PHYS 231 Lecture Notes – Week 1

Reading from Maoz (2nd edition):

- **Chapter 1 (mainly background, not examinable)**
- **Sec. 2.1**
- **Sec. 2.2.1**
- **Sec. 2.2.3**
- **Sec. 2.2.4**

Much of this week's material is quite descriptive, but some mathematical and physical points may benefit from expansion here. I provide here expanded coverage of material covered in class but not fully discussed in the Maoz text. For material covered in the text, I'll mostly just give some generalities and a reference. References to slides on the web page are in the format "slidesx.y/nn," where x is week, y is lecture, and nn is slide number.

1.1 Kepler's Laws and Planetary Orbits

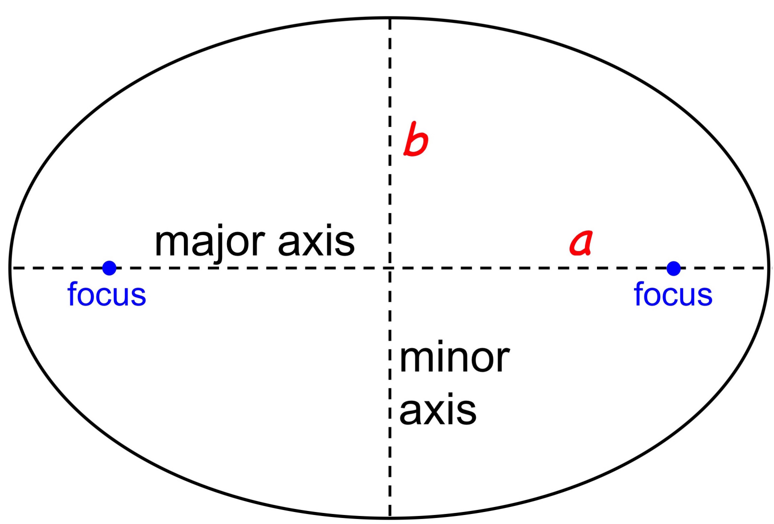
Kepler's laws describe the motions of the planets:

I Planets move in elliptical orbits with the Sun at one focus.

II Planets obey the equal areas law, which is just the law of conservation of angular momentum expressed another way.

III The square of a planet's orbital period is proportional to the cube of its semimajor axis.

The figure below sketches the basic parts of an ellipse; the following table lists some basic planetary properties. (We'll return later to how we measure the masses of solar-system objects.)



The long axis of an ellipse is by definition twice the semi-major axis a. The short axis—the "width" of the ellipse, perpendicular to the long axis—is twice the semi-minor axis, b. The eccentricity is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

Thus, an eccentricity of 0 means a perfect circle, and 1 is the maximum possible (since $0 < b \leq a$). Less obvious from this definition, but important when actually working with orbital properties, the distance from the focus to the center of the ellipse (the intersection of the major and minor axes) is ae. Note that all planetary eccentricities are small enough that the orbits are almost circular. The largest eccentricity is that of Mercury, and even in this case, some quick math shows that $b_{Mercury} \approx 0.98\, a_{Mercury}$.

Planet	semimajor axis (AU)	period (yr)	eccentricity	inclination (degrees)	mass (Earth)	radius (Earth)
Mercury	0.387	0.241	0.206	7.0	0.0553	0.383
Venus	0.723	0.615	0.0068	3.4	0.815	0.950
Earth	1	1	0.0167	0	1	1
Mars	1.52	1.88	0.0934	1.9	0.107	0.532
Jupiter	5.20	11.9	0.0485	1.3	318	11.0
Saturn	9.54	29.5	0.0557	2.5	95.2	9.14
Uranus	19.2	84.0	0.0472	0.77	14.5	3.98
Neptune	30.1	165	0.00858	1.8	17.2	3.87

Kepler's laws were derived for the inner planets (Mercury through Mars), and the fact that they hold for the outer planets too means that they are more than simple empirical fits. They have predictive power too, even though Kepler did not understand *why* planets orbit the Sun—that understanding would have to wait nearly 7 decades for Isaac Newton to develop his Laws of Motion and Gravity, along with the mathematical tools to solve the problem.

The planets obey Kepler's laws because they have to. Newtonian mechanics and gravity demand it. Perhaps more interesting are the planetary dynamical properties that aren't mandated by Newton.

- All planets move in nearly circular orbits.
- All planets orbit in nearly the same plane.
- That plane is also the equatorial plane of the Sun.
- All planets orbit the Sun in the same sense—counterclockwise when viewed from above the north pole.
- The Sun rotates in the same sense.

