Equations of Stellar Structure

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}$$ P, M, ρ

$$\frac{dM}{dr} = 4\pi r^2 \rho$$ M, ρ

$$\frac{dL}{dr} = 4\pi r^2 \rho\,(\epsilon - \epsilon_\nu)$$ L, ρ

$$\frac{dT}{dr} = -\frac{3\kappa L \rho}{16\pi ac\, r^2\, T^3}$$ T, L, ρ

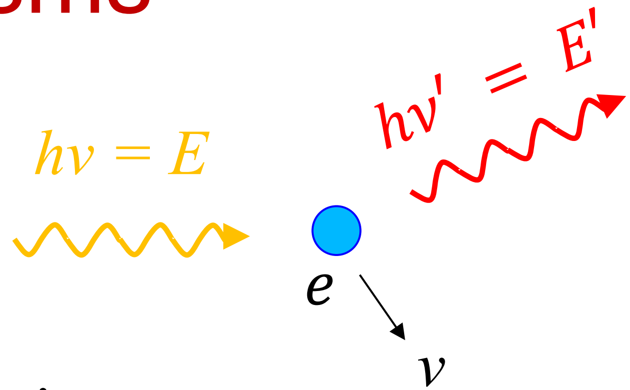
equation of state: $P(\rho, T, X, Y)$

Opacity

$$\kappa_{es} \;=\; 0.02\,(1+X)\;[\mathrm{m}^2/\mathrm{kg}]$$

$$\begin{aligned} n_e &= n_H + 2n_{He} + \sum_A \frac{1}{2} A\, n_A \\ &= \frac{\rho}{m_H}\left(X + \frac{1}{2}Y + \frac{1}{2}Z\right) \\ &= \frac{\rho}{2m_H}(1+X) \end{aligned}$$

Opacity Mechanisms

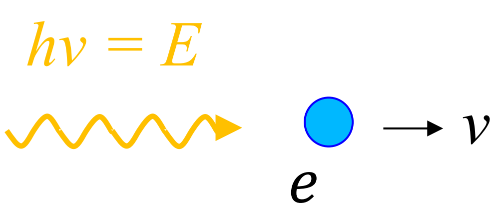


electron scattering

bound-bound = line formation

bound-free = photoionization

free-free

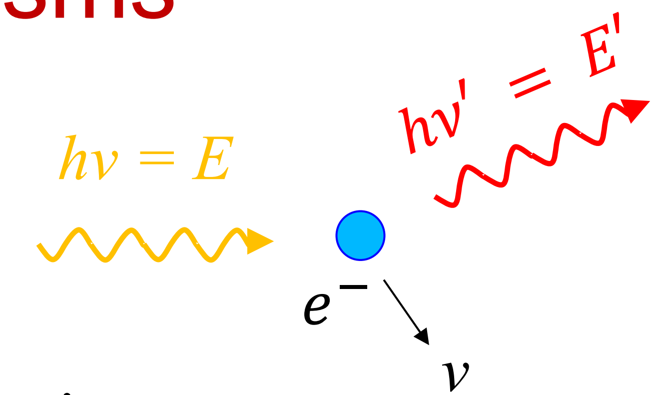


energy: $\frac{1}{2} m_e v^2 = E$

momentum: $m_e v = E/c$

$\Longrightarrow \; v = 2c$!

Opacity Mechanisms

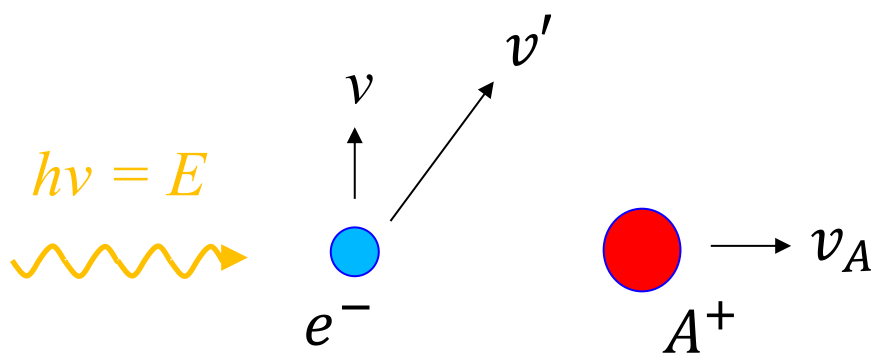


electron scattering

bound-bound = line formation

bound-free = photoionization

free-free



Opacity

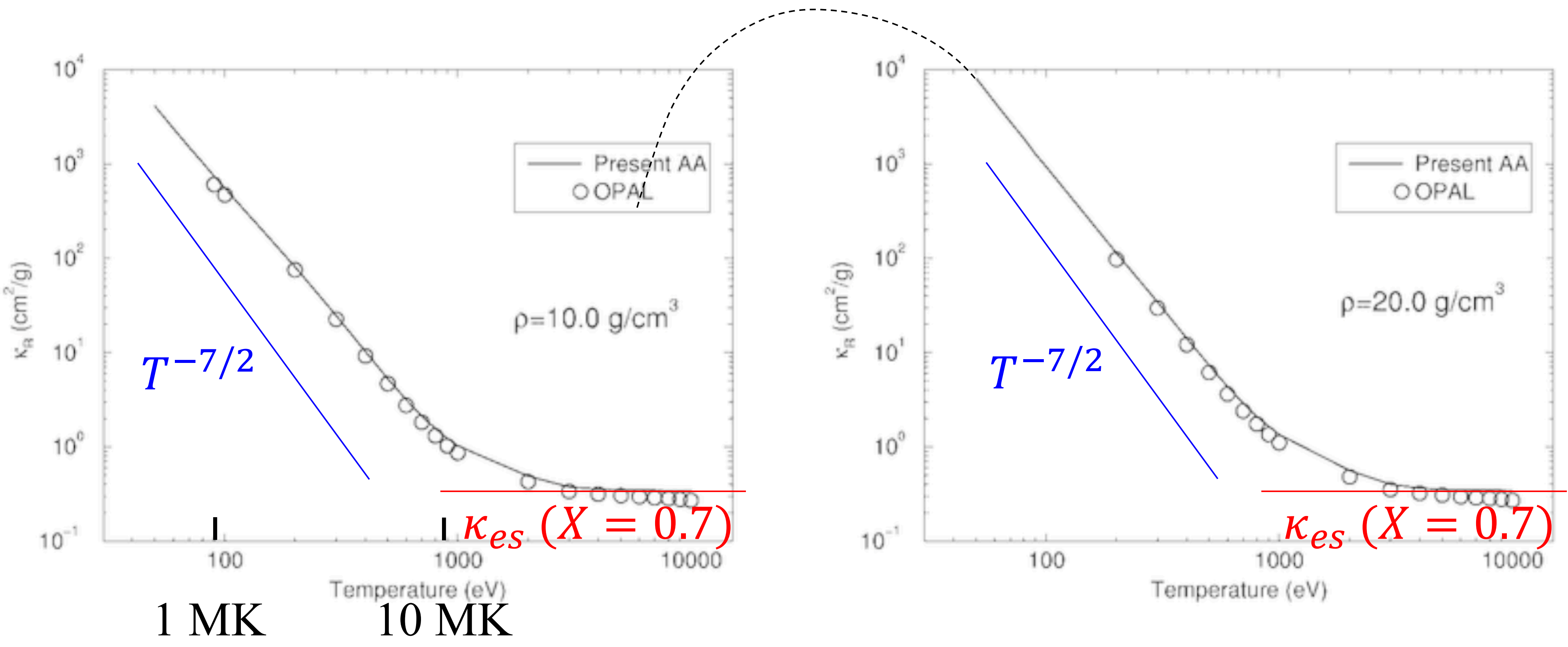
$$\kappa_{es} = 0.02\,(1+X)\ [\mathrm{m}^2/\mathrm{kg}]$$

$$\kappa_{ff} = 1.2\times 10^4\,(1-Z)(1+X)\left(\frac{\rho}{10^3\ \mathrm{kg/m^3}}\right)\left(\frac{T}{10^5\ \mathrm{K}}\right)^{-7/2}$$

$$\kappa_{bf} = 1.4\times 10^4\,Z(1+X)\left(\frac{\rho}{10^3\ \mathrm{kg/m^3}}\right)\left(\frac{T}{10^5\ \mathrm{K}}\right)^{-7/2}$$

