}\ \mathrm{K},$$

so the radiation field is extremely close to blackbody. Blackbody radiation is isotropic, and so no net energy transport would occur if the field were precisely blackbody. However, the small anisotropy

$$\frac{\Delta T}{T} \;\approx\; 10^{-11}$$

is enough to drive the entire energy flux of the Sun through the opaque interior.

As the photon energy bounces around in the solar interior, it slowly diffuses out toward the surface. We can better understand the physics by making explicit the connection between the microscopic (mean free path) and macroscopic (diffusion process) descriptions of the problem. Here's how. Consider a gas of particles of number density n, with isotropic velocities and average speed $\bar{v}$, and imagine a surface of unit area, in the $x-y$ plane, so z is in the direction perpendicular to the surface — let's think of this as the outward direction in a star. The number of particles crossing the surface in the positive z direction, per unit area, per unit time, is

$$f_+ \;=\; \tfrac{1}{6} n\bar{v}.$$

The factor of $\frac{1}{6}$ comes from the facts that (i) only half the particles are traveling in the positive z direction, and (ii) for an isotropic distribution, the average velocity of those particles in the z direction is $\int_0^{\pi/2} \bar{v}\cos^2\theta \sin\theta d\theta = \frac{1}{3}\bar{v}$.

These particles are traveling from a region of slightly higher energy density u into a region of lower energy density. Energy density u is energy per unit volume, so the energy per particle is u/n. The excess energy of the outgoing particles relative to their new surroundings is

$$\delta u = -\frac{du}{dz}\ell,$$

since the distance traveled is ℓ and $du/dz < 0$, so the energy flux they transport across the surface (watts per square meter) is

$$F_+ \;=\; \tfrac{1}{6} n\bar{v}\frac{\delta u}{n} \;=\; \tfrac{1}{6}\bar{v}\delta u.$$

Similarly, particles crossing in the negative z direction, from cooler to hotter, transport negative excess energy $-\delta u$ inward and therefore also contribute an outward flux

$$F_- \;=\; \tfrac{1}{6}\bar{v}\delta u.$$

Thus, the net energy flux transported outward by this process is

$$F = -\tfrac{1}{3}\bar{v}\ell \, \frac{du}{dz}.$$

Now let's apply this to photons in a star, where z becomes r, the flux is

$$F = \frac{L}{4\pi r^2},$$

$\bar{v} = c$, $u = aT^4$, and $\ell = 1/\kappa\rho$. Substituting in, we find

$$\begin{aligned} \frac{L}{4\pi r^2} &= -\tfrac{1}{3}\frac{c}{\kappa\rho}\frac{du}{dr} \\ &= -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}, \end{aligned}$$

so

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi acr^2T^3}.$$

4.6 Energy Conservation Equation

See Maoz §3.4.

4.7 Equations of Stellar Structure

See Maoz §3.5. The four basic equations are

$$\begin{aligned} \frac{dP}{dr} &= -\frac{GM\rho}{r^2} \\ \frac{dM}{dr} &= 4\pi r^2\rho \\ \frac{dT}{dr} &= -\frac{3\kappa\rho L}{16\pi acr^2T^3} \\ \frac{dL}{dr} &= 4\pi r^2\rho(\varepsilon - \varepsilon_\nu) \end{aligned}$$

If we regard radius r as an independent variable, we have 4 equations in 5 dependent variables: $P(r), \rho(r), T(r), M(r)$, and $L(r)$. The final equation is the equation of state,

$$P(r) = P(\rho, T, X, Y).$$

In addition, we have 3 additional functions, in principle knowable from elementary considerations, which describe the opacity $\kappa(\rho, T, X, Y)$, the total energy generation $\varepsilon(\rho, T, X, Y)$, and the neutrino energy loss rate $\varepsilon_\nu(\rho, T, X, Y)$.

With these functions accounted for, we have, in principle, enough equations to solve for all the unknowns. The result, calibrated to the observed properties of the Sun, is known as the *Standard Solar Model*.

4.8 Opacity

See Maoz §3.7.