These properties are telling us a lot about how the solar system formed, not about the basic gravitational physics that holds it together.

1.2 The Scale of the Solar System

Kepler's laws provide us with a map of the solar system without a scale marker. We can determine the shapes and orientations of the planets' orbits, but we only know their *relative* sizes. Measuring a planet's orbital period in years only tells us its orbital size relative to Earth's orbit, which is not itself determined:

$$\frac{a}{a_\oplus} = \left(\frac{P}{P_\oplus}\right)^{2/3}.$$

We call Earth's orbital period $P_\oplus$ a year (1 yr). Earth's semimajor axis $a_\oplus$ is called an astronomical unit (1 AU). But how to measure it? We need an independent distance measurement to accomplish this. Once we have done that, all planetary scales in the solar system are known.

There are many ways to make such an independent measurement. Perhaps the simplest (today) is to use radar ranging to determine the distance to a nearby planet—Venus. Imagine for simplicity an instant when Earth and Venus are exactly aligned in their (circular) orbits with the Sun. Venus has a semimajor axis of 0.7 AU, Earth 1.0 AU. The distance between them is therefore 0.3 AU. A radar beam from a large instrument on Earth (such as the 300 m disk at Arecibo, Puerto Rico) will strike Venus, reflect off, and some of it will be detected at Earth roughly 5 minutes later, after an 0.6 AU round trip at speed $c = 3 \times 10^8$ m/s. Hence we can say

$$0.6\text{AU} = 300\,\text{s} \times 3 \times 10^8\,\text{m/s} \;=\; 9 \times 10^{10}\,\text{m},$$

and so $1\,\text{AU} = 1.5 \times 10^{11}$ m.

1.3 Sizing Up the Sun

Now that we know the value of 1 AU, we can figure out some basic properties of the Sun.

1. The Sun's angular diameter is about 0.53 degrees, or 0.0092 radians. At a distance of 1 AU, simple geometry tells us that this corresponds to a distance of $9.2 \times 10^{-3} \times 1.5 \times 10^{11}\,\text{m} = 1.4 \times 10^8$ m. A more accurate calculation gives a radius of $R_\odot = 696,000\,\text{km}$.

2. Knowing Earth's $a = 1\,\text{AU}$ and orbital period $P = 1\,\text{yr}$, we can determine the mass of the Sun, much as we did for the Moon. The result is