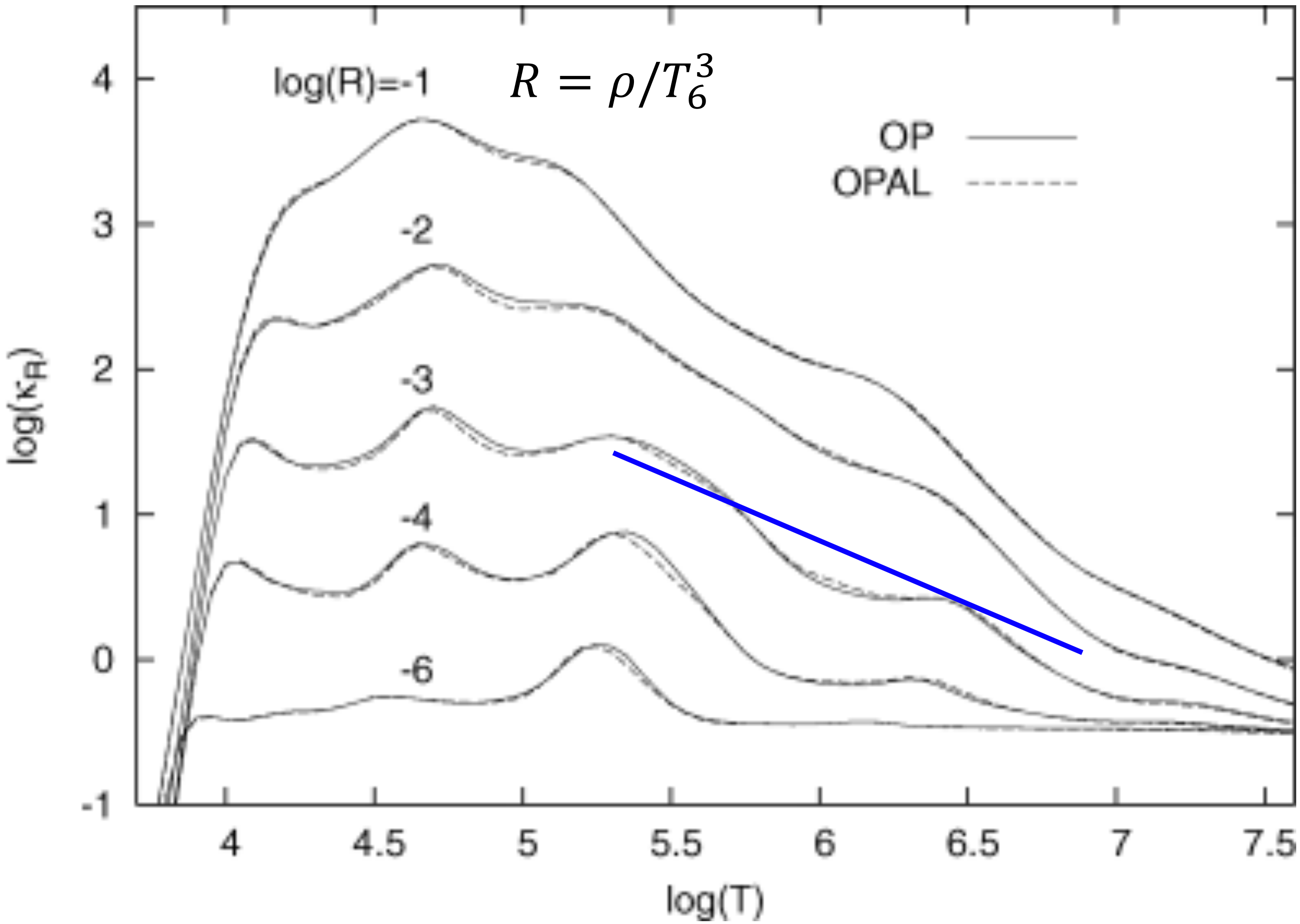
Kramers Law: $\kappa \propto \rho\, T^{-7/2}$

Opacity

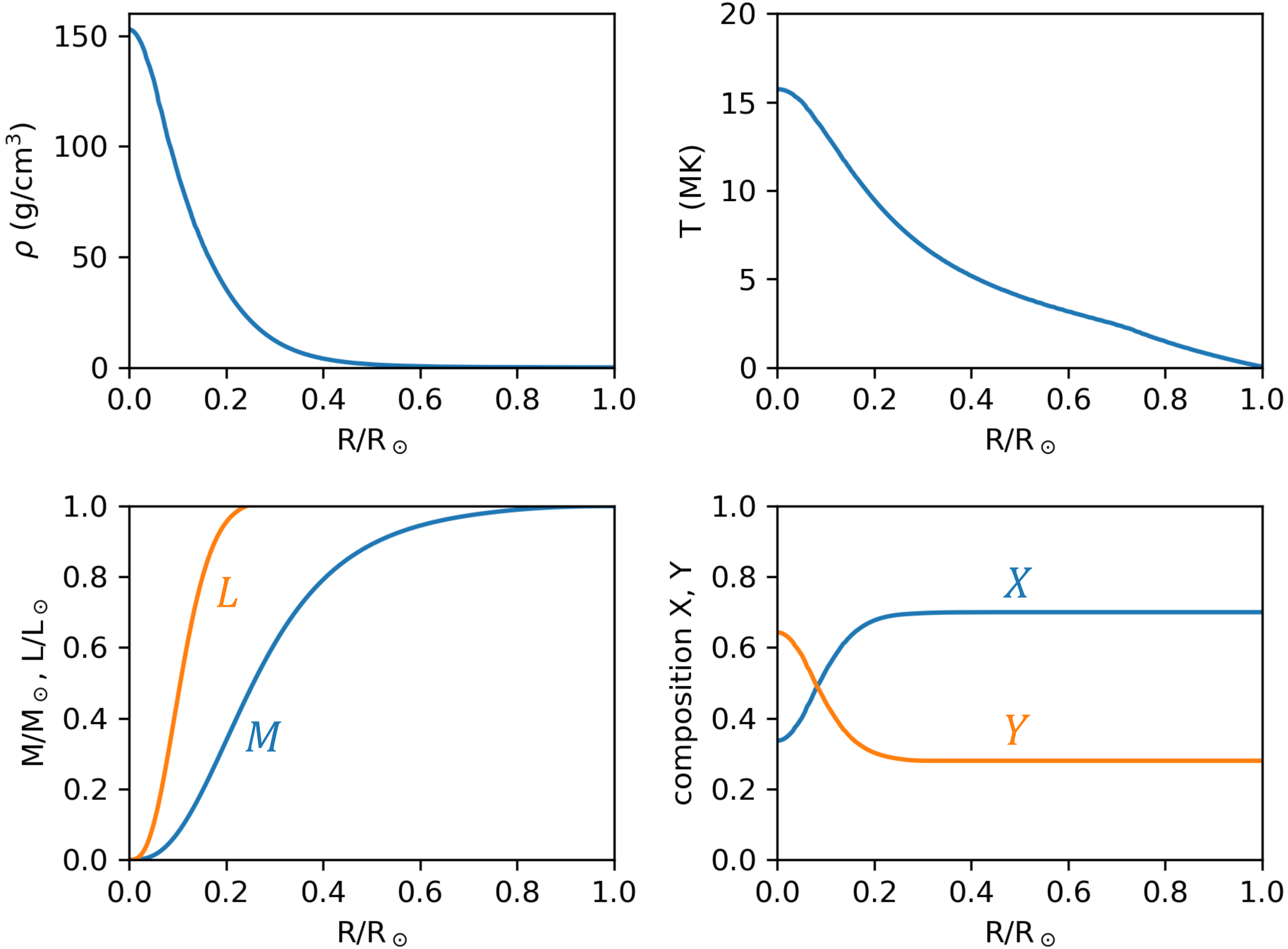


Present AA
OPAL
ρ=10.0 g/cm³
$T^{-7/2}$
$\kappa_{es}\ (X = 0.7)$
κR (cm²/g)
Temperature (eV)
100
1000
10000
1 MK
10 MK
ρ=20.0 g/cm³

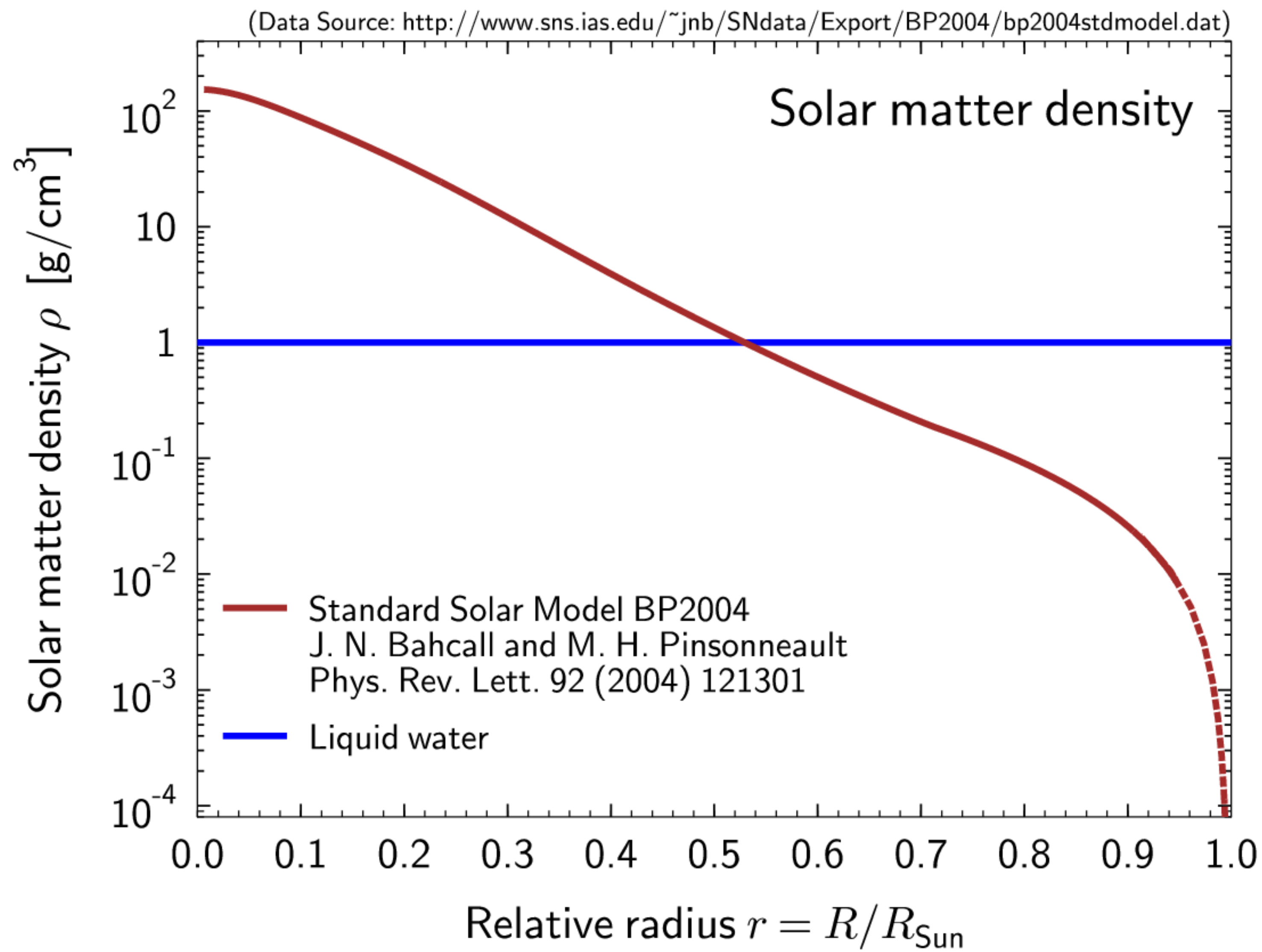
Figure 2 Rosseland-mean opacities from the OP and OPAL for the S92 mix.