Bottom line: the opacity law in the fully ionized regime is also very simple.

$$\kappa_{es} = 0.02\,(1+X) \quad \text{m}^2/\text{kg}.$$

This is just the Thomson scattering cross section in another form. At lower temperatures, other processes — bound-free and free-free absorption — become important. Both scale as $\rho/T^{7/2}$, and both (depending on the density) can be much larger than the electron scattering opacity for low temperatures — close to the surface of the star.

$$\begin{aligned}\kappa_{ff} &= 1.2\times 10^4\,(1-Z)(1+X)\left(\frac{\rho}{10^3\ \text{kg/m}^3}\right)\left(\frac{T}{10^5\ \text{K}}\right)^{-7/2}\ \text{m}^2/\text{kg}\\ \kappa_{bf} &= 1.4\times 10^4\,Z(1+X)\left(\frac{\rho}{10^3\ \text{kg/m}^3}\right)\left(\frac{T}{10^5\ \text{K}}\right)^{-7/2}\ \text{m}^2/\text{kg}\end{aligned}$$

4.9 Scaling Relations

See Maoz §3.8.
The scaling relations derived from the equations of stellar structure are:

$$\begin{aligned} P &\sim \frac{M\rho}{R} \\ M &\sim \rho R^3 \quad \Rightarrow \quad P \sim \frac{M^2}{R^4} \\ L &\sim \frac{T^4 R}{\kappa\rho}. \end{aligned}$$

We can extract mass-luminosity and mass-radius relations by exploring these equations for different combinations of the equation of state and the opacity law. For the equation of state, we can consider

1. the ideal nonrelativistic gas law, $P \sim \rho T$,
2. radiation pressure, $P \sim T^4$.

For the opacity, we have

A. electron scattering, $\kappa =$ constant,
B. Kramers law, $\kappa \sim \rho T^{-7/2}$.

Not all combinations make sense—specifically, radiation pressure only dominates at high temperatures, while the Kramers law is appropriate at low temperatures. The other combinations are as follows.

1A: $P \sim \rho T$, $\kappa =$ constant. This implies

$$\begin{aligned} T &\sim \frac{M}{R} \\ L &\sim \frac{T^4 R}{\rho} \sim M^3, \end{aligned}$$

so the mass-luminosity relation is similar to that observed in upper main sequence stars. If the central temperature T is taken to be constant (it actually varies by just a factor of 4 over a factor of 100 in M), we find $R \sim M$. Including the small variation in T as a power law in M, $T \sim M^0.3$, we obtain $R \sim M^{0.7}$, as observed. See slides4.2/16.

2A: $P \sim T^4$, $\kappa \sim$ constant. This implies

$$\begin{aligned} T &\sim \frac{M^{1/2}}{R} \\ L &\sim \frac{M^2}{R^4}\frac{R^4}{M} \sim M, \end{aligned}$$

as observed for high-mass stars.

1B: $P \sim \rho T$, $\kappa \sim \rho T^{-7/2}$. Now

$$\begin{aligned} T &\sim \frac{M}{R} \\ L &\sim \frac{M^4}{R^4}\frac{R^7}{M}\left(\frac{M}{R}\right)^{7/2} \sim M^{11/2}R^{-1/2}. \end{aligned}$$

If $T \sim$ constant, this implies $L \sim M^5$; if instead we let $T \sim M^{0.3}$ we get $L \sim M^{5.15}$, consistent with the steepening of the mass luminosity for lower-mass stars.

In general, the temperature gradient T/R decreases as M decreases. Consequently, at the bottom of the main sequence, for $M < 0.5M_\odot$, stars are fully convective. The effect of confection is that energy can reach the surface more easily than by radiation alone, the stellar radius shrinks relative to a radiative star, the internal temperature increases, and the luminosity is higher than would be predicted for a radiative star. As a result, the mass-luminosity relation becomes less steep at low masses. See slides4.2/16.

4.10 Convection

See Maoz §3.12.

The criterion for convective instability is that the temperature gradient becomes too steep (i.e. T is decreasing too rapidly with r):

$$\left|\frac{dT}{dr}\right| > \frac{\gamma - 1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|.$$

When part of a star becomes convectively unstable, the energy transport mechanism switches from radiation to convective motion, driven by the physical upwelling of gas. The result is that hot gas moves up and cool gas moves down, effectively reducing the temperature gradient to the point where stability is restored. The result is, in a convectively unstable region, convection replaces the radiative temperature gradient equation with

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr}.$$

4.11 Nuclear Fusion

See Maoz §§3.9, 3.10.
The book goes into more detail than we really need, so here are the key points to take from the discussion.