$$M_\odot = \frac{4\pi^2 r^3}{GP^2} \;=\; 2.0 \times 10^{30}\,\text{kg}.$$

1.4 Distances to Nearby Stars

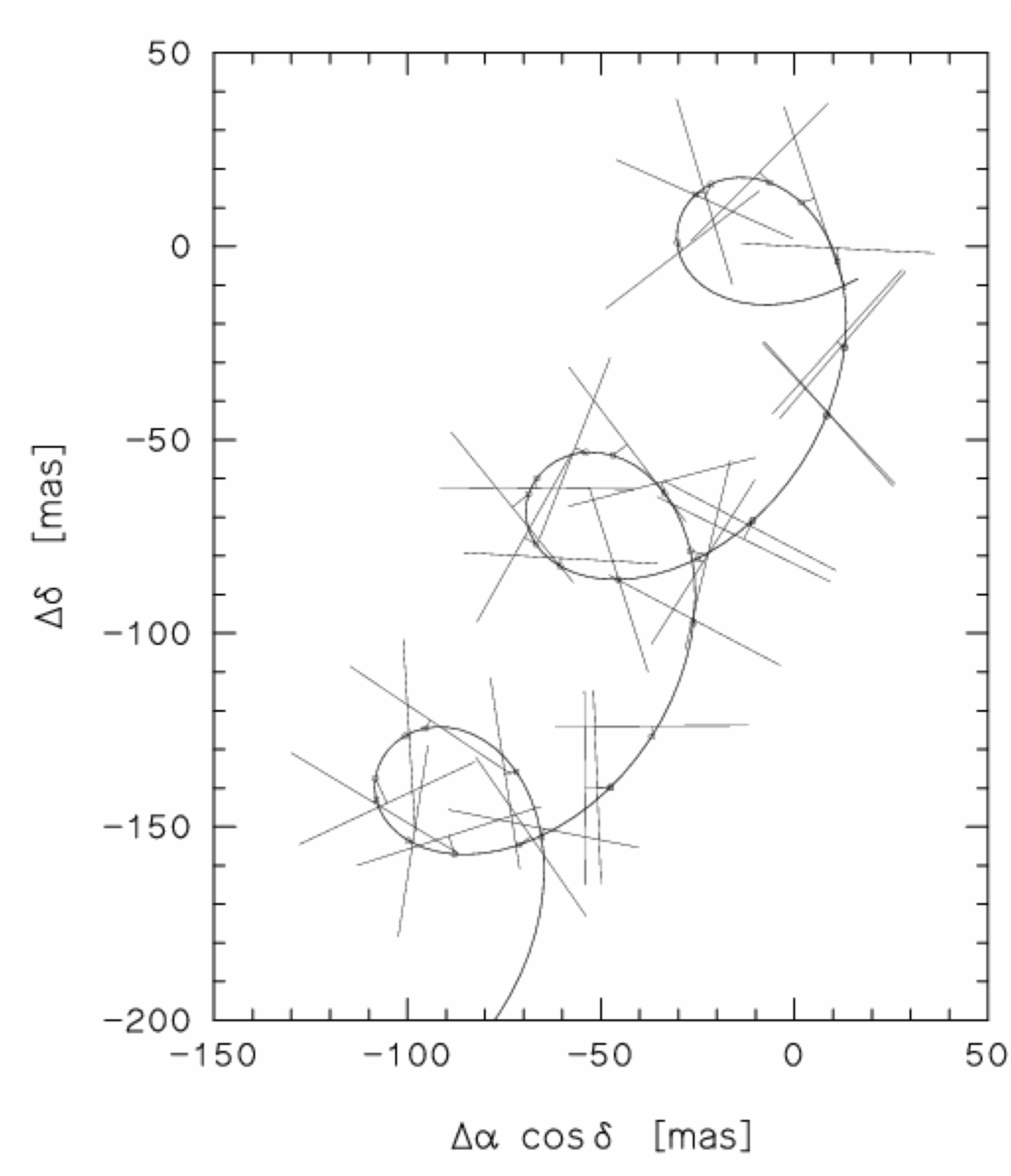
Once we know the scale of the solar system, we can use elementary geometry to measure the distances to nearby stars. As discussed in the Lecture 1 slides (slides1.1/15–19), and in §2.2.1 of the Maoz text, nearby stars appear to move relative to distant ones as Earth orbits the Sun. This allows us to construct a long, skinny triangle right triangle with Earth, the Sun, and the distant star at the three vertices. The parallactic angle p is at the star, the opposite side is 1 AU, and the adjacent side is the distance D to the star. The small angle formula then says

$$p(\text{radians}) = \frac{1\text{AU}}{D}.$$

Since p is generally very small, it is normally measured in arcseconds, where 1 arcsecond $(1'') = \frac{1}{3600}^\circ = \frac{1}{206000}$ radians. If we plug in $p = 1''$ we find $D = 206,000$ AU, a distance called 1 parsec that defines the distance unit used throughout astronomy. If we measure distances in parsecs, the above formula takes on a particularly simple form:

$$p(\text{arcsec}) = \frac{1}{D(\text{pc})}.$$

Parallaxes as small as $\sim 0.01''$ can be reliably measured from the ground. (Even though stellar images are generally much larger than this, it is possible to measure a shift in the position of a blob that is much smaller than the blob's size.) This corresponds to distances of ~ 100 pc.



The Hipparcos satellite, launched in 1989, could accurately measure parallaxes as small as $1 - 2 \times 10^{-3}$ arcsec. Again, the errors in individual measurements were considerably larger than this (tens of milliarcseconds), but by combining multiple measurements and fitting a curve to the results, much higher accuracy could be obtained. The above figure (from M. Perryman, the

Hipparcos project leader) shows how this works for one particular (unnamed) star. Each straight line in the graph indicates the observed position of the star at some particular time. (The precise location along this line was not determined by the satellite.) The curve is the stellar path that best fits all the measurements. The amplitude of the "loops" gives the star's parallax, due to Earth's motion. The superimposed linear motion is the star's true real spatial motion transverse to the line of sight (called *proper motion* in astronomical jargon).

The Gaia mission, launched by the European Space Agency in December 2013, has been returning high-quality data since 2015. It can measure parallactic angles with errors as small as 10 microarcseconds, extending the useful distance range to ∼10 kpc. The current data release contains position, velocity, brightness and color data for almost 1.7 billion stars, spanning a large portion of the Milky Way Galaxy.