OXFORD UNIVERSITY PRESS

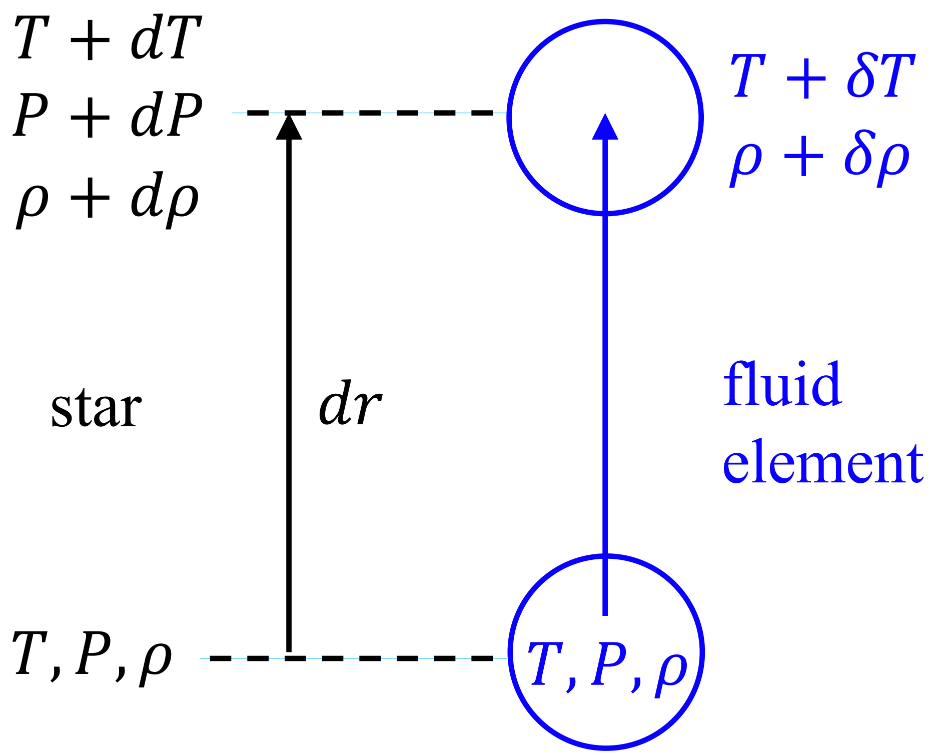


150
100
50
0
ρ (g/cm^3)
0.0
0.2
0.4
0.6
0.8
1.0
R/R$_\odot$
20
15
10
5
0
T (MK)
R/R$_\odot$
1.0
0.8
0.6
0.4
0.2
0.0
M/M$_\odot$, L/L$_\odot$
L
M
R/R$_\odot$
composition X, Y
X
Y
R/R$_\odot$



(Data Source: http://www.sns.ias.edu/~jnb/SNdata/Export/BP2004/bp2004stdmodel.dat)
Solar matter density
Solar matter density ρ [g/cm^3]
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
Relative radius $r = R/R_{\text{Sun}}$
Standard Solar Model BP2004
J. N. Bahcall and M. H. Pinsonneault
Phys. Rev. Lett. 92 (2004) 121301
Liquid water

Convection



if $\delta\rho > d\rho$, element will sink $\Longrightarrow$ stable

otherwise, element will continue to rise $\Longrightarrow$ convectively unstable

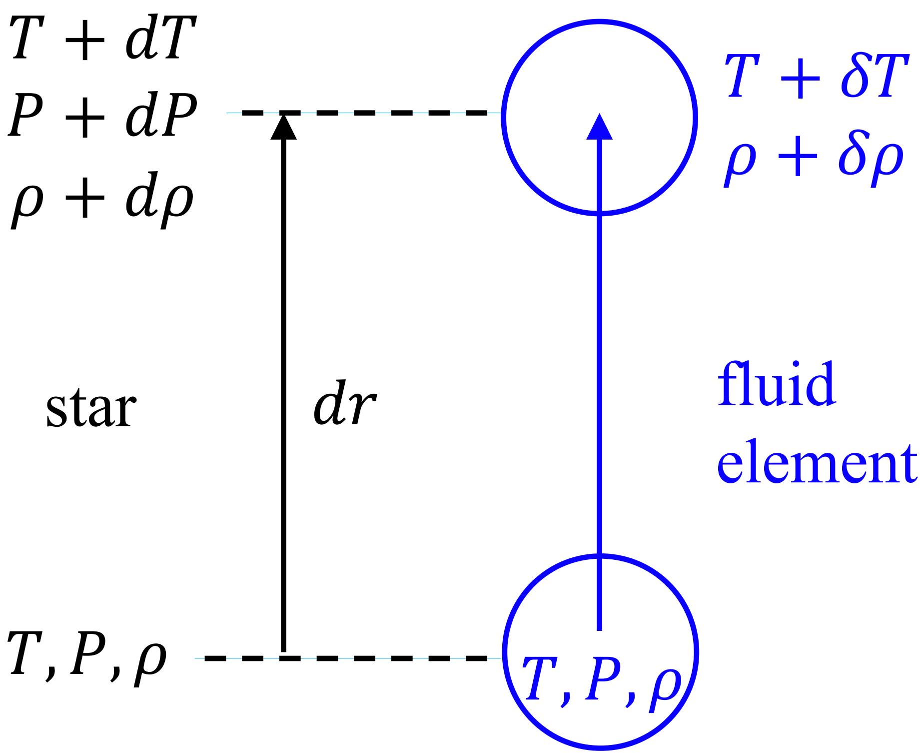
star: $\rho \propto \frac{P}{T} \;\Rightarrow\; \frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}$

element: adiabatic $P \propto \rho^{\gamma}$

$\left(\gamma = \frac{5}{3}\right) \quad \Longrightarrow \frac{\delta P}{P} = \gamma \frac{\delta\rho}{\rho}$

pressure: $\delta P = dP$

Convection



unstable $\Longrightarrow \dfrac{\delta\rho}{\rho} < \dfrac{d\rho}{\rho}$

$$\Longrightarrow \frac{1}{\gamma}\frac{\delta P}{P} < \frac{dP}{P} - \frac{dT}{T}$$

$$\Longrightarrow \frac{dT}{T} < \frac{\gamma-1}{\gamma}\frac{dP}{P}$$

$$\Longrightarrow \frac{dT}{dr} < \frac{\gamma-1}{\gamma}\frac{T}{P}\frac{dP}{dr}$$

$$(< 0)$$

$$\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dr}$$

Equations of Stellar Structure

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2} \qquad P, M, \rho$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \qquad M, \rho$$

$$\frac{dL}{dr} = 4\pi r^2 \rho\,(\epsilon - \epsilon_\nu) \qquad L, \rho$$

$$\frac{dT}{dr} = -\frac{3\kappa L\rho}{16\pi ac\, r^2\, T^3} \qquad \mathrm{T}, L, \rho$$

equation of state: $P(\rho, T, X, Y)$

$$\frac{P}{R} \sim \frac{M\rho}{R^2} \Longrightarrow P \sim \frac{M\rho}{R}$$

$P \sim M^4/R^2$

$$\frac{M}{R} \sim R^2\rho \Longrightarrow M \sim R^3\rho$$

$$\frac{T}{R} \sim \frac{\kappa\rho L}{R^2T^3} \Longrightarrow L \sim \frac{RT^4}{\kappa\rho}$$

Upper main sequence: $P \sim \rho T, \quad \kappa \sim \text{constant}$

$$\Longrightarrow \; T \sim \frac{M}{R}$$

$$L \sim \frac{T^4 R}{\rho} \sim \frac{\left(\frac{M}{R}\right)^4 R^4}{M} \sim M^3$$

Very high mass: $P \sim T^4, \quad \kappa \sim \text{constant}$

$$\Longrightarrow \; P \sim \frac{M^2}{R^4} \sim \mathrm{T}^4 \; \Longrightarrow \; T \sim \frac{M^{1/2}}{R}$$

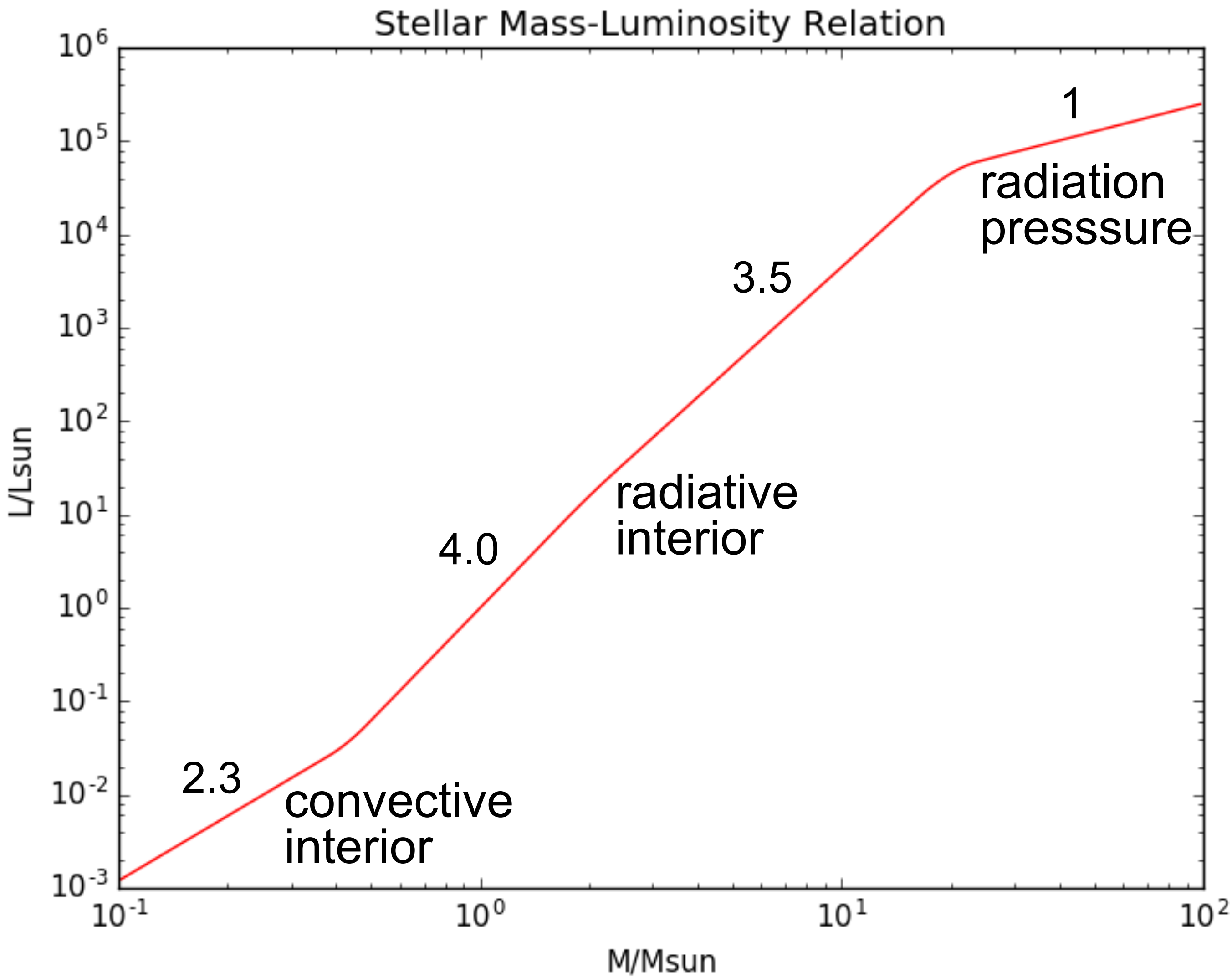
$$L \sim \frac{\frac{M^2}{R^4} R^4}{M} \sim M$$