1. The Kelvin-Helmholtz time scale is $\tau_{KH} = E_{th}/L \approx 1.6 \times 10^7$ yr for the Sun. This is the time scale on which the Sun could radiate away its current thermal energy. The fact that it is short compared to the age of the Sun, and that the solar luminosity is known from geological measurements to have been roughly constant over much longer time scales, tell us that an energy source is needed to power the Sun over billions of years.

2. We can get a handle on the overall energetics by noting that the Sun radiates $L_\odot/M_\odot = 2 \times 10^{-4}$ W/kg. This is quite a modest requirement—burning wood or gasoline, for example, would liberate far more power per unit mass—but when we take into account the fact that the Sun will sustain this output for 10 billion years, we find a total energy requirement of $\sim 6 \times 10^{13}$ J/kg. This is a lot, and only nuclear processes like fusion and fission even come close. There is very little uranium or plutonium in the Sun, so fusion it it.

3. Looking at the basic proton-proton reaction (slides4.2/18), where burning 4 protons liberates 26.2 MeV $= 4.2 \times 10^{-13}$ J of electromagnetic energy, that the energy generated per unit mass is 6.2×10^{13} J/kg, in line with the above estimates.

4. The solar core is a very low energy nuclear reactor. Because of electrostatic repulsion, nuclei of atomic numbers Z_A and Z_B would need an energy of $E_C \approx Z_A Z_B$ MeV to overcome the coulomb barrier and approach within the $\sim 10^{-15}$ m needed for the strong force to bind them. In the core of the Sun, $kT \approx 1$ keV, so classically, no protons have enough energy to do this.

5. Quantum mechanics to the rescue! Protons can tunnel through a barrier too high for them to overcome classically. A detailed calculation says that the probability of penetrating the coulomb barrier (and hence fusing) is

$$g(E) = e^{-(E_G/E)^{1/2}},$$

   where

$$E_G = 493\,\mathrm{keV}\,(Z_A Z_B)^2 \left(\frac{\mu}{\frac{1}{2}m_p}\right)$$

   is the Gamow energy (see Maoz Eq. 3.117). In the Sun's core, $g \approx 10^{-10}$—still small, but not zero!

6. Once we have this key piece of physics, we can derive an expression for the nuclear reaction rates. The number of protons declines with increasing energy E as $e^{-E/kT}$, while the Gamow factor falls off exponentially with decreasing E for $E \ll E_G$. The overlap between the two factors peaks at an energy of about 5 keV (see slides4.2/21). Integrating over the various probabilities, and including the nuclear fusion cross section, the final result for the emissivity (W/kg) is

$$\epsilon = \frac{2^{5/3}\sqrt{2}}{\sqrt{3}} \frac{\rho X_A X_B}{m_H^2 A_A A_B \sqrt{\mu}} Q S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} e^{-3(E_G/4kT)^{1/3}}.$$

   This looks complicated, but it is applicable to all nuclear reactions in *any* star. The factor S_0 gives the cross section for the particular reaction under consideration, and Q is the energy it releases. X_A and X_B are the abundances, per unit mass, of the reactants A and B.

7. The pp chain has some alternate branches (see slides4.2/23). They accomplish the same net reaction, but differ in the numbers and types of neutrinos they emit.

8. In stars more massive than the Sun, the CNO cycle (see slides4.2/24) is more efficient than the pp chain in converting hydrogen into helium. Carbon, nitrogen, and oxygen act as catalysts that accelerate the rate of helium production over the pp reactions.

9. It can be shown (see the text and Homework 4) that these reactions are very sensitive to T. The pp chain scales as T^4, while the CNO cycle scales as T^{18}. This sensitivity is the main reason why there is such a small temperature spread in stellar cores—a small increase in T can produce a huge increase in luminosity.

10. The neutrinos produced in these reactions stream out of the solar core without any further interactions. The neutrino interaction cross section at these energies is a few times 10^{-50} m^2, so the mean free path is about 1 light year. The neutrino flux at Earth is about 6.6×10^{14} m^{-2}s^{-1}. Remarkably, despite their tiny interaction cross sections, these neutrinos have been detected in numbers completely consistent with the standard solar model.

4.12 Leaving the Main Sequence

See notes for week 6.