1.5 Stellar Velocities

Let's continue our inventory of stellar properties. We have seen how simple geometry (trigonometric parallax) can be used to measure the distances to many stars. We can use similar geometric reasoning to measure a star's transverse velocity relative to the Sun—that is, the component of its relative velocity that lies perpendicular to the line of sight (see the notes for lecture 3 for a diagram). In time Δt the star moves a transverse distance $v_t \Delta t$, so, seen from Earth, it moves through an angle

$$\Delta\theta = \frac{v_t \Delta t}{D}$$

so

$$\frac{\Delta\theta}{\Delta t} = \frac{v_t}{D}.$$

The quantity on the left hand side is the proper motion, or angular speed of the star across the sky, usually denoted by the symbol μ and expressed in arcsec/yr. Thus if the distance D is known, we can use the proper motion to determine the transverse velocity. In "astronomical" units, we have

$$v_t\,(\text{km/s}) = 4.74\, D\,(\text{pc})\ \mu\,(\text{arcsec/yr}).$$

Proper motions of stars are generally very small—a star at a distance of 100 pc moving with a transverse velocity of 30 km/s relative to the Sun (pretty typical for our Galactic neighborhood) would have a proper motion of just 0.06 arcsec/yr. By and large, proper motions, and hence transverse velocities, are well known only for relatively nearby stars, although HST and Gaia have recently pushed the useful range out to several kiloparsecs.

What about the other component of the star's motion? Often, if the star is bright enough, its radial velocity can be accurately measured from the *Doppler shift* of its spectrum. Recall from your intro Physics class that when a wave source is moving with respect to an observer, the observer sees a longer or shorter wavelength than was emitted, depending on its motion away from or toward the source. For sound waves, where you probably first saw the derivation, the wavelength shift depends in slightly different ways on the speed of the source and the speed of the observer relative to the medium through which the wave moves. Light is actually much simpler. The Doppler shift depends only on the relative radial motion v of the source and the observer, and (in terms of wavelength) takes the form

$$\lambda_{observed} = \lambda_{source} \sqrt{\frac{1 + v/c}{1 - v/c}},$$

where by convention $v > 0$ when the observer is moving away from the source. For $v \ll c$, this simplifies, to first order (i.e. neglecting terms of order v^2/c^2 and higher), to

$$\lambda_{observed} = \lambda_{source}\left(1 + \frac{v}{c}\right).$$

In terms of the change in wavelength $\Delta\lambda$ (or frequency $\Delta\nu/\nu$), we have

$$\frac{\Delta\lambda}{\lambda} \;=\; -\frac{\Delta\nu}{\nu} \;=\; \frac{v}{c},$$

Thus, all spectral features (see below) observed in a star with a radial velocity of +30 km/s away from the Sun will be Doppler shifted by $v/c = 30/300000 = 10^{-4}$, so a 500 nm feature will be observed at 500.05 nm—a small shift, but easily measurable with modern instruments.

We'll see in a moment that light has both wave and particle properties. But even in particle mode (photons), the wavelength of light is subject to the Doppler effect, so the above discussion still applies.

It's worth noting that, when we know the distance to a star, we can determine its three-dimensional location in space, since we know all three coordinates in spherical polar coordinates relative to the Sun. For many of the same stars, whose proper motions and Doppler shifts can be measured, we can also reconstruct their three-dimensional spatial velocities. Thus, for an increasing number of stars in our Galaxy, we have full six-dimensional dynamical information on their positions and velocities, a fact that will ultimately be critical to our understanding of the dynamics of our Galaxy.

1.6 Gravitational Acceleration at Earth's Surface

Now let's turn to measurement of masses in astronomy. Newton's laws provide the key.

Newton's law of gravity says that the gravitational force between two bodies of masses M (Earth, say) and m (you, perhaps) is

$$F_{grav} = \frac{GMm}{r^2}$$

where r is the distance between the bodies and G is a fundamental constant. The force is attractive, and is directed along the line between the two bodies.

This force produces an acceleration a. By Newton's second law,

$$F_{grav} = ma$$

so

$$a = \frac{GM}{r^2}.$$

Note that a is independent of m. Actually, this is because we did some sleight of hand in the two F_{grav} equations above—we assumed that we could use the same m in each. In other words, we assumed that the m that couples to gravity is the same m that gives matter inertia. This turns out to be true—at least, no experiment has ever succeeded in finding a difference between the two. The equality of the so-called gravitational mass and the inertial mass turns out to be a very fundamental statement of physics—called the equivalence principle—that forms the basis for Einstein's general theory of relativity. But more on that later!