Lower main sequence: $P \sim \rho T, \quad \kappa \sim \rho T^{-7/2}$

$$\Longrightarrow \; T \sim \frac{M}{R}$$

$$L \sim M^{11/2} R^{-1/2}$$

$$\sim M^{5} \quad \text{if } T \sim \text{constant}$$



Stellar Mass-Luminosity Relation
L/Lsun
M/Msun
2.3
convective
interior
4.0
radiative
interior
3.5
1
radiation
presssure

Solar Energy Production

Kelvin-Helmholtz time scale

$$t_{KH} \sim E_{th}/L$$

$$\sim \frac{GM_\odot^2}{2R_\odot L_\odot}$$

$$\sim 1.6\times10^7\,\mathrm{yr}$$

Luminosity per unit mass

$$\frac{L_\odot}{M_\odot} \sim 2\times10^{-4}\ \mathrm{W/kg} \qquad \rightarrow \quad 3\times10^{13}\ \mathrm{J/kg}$$

Main Sequence nuclear reactions

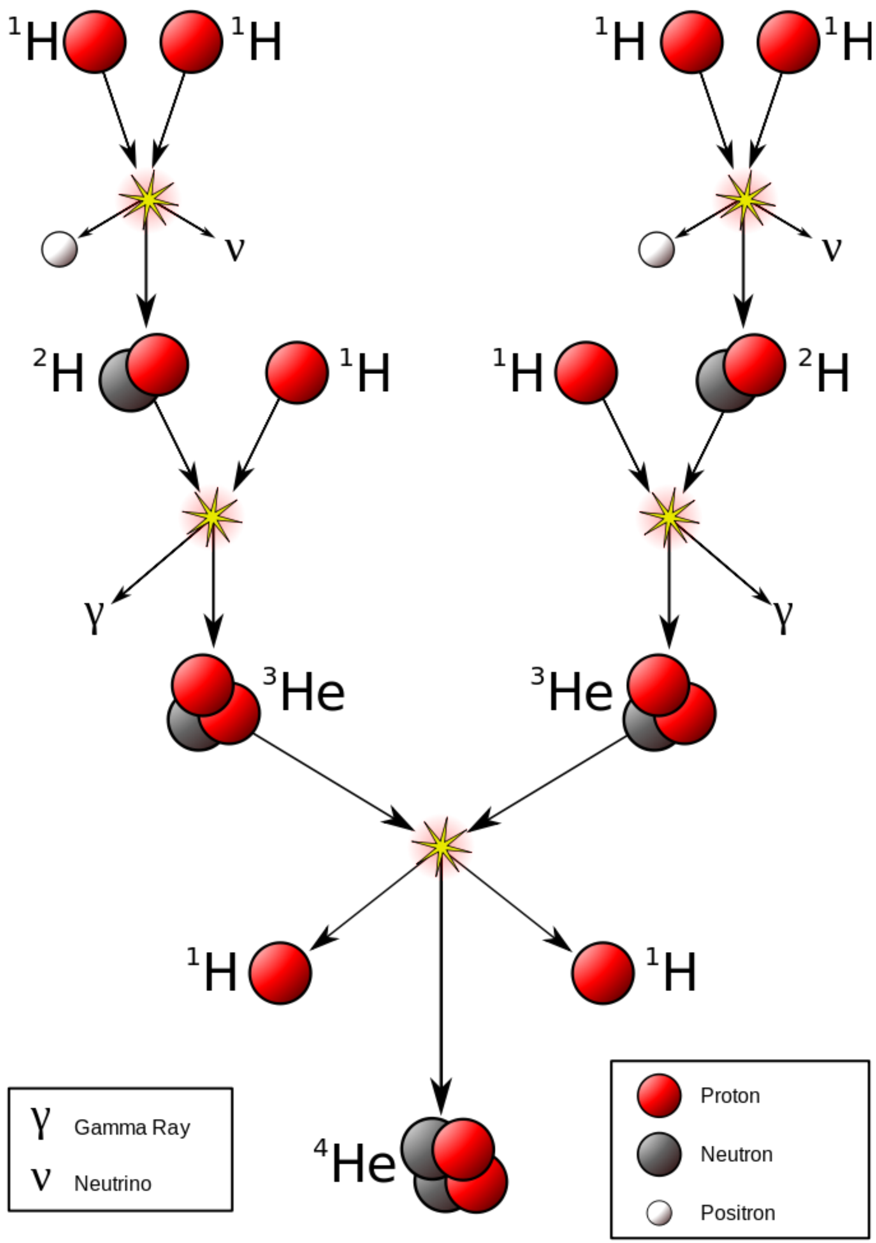
Proton-proton chain

$^{1}_{1}H + ^{1}_{1}H \rightarrow ^{2}_{2}He \rightarrow ^{2}_{1}H + e^{+} + \nu_e$

$^{2}_{1}H + ^{1}_{1}H \rightarrow ^{3}_{2}He$

$^{3}_{2}He + ^{3}_{2}He \rightarrow ^{4}_{2}He + 2\,(^{1}_{1}H)$

$4\,(^{1}_{1}H) \rightarrow ^{4}_{2}He + 2\,e^{+} + 2\nu_e + 24\ \mathrm{MeV}$



Proton-proton chain

$$\underbrace{4\mathrm{p} \rightarrow {}^4He}_{\Delta m\, c^2 \;=\; 25.71\ \mathrm{MeV}} + 2e^+ + 2\nu_e + \underbrace{2\gamma + KE}_{24.17\ \mathrm{MeV\ local}}$$

1.02 MeV

$2e^+ + 2e^- \rightarrow 4\gamma$

extra 1.02 MeV

local

0.52 MeV

escapes

total energy released = 26.73 MeV

total local heating = 26.21 MeV

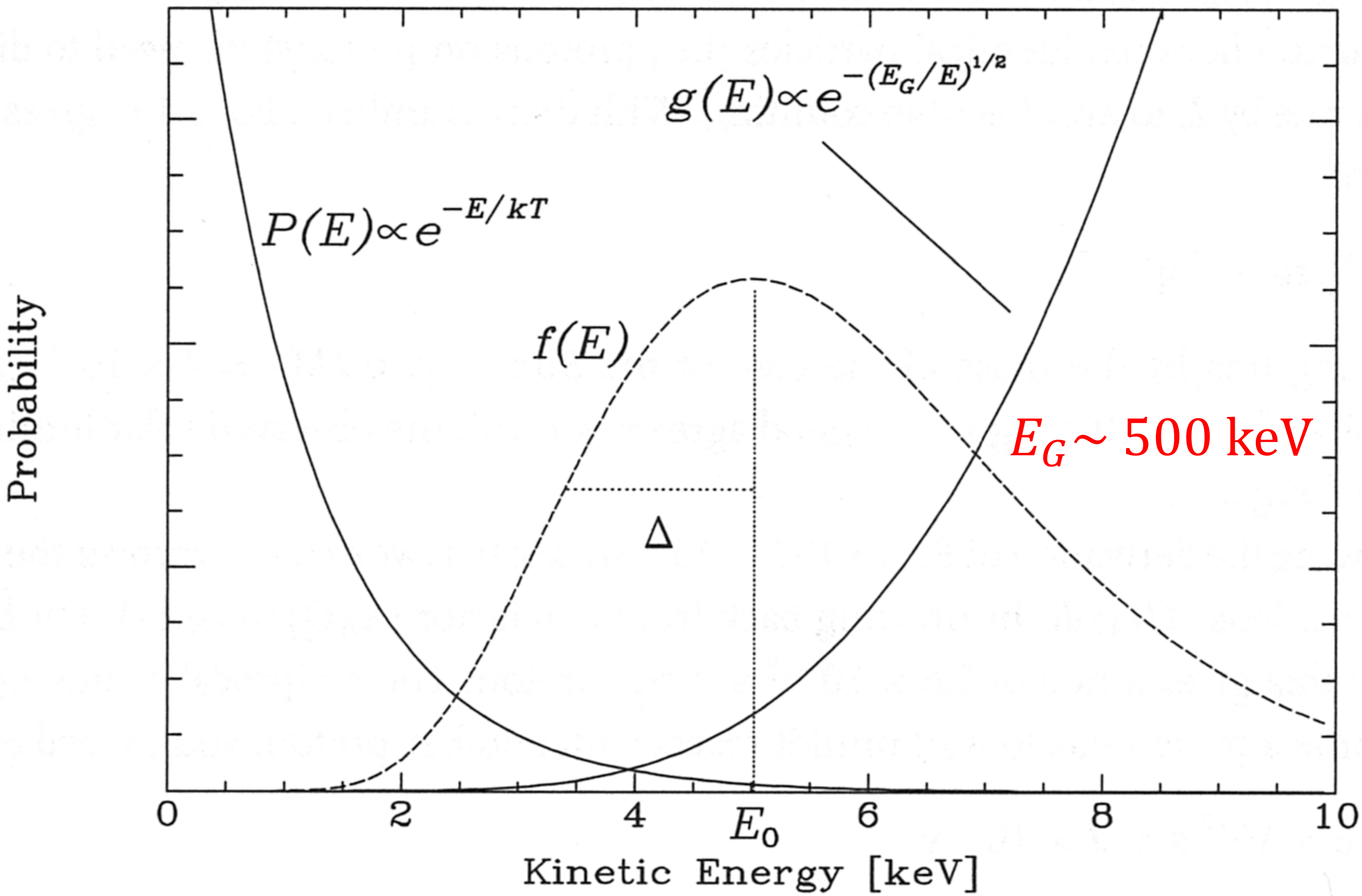
Proton-proton chain

$$4\mathrm{p} \rightarrow {}^4He + 2e^+ + 2\nu_e + 2\gamma + KE$$

$$\begin{aligned} \text{total energy released} &= 26.73 \text{ MeV} \\ \text{total local heating} &= 26.21 \text{ MeV} \end{aligned}$$

$$\begin{aligned} \text{total mass of hydrogen} &= 4\, m_H = 6.68 \times 10^{-27} \text{ kg} \\ \text{total mass lost} &= 4.58 \times 10^{-29} \text{ kg} \;\; (0.7\%) \\ \text{total local heating} &= 4.20 \times 10^{-12} \text{ J} \end{aligned}$$

$$\text{energy/mass} = 6.28 \times 10^{14} \text{ J/kg}$$



$g(E) \propto e^{-(E_G/E)^{1/2}}$
$P(E) \propto e^{-E/kT}$
$f(E)$
Δ
$E_G \sim 500$ keV
Probability
0
2
4
E_0
6
8
10
Kinetic Energy [keV]

$$\epsilon = \frac{2^{5/3}\sqrt{2}}{\sqrt{3}} \frac{\rho X_A X_B}{m_H^2 A_A A_B \sqrt{\mu}} \, Q S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} e^{-3(E_G/4kT)^{1/3}}$$

$$\mathrm{E}_G \;\approx\; 500\ \mathrm{keV}\;\; Z_A^2 Z_B^2$$

$$^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e$$

$$^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He} + \gamma$$

69%

$$^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H}$$

(PP I)

31%

$$^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \rightarrow {}^{7}_{4}\mathrm{Be} + \gamma$$

99.7%

$$^{7}_{4}\mathrm{Be} + e^{-} \rightarrow {}^{7}_{3}\mathrm{Li} + \nu_e$$

$$^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \rightarrow 2\,{}^{4}_{2}\mathrm{He}$$

(PP II)

0.3%

$$^{7}_{4}\mathrm{Be} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{8}_{5}\mathrm{B} + \gamma$$

$$^{8}_{5}\mathrm{B} \rightarrow {}^{8}_{4}\mathrm{Be} + e^{+} + \nu_e$$

$$^{8}_{4}\mathrm{Be} \rightarrow 2\,{}^{4}_{2}\mathrm{He}$$

(PP III)

Main Sequence nuclear reactions

CNO cycle

$^{12}{}_{6}C + {}^{1}{}_{1}H \rightarrow {}^{13}{}_{7}N + \gamma$

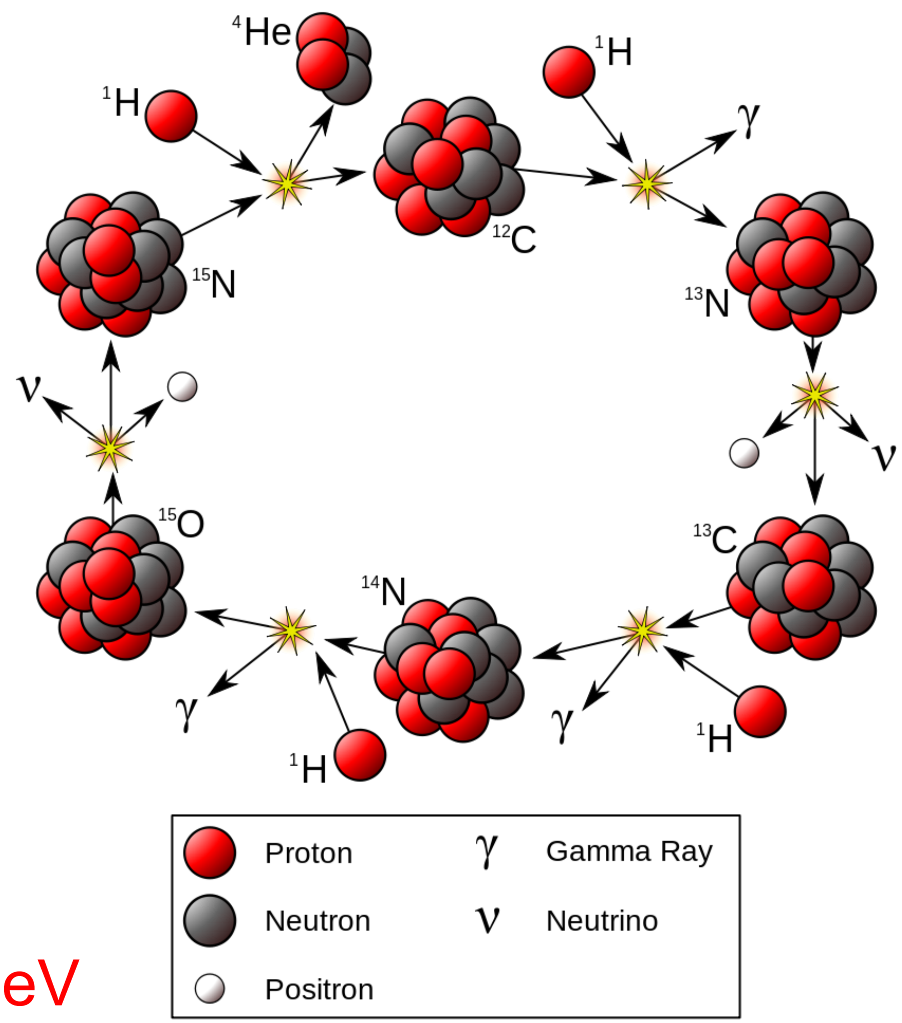
$^{13}{}_{7}N \rightarrow {}^{13}{}_{6}C + e^{+} + \nu_e$

$^{13}{}_{6}C + {}^{1}{}_{1}H \rightarrow {}^{14}{}_{7}N + \gamma$

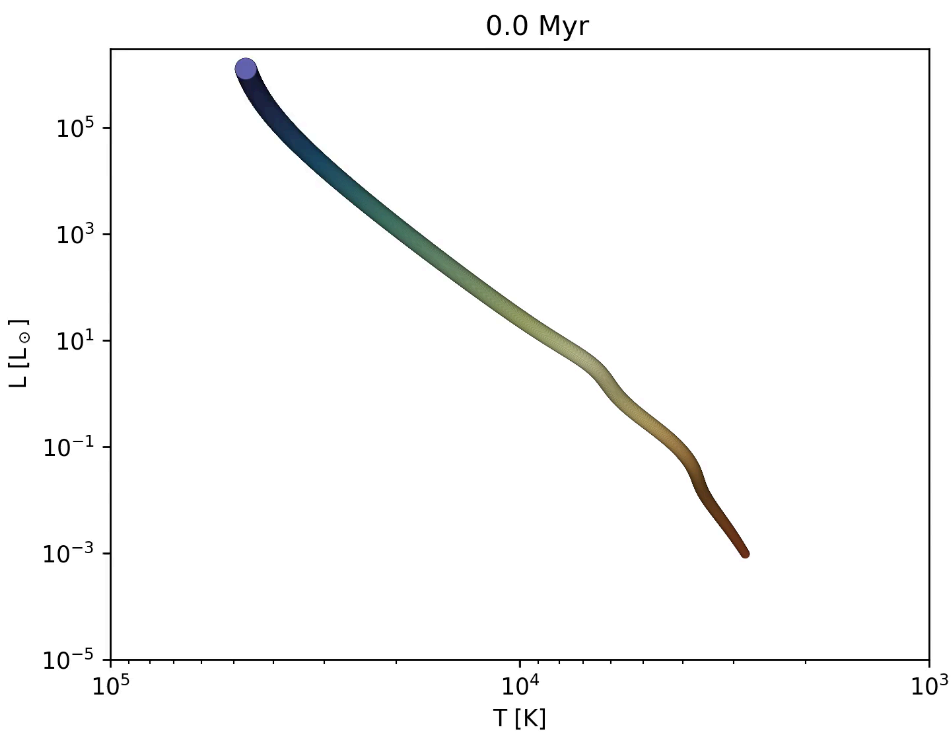
$^{14}{}_{7}N + {}^{1}{}_{1}H \rightarrow {}^{15}{}_{8}O + \gamma$