On Earth's surface (at distance R from Earth's center), the gravitational acceleration is easily measured to be

$$g \approx 9.80\,\mathrm{m/s^2}$$

so we can write

$$g = \frac{GM}{R^2}.$$

Newton never knew the value of G. The best he could do was to determine the combination GM. However, G has subsequently been measured in numerous laboratory experiments (carried out since the late 19th century) as $G = 6.67 \times 10^{-11}\,\mathrm{m^3/kg\,s^2}$. Knowing G, we can turn the equation around to say

$$M = \frac{gR^2}{G}.$$

In other words, we can use our knowledge of gravity to weigh our planet! Plugging in g, G, and $R = 6400\,\mathrm{km}$, we find

$$M_\oplus = 5.97 \times 10^{24}\,\mathrm{kg}$$

(the subscript $\oplus$ is the conventional astronomical symbol for Earth).

This is essentially how *all* masses in astronomy are measured. We have to see some sort of gravitational interaction between two (or more) bodies—and assume the equivalence principle—in order to say anything about the masses involved.

1.7 The Moon and the Inverse Square Law

The apple that may or may not have landed on Newton's head gets all the credit for his law of gravity, but his laws of motion really deserve the honor. Newton was trying to understand what causes objects to fall toward Earth, and in particular how it (whatever it is) scales with distance. He postulated that the force responsible for the apple's acceleration at distance $r = R_\oplus$ is also responsible for keeping the Moon in orbit around Earth, at distance $r_M = 384,000\,\mathrm{km}$.

The Moon orbits Earth every 27.3 days, so (assuming a circular orbit—not quite true), its orbital speed is

$$v_M = \frac{2\pi r_M}{27.3\,\mathrm{days}} \;=\; 1.02\,\mathrm{km/s}.$$

From Physics I, the moon's acceleration therefore is

$$a_M = \frac{v_M^2}{r_M} \;=\; 2.7 \times 10^{-3}\,\mathrm{m/s^2}.$$

Thus $a_M/g = 2.8 \times 10^{-4}$, which is the same as $R_\oplus^2/r_M^2$, setting Newton on course for the critical scaling of his law of gravity, $a \propto r^{-2}$. The accelerations of the planets, as described by Kepler's laws, are also consistent with this inverse-square fall-off, indicating to Newton that his law of gravitation really might be universal in nature.

1.8 Orbital Motion

Now consider an object (a spacecraft or a satellite) in orbit at distance r around our mass M. For simplicity, let's take the orbit to be circular. If the orbital speed is v, then the acceleration is

$$a = \frac{v^2}{r}.$$

Equating this to the acceleration due to gravity at that distance, we find

$$\frac{v^2}{r} = \frac{GM}{r^2},$$

so

$$v = \sqrt{\frac{GM}{r}}.$$

The orbital period P is just distance divided by speed, so

$$P = \frac{2\pi r}{v} \;=\; 2\pi\sqrt{\frac{r^3}{GM}}.$$

Note that $P^2 \propto r^3$, which is Kepler's third law of planetary motion.
A few examples:

1. The *Hubble Space Telescope* orbits about 100 km above Earth's surface, at $r \approx 6500\,\text{km}$. Using the above numbers we find, for *Hubble*,

$$\begin{aligned} v &= 7.83\,\text{km/s} \\ P &= 1.45\,\text{hours.} \end{aligned}$$

2. GPS satellites have orbital periods of exactly 12 hours. Turning around the above relation for the period, we find that they have $r = 26,600\,\text{km}$.

3. Geosynchronous satellites have $P = 24\,\text{hours}$, a factor of 2 larger, so from the above proportionality they lie $2^{2/3} = 1.59$ times farther out, at $r = 42,200\,\text{km}$. Because it orbits at exactly the same rate as Earth rotates, a geosynchronous satellite orbiting in Earth's equatorial plane always stays above the same point on the planet, and so appears fixed in the sky.