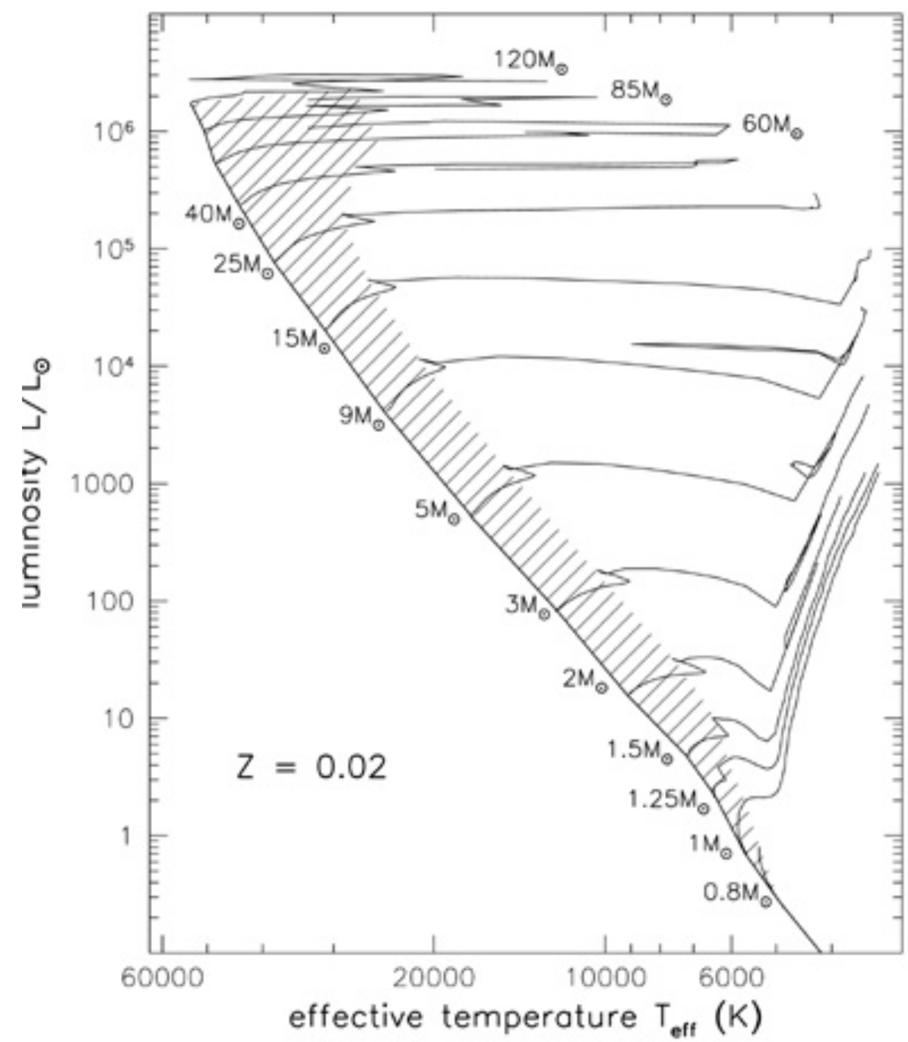
$^{15}{}_{8}O \rightarrow {}^{15}{}_{7}N + e^{+} + \nu_e$

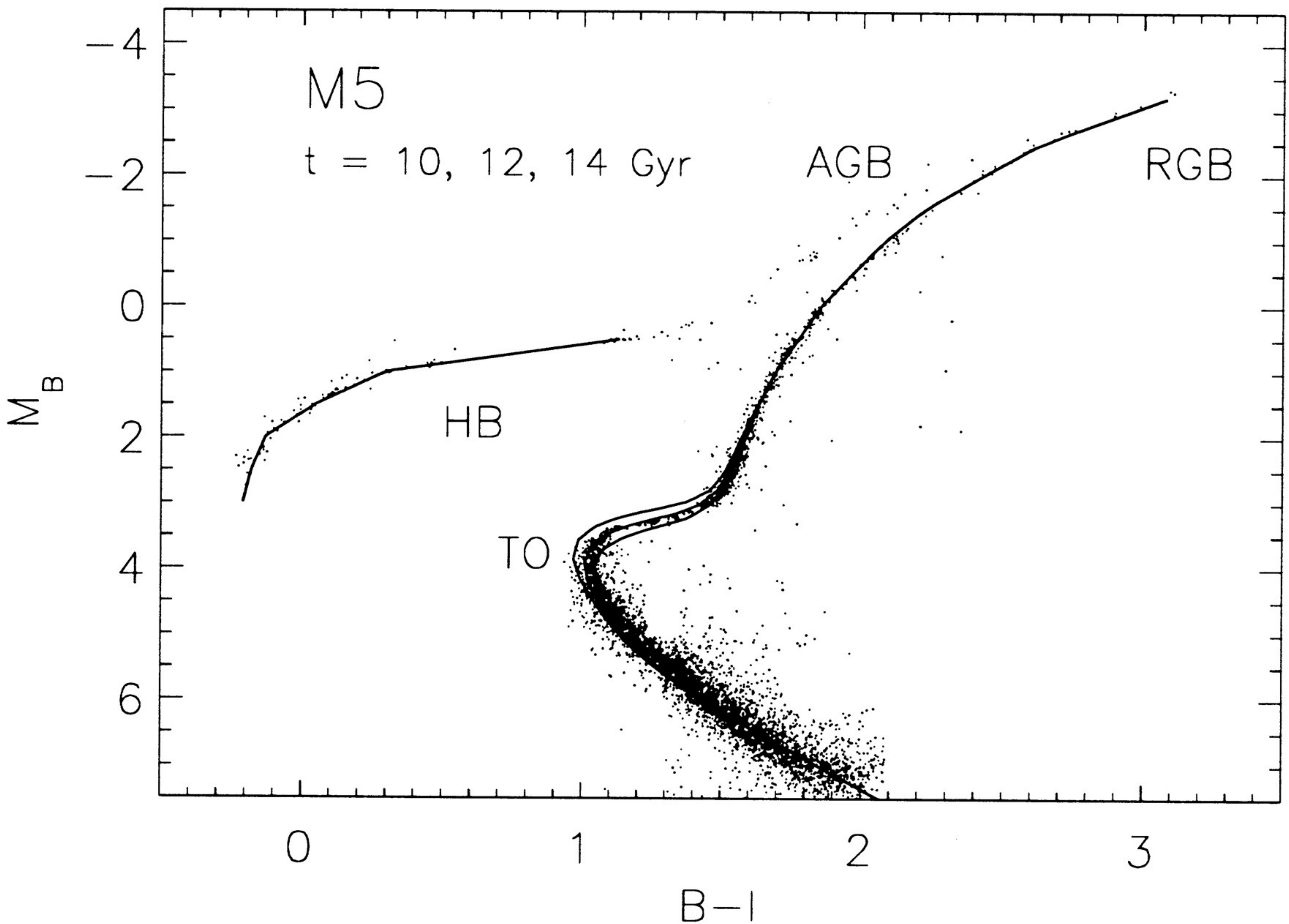
$^{15}{}_{7}N + {}^{1}{}_{1}H \rightarrow {}^{12}{}_{6}C + {}^{4}{}_{2}He$



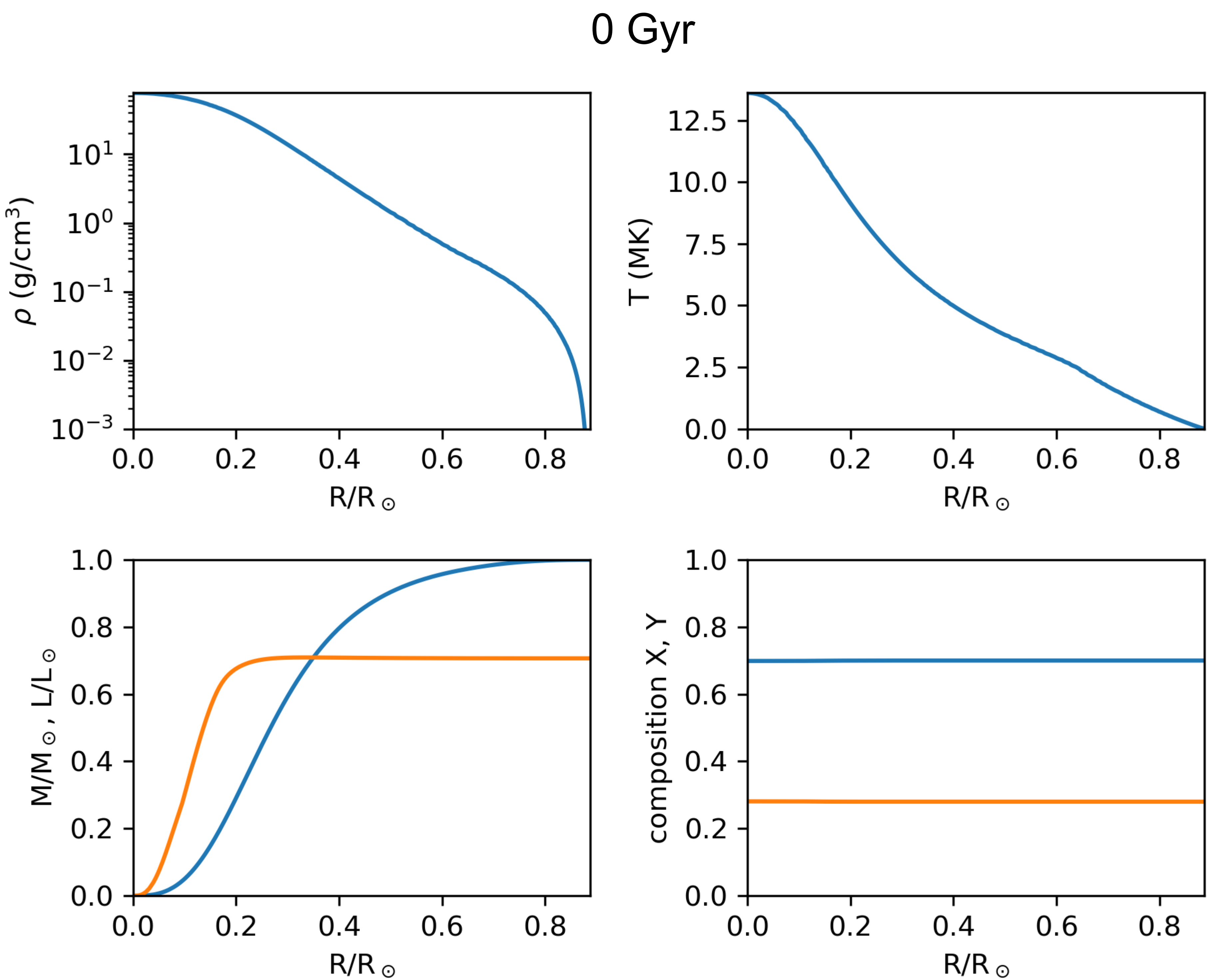
$4\ ({}^{1}{}_{1}H) \rightarrow {}^{4}{}_{2}He + 2\,e^{+} + 2\nu_e + 24\ MeV$