1.9 The Two-Body Problem

In the previous sections we studied Kepler's laws and Newtonian gravity as they relate to the *one-body problem*—a test particle of negligible mass orbiting a massive body such as the Sun. We saw (at least for a circular orbit, but the result is general) that Newton's version of Kepler's third law is

$$P^2 = \frac{4\pi^2\, a^3}{GM},$$

where P is period, a is semimajor axis, and M is the central mass. If we measure P in years, a in AU, and M in solar masses, the relation is even simpler:

$$P^2 = \frac{a^3}{M}.$$

Now let's consider the *two-body problem*: two bodies of masses m_2 and m_2 interacting gravitationally, where now the masses are arbitrary, and may be comparable. Let the bodies have position vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The accelerations of the two bodies are

$$\begin{aligned} \boldsymbol{a}_1 &= \frac{Gm_2\,(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{r^3} \\ \boldsymbol{a}_2 &= \frac{Gm_1\,(\boldsymbol{x}_1 - \boldsymbol{x}_2)}{r^3} \end{aligned}$$

where $r = |\boldsymbol{x}_2 - \boldsymbol{x}_1|$ (and note that the forces $\boldsymbol{F}_1 = m_1\boldsymbol{a}_1$ and $\boldsymbol{F}_2 = m_2\boldsymbol{a}_2$ are indeed equal and opposite, as required by Newton's third law).

The center of mass of the system is

$$\boldsymbol{x}_{CM} = \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{m_1 + m_2},$$

and the center of mass acceleration is

$$\begin{aligned}\boldsymbol{a}_{CM} &= \frac{m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2}{M} \\ &= \frac{\boldsymbol{F}_1 + \boldsymbol{F}_2}{M} \\ &= \boldsymbol{0}.\end{aligned}$$

Here $M = m_1 + m_2$ is the total mass of the system. In other words, the center of mass moves with constant velocity. If the particles were each subject to an external force $\boldsymbol{F}$, the center of mass would have acceleration $\boldsymbol{F}/M$—it would move through space like a single particle of mass M.

Now let's consider the relative motion of the two bodies. Let $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ and $\boldsymbol{a} = \boldsymbol{a}_2 - \boldsymbol{a}_1$. Then

$$\begin{aligned}\boldsymbol{a} &= -\frac{Gm_1\boldsymbol{r}}{r^3} - \frac{Gm_2\boldsymbol{r}}{r^3} \\ &= -\frac{GM\boldsymbol{r}}{r^3}.\end{aligned}$$

In other words, the relative motion is exactly the same as one-body motion around a mass M. If we let

$$\begin{aligned}\boldsymbol{r}_1 &= \boldsymbol{x}_1 - \boldsymbol{x}_{CM} \\ \boldsymbol{r}_2 &= \boldsymbol{x}_2 - \boldsymbol{x}_{CM}\end{aligned}$$

with similar expressions for the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then it is easily shown that

$$\begin{aligned}m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 &= 0 \\ \boldsymbol{r}_1 &= \frac{m_2}{M}\boldsymbol{r} \\ \boldsymbol{r}_2 &= -\frac{m_1}{M}\boldsymbol{r},\end{aligned}$$

and similarly for the velocities. Thus the relative motion is an ellipse described by Kepler's third law as above, and the component stars execute similar ellipses with the center of mass as the common focus, with the same period and semimajor axes

$$\begin{aligned}a_1 &= \frac{m_2}{M}a \\ a_2 &= \frac{m_1}{M}a.\end{aligned}$$

Problem solved!

1.10 Conservation Laws

Going back to the one-body problem,

$$\boldsymbol{a} = -\frac{GM\boldsymbol{r}}{r^3},$$

we can straightforwardly derive a couple of important conservation laws. Gravity is a conservative force, meaning that the acceleration can be written as the gradient of a potential ϕ, where

$$\phi(r) = -\frac{GM}{r}.$$

You can verify for yourself that $a = -\phi'(r)$. We have conventionally taken ϕ to go to zero as $r \to \infty$. The sum of the kinetic and potential energies (expressed here per unit mass, for convenience)—the energy—is a conserved quantity of the motion:

$$E = \tfrac{1}{2}v^2 - \frac{GM}{r}.$$

Angular momentum is also conserved. Let $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{v}$. Then

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{v} + \boldsymbol{r} \times \frac{d\boldsymbol{v}}{dt} \\ &= \boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{a} \\ &= \mathbf{0}. \end{aligned}$$

Since the angular momentum defines the normal to the orbit plane, this also implies that the orbit is planar.

1.11 Bound and Unbound Orbits

It should be clear that positive and negative energies describe two qualitatively different types of orbit. If $E > 0$, then it is possible for the distance r to become arbitrarily large and for the velocity to remain positive. In other words, the particle can escape. If the velocity at infinite separation is denoted by v_∞, then

$$E = \tfrac{1}{2}v_\infty^2.$$

If $E < 0$, r can't go to infinity because that would mean $v^2 = 2E < 0$, which is not physical. This is the bound elliptical orbit we have been discussing up to now. Note that, for a circular orbit, with $v^2 = GM/r$, we have

$$E_{circ} = -\frac{GM}{2r}.$$

The dividing line, $E = 0$ is a special case where the particle just makes it to infinity, but with zero velocity. For a given initial separation r, the critical speed that separates bound and unbound motion is called the escape speed, v_{esc}. It is given by

$$v_{esc}^2 = \frac{2GM}{r}.$$

The orbit in the positive energy case is a geometric shape called a *hyperbola* that is closely related to the ellipse by the theory of conic sections, which dates back more than two millennia to the ancient Greeks. The orbit in the intermediate zero-energy case is a *parabola*.

1.12 Measuring Stellar Masses

These basic considerations are enough to determine the masses of stars in many observed binary systems. See §2.2.4 of Maoz. Briefly, the parameters we'd like to know are the semimajor axis, eccentricity, and inclination of the binary orbit, as well as the masses of the component stars. For nearby binaries we might know the distance and we may be able to resolve the individual orbits of the stars. For more distant binaries, we may only have limited velocity information on one or both components. Broadly speaking,

- If we can resolve a binary and know its distance or have velocity information, then we can determine all the orbital parameters and the component masses.
- If we can't resolve the binary, but have spectroscopic radial velocity information on both components, then we can measure the mass ratio M_1/M_2, but only $M \sin i$, where the inclination i is the angle between the line of sight and the normal vector to the binary plane. Thus we can only determine the masses if the inclination is known. An important example of this is an eclipsing binary, where we see the orbit edge-on and $i = 90^\circ$, so $\sin i = 1$.
- If we only have radial velocity information on one component (M_1, say), then only the so-called *mass function*

  $$f(M_1, M_2, i) = \frac{M_2^3 \sin^3 i}{(M_1 + M_2)^2}$$

  can be determined.

1.13 Light, Particles, and Waves

Wave-particle duality in physics is a huge topic, and one that we can't deal with in any detail here. Suffice it to say that the pendulum has shifted from particles to waves and (almost) back over the last 400 years, and our current "wavicles" understanding is much more nuanced than either.

Simply put, back in the day, the "particle" school of thought held that light was made up of particles, with (somehow) different particles corresponding to different colors. The competing "wave" theory viewed light as a wave phenomenon, with color corresponding to wavelength ν or frequency λ, the two being related by the well known wave relation

$$\lambda\nu = c,$$

where $c = 3 \times 10^8$ m/s is the speed of light.

In the late 17th century, Newton was a strong proponent of the particle theory. His double prism experiment indicated that, once white light had been split into a spectrum by a prism, once a color of light was isolated, passing it through another prism produced no further splitting, consistent with the view that like particles could not be separated again. Of course, the wave theory, which viewed different colors as different wavelengths of light, could also account for this observation by asserting that the bending of light by a prism was a wavelength-dependent process.

By the 19th century, the weight of scientific opinion was firmly on the side of the wave theory. Measurements of the diffraction of light as it passed through an opening, the Young double slit experiment demonstrating interference of light, and the development of Maxwell's equations, which unified all known electromagnetic phenomena, and in addition predicted electromagnetic radiation, all seemed to indicate conclusively that waves had won the day. Slide 1.2/22 illustrates the range of properties of electromagnetic waves.

However, the story wasn't over. Toward the end of the 19th century, a series of experiments by Hertz, Lenard, and others established the reality of what is now called the *photoelectric effect*: A beam of light can dislodge electrons from a metal surface, but in an unexpected way. Below a critical frequency that depends on the metal in question, no electrons are produced. Above that critical frequency, the energy of the electrons depends on the frequency of the light, but increasing the intensity of the beam at constant frequency increases the number of electrons emitted, but not their energies. None of these results could be explained by the wave theory.

In 1905 Einstein published an explanation of the photoelectric effect that effectively defined the modern view of radiation. To explain the experimental results, he proposed that light consisted of photons—packets of electromagnetic waves—and that the energy E of a photon is proportional to its frequency ν:

$$E = h\nu,$$

where $h = 6.63\times10^{-34}$ J.s is Planck's constant. The energy of a photon is transferred to an electron. The frequency threshold was explained by positing that a fixed amount of energy is needed for the electron to escape from the metal's surface. Higher frequency means more energy, higher intensity means more photons, as observed. This insight won Einstein the 1923 Nobel Prize.

As a rule of thumb, light behaves as a wave on large scales, but as a particle (photon) on small scales. A corresponding statement can be made about all known elementary matter particles, which also exhibit both particle and wave properties, depending on circumstances.