0.0 Myr
L [L⊙]
T [K]
10^5
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
10^5
10^4
10^3

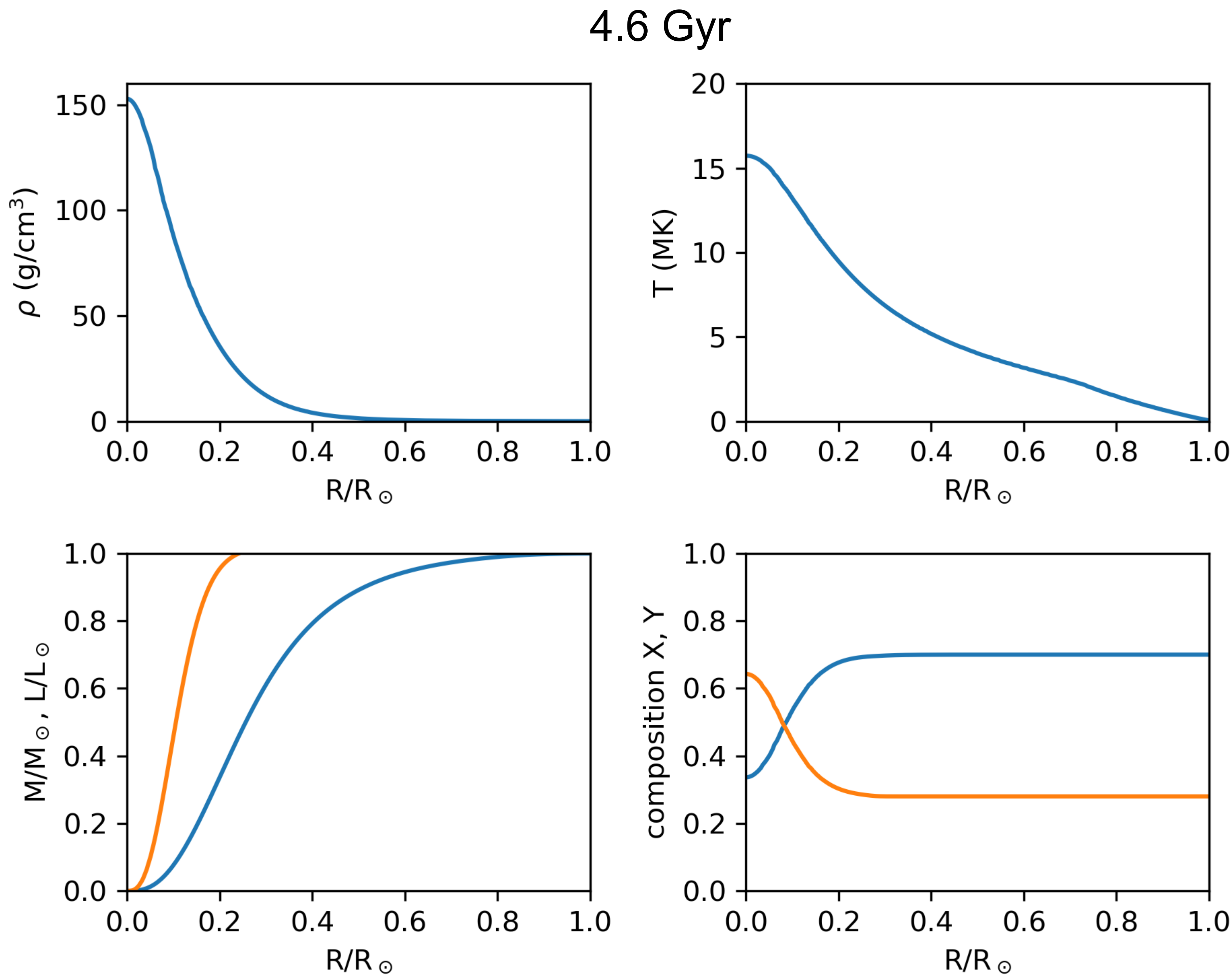

120M⊙
85M⊙
60M⊙
40M⊙
25M⊙
15M⊙
9M⊙
5M⊙
3M⊙
2M⊙
1.5M⊙
1.25M⊙
1M⊙
0.8M⊙
Z = 0.02
luminosity L/L⊙
10^6
10^5
10^4
1000
100
10
1
60000
20000
10000
6000
effective temperature T_{eff} (K)



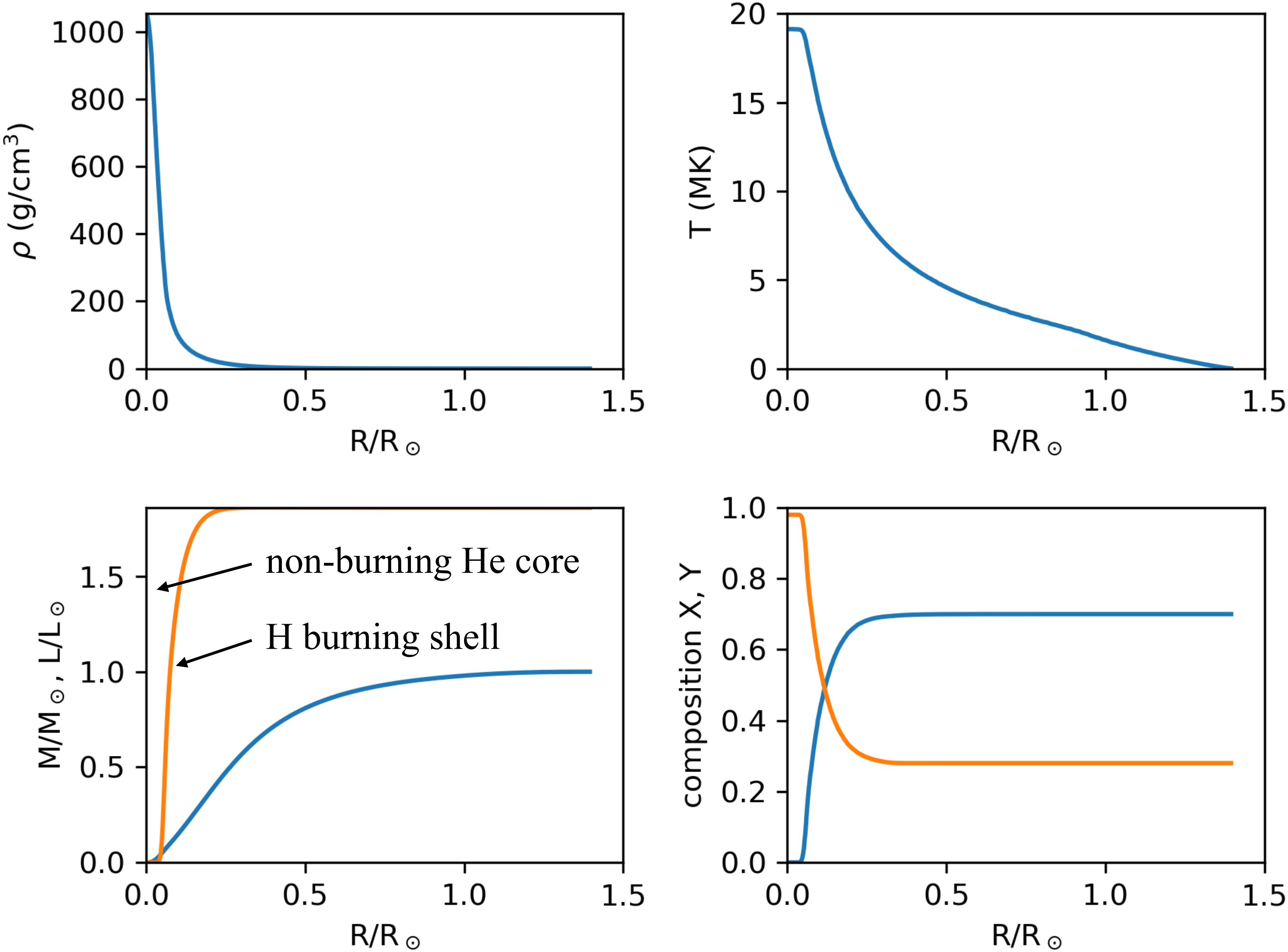
M5
t = 10, 12, 14 Gyr
AGB
RGB
HB
TO
M_B
B−I
−4
−2
0
2
4
6
0
1
2
3



0 Gyr
ρ (g/cm^3)
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
0.0
0.2
0.4
0.6
0.8
R/R$_\odot$
T (MK)
12.5
10.0
7.5
5.0
2.5
0.0
R/R$_\odot$
M/M$_\odot$, L/L$_\odot$
1.0
0.8
0.6
0.4
0.2
0.0
R/R$_\odot$
composition X, Y
R/R$_\odot$