1.14 Flux, Luminosity, and the Inverse-Square Law

Let's imagine a body of radius R radiating isotropically—uniformly in all directions. The amount of energy it radiates per unit time is its *luminosity* L (unit: watts).

Now imagine this energy spreading out through space as it travels away from the star. Assuming no energy is lost from or added to the radiation field as it goes, the amount of energy crossing a surface of radius D centered on the star must be the same—L. Hence the flux of radiation at distance D is

$$f(D) = \frac{L}{4\pi D^2}.$$

The flux decreases as an *inverse-square law*. This relation between f, L, and D provides us with a means of measuring the third quantity once the other two are known. For the Sun, we measure $f = 1.4\mathrm{kW/m^2}$ and know $D = 1$ AU, so the solar luminosity is $L_\odot = 3.96 \times 10^{26}$ W. For nearby Sirius we measure $f = 1.2 \times 10^{-7}$ W/m^2 and $D = 2.6$ pc, so $L = 1.0 \times 10^{28}$ W $= 25L_\odot$. But if we know the luminosity by some other means (for example, if we recognize a distant object to be of a type we know from nearby observations), we can turn the inverse-square law around to get

$$D = \sqrt{\frac{L}{4\pi f}}.$$

Such objects, collectively referred to as *standard candles*, are very useful to astronomers, since they allow us to extend the distance scale far beyond objects whose parallaxes can be measured. We will see quite a few examples throughout the course.

1.15 Blackbody Radiation

Few objects (other than masers and lasers) emit radiation at a single frequency. In most cases, the energy is spread out over a range of frequencies. Some typical spectra of stars are shown in slides1.2/30. All of this material is discussed in more detail in Maoz, §2.1.

In general, we can characterize the spread of energy contained in a radiation field by its energy density, u_ν, the amount of energy per unit volume per unit frequency (unit: $\mathrm{J\,m^{-3}\,Hz^{-1}}$). The amount of energy per unit volume with frequency between ν and $\nu + d\nu$ is $u_\nu d\nu$. The intensity, I_ν, is defined as the amount of energy per unit time per unit area per unit frequency per unit solid angle ($\mathrm{W\,m^{-2}\,Hz^{-1}\,ster^{-1}}$). That is, if the beam has intensity I_ν, then the total power per unit area carried in solid angle $d\Omega$ in frequency range $[\nu, \nu + d\nu)$ is

$$I_\nu\, d\nu d\Omega \;\; \mathrm{W/m^2}.$$

For an isotropic radiation field (uniformly spread over all directions, covering 4π steradians), the intensity is related to the energy density by

$$I_\nu \;=\; \frac{c}{4\pi} u_\nu.$$

A *blackbody* spectrum is a thermodynamic idealization that approximates fairly well the overall energy emission of many stars. The detailed distribution, which comes from statistical mechanics, is

$$I_\nu \;=\; B_\nu \;=\; \frac{2h}{c^2}\,\frac{\nu^3}{e^{h\nu/kT}-1}.$$

We could just as well work in terms of B_λ, which is defined per unit wavelength rather than per unit frequency. Note that B_ν and B_λ depend only on a single physical variable—the temperature T.

The energy flux f_ν for an isotropic blackbody field is given by

$$f_\nu = \pi B_\nu \quad \mathrm{W\,m^{-2}\,Hz^{-1}},$$

and the total flux is obtained by integrating over all frequencies:

$$f \;=\; \frac{2\pi^5 k^4}{15c^2h^3}\,T^4 \;=\; \sigma T^4.$$

This is the *Stefan-Boltzmann* Law; the quantity $\sigma = 5.7\times10^{-8}$ $\mathrm{Wm^{-2}K^{-4}}$ is the Stefan-Boltzmann constant.

We can also look at how the peak of the blackbody distribution depends on temperature. Differentiating B_ν with respect to ν (or B_λ with respect to λ), we can find where the intensity has a maximum. The result is the *Wien displacement law*, which may be written for B_ν as

$$h\nu_{max} = 2.8\,kT,$$

and for B_λ as

$$\lambda_{max}T = 0.0029\ \mathrm{m\cdot K}.$$

Note that, as the temperature increases, the peak of the emission moves to higher frequencies (proportional to T) and shorter wavelengths (inversely proportional to T)—from infrared (warm) to red-hot to white-hot to blue-hot and beyond. Measurement of this peak gives us our first estimate of the temperature of a star. Note, however, that most stellar spectra are only roughly approximated by a blackbody (see slides1.2/30).