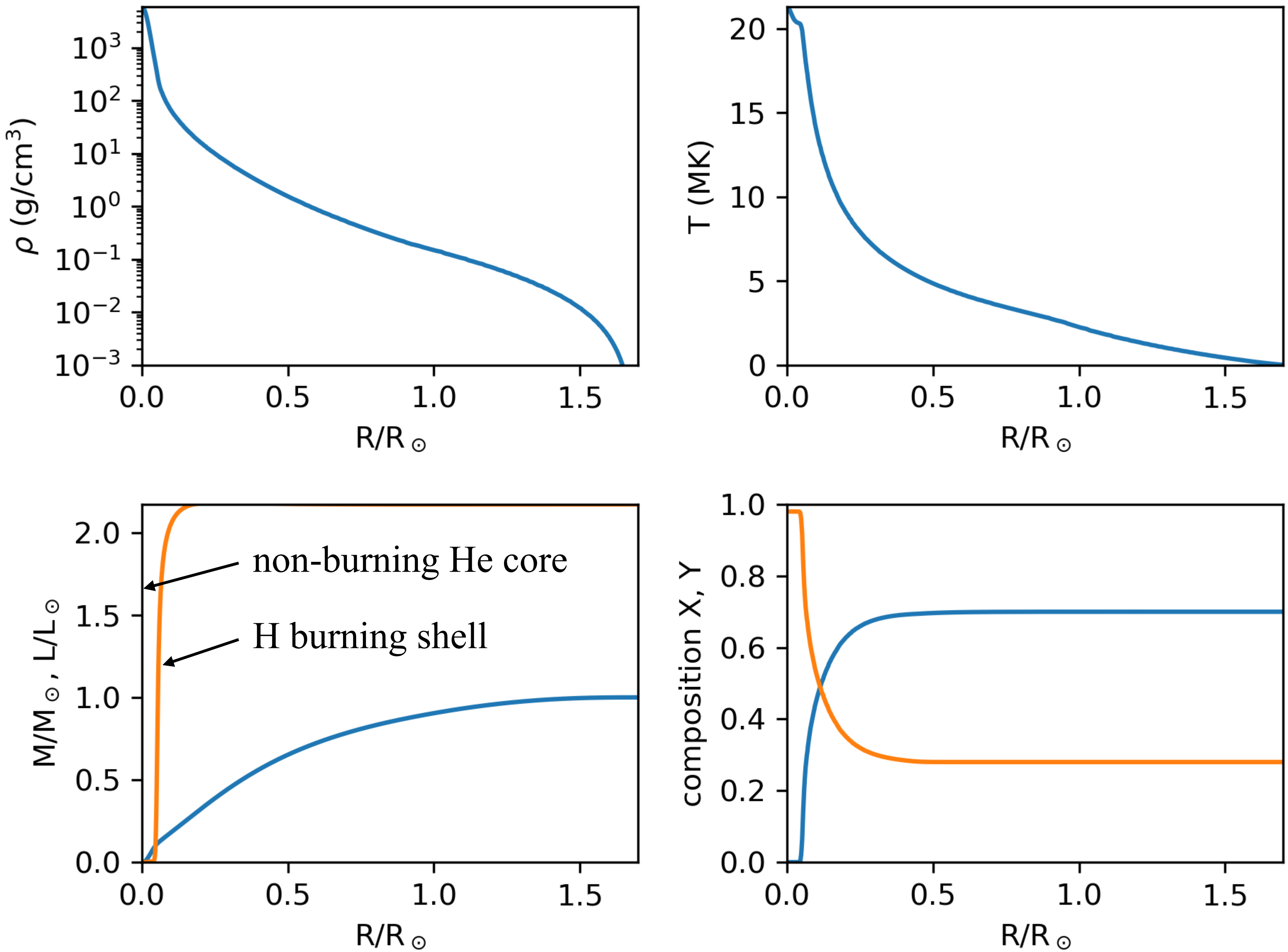

4.6 Gyr
ρ (g/cm³)
R/R ⊙
T (MK)
R/R ⊙
M/M ⊙ , L/L ⊙
R/R ⊙
composition X, Y
R/R ⊙

10 Gyr



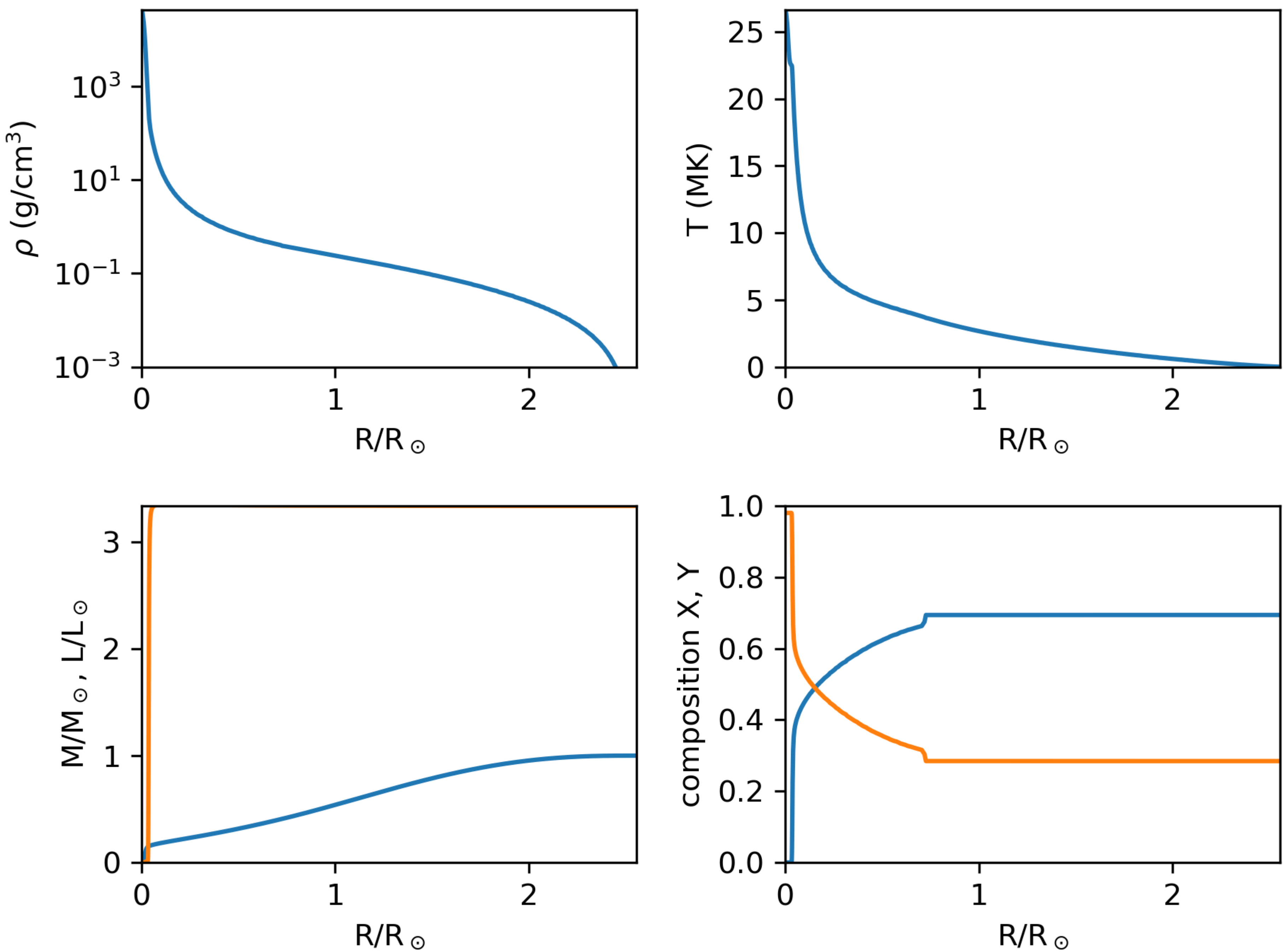
ρ (g/cm³)
1000
800
600
400
200
0
0.0
0.5
1.0
1.5
R/R☉
T (MK)
20
15
10
5
0
M/M☉, L/L☉
non-burning He core
H burning shell
composition X, Y
1.0
0.8
0.6
0.4
0.2

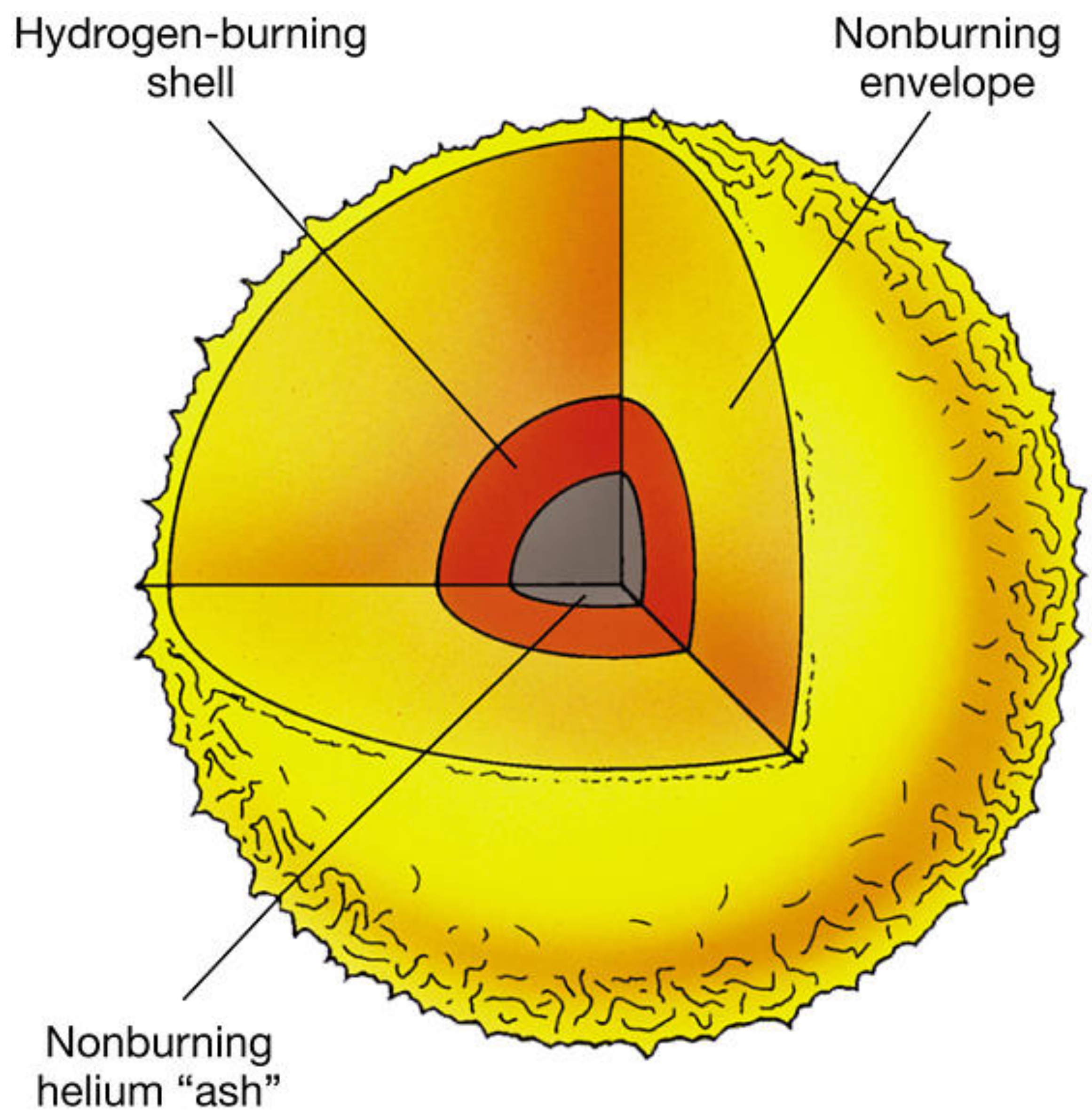
11 Gyr



ρ (g/cm³)
R/R⊙
T (MK)
R/R⊙
M/M⊙, L/L⊙
non-burning He core
H burning shell
R/R⊙
composition X, Y
R/R⊙

11.5 Gyr




Hydrogen-burning
shell
Nonburning
envelope
Nonburning
helium “ash”