# Interstellar Masers

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**Abstract**

This report is on interstellar masers with a focus on the quantum mechanical description. First it is discussed how atomic hydrogen in interstellar space would reach an ionization equilibrium. Then it is shown how the excited hydrogen atoms in this equilibrium will ultimately end up in a saturated state. The lifetime of these saturated states is then computed showing that there will be a population inversion that will create laser activity. Finally the major discoveries in maser research are outlined.

## Equilibrium

A cloud of atomic hydrogen in interstellar space would reach an ionization equlibrium

$$H \leftrightarrows p^+ + e^- \tag{1}$$

This equilibrium is governed by the Saha equation

$$\frac{n_e n_{i+1}}{n_i} = \frac{2g_{i+1}}{\Lambda^3 g_i} e^{-\left(\frac{\epsilon_{i+1}-\epsilon_i}{kT}\right)} \tag{2}$$

where $n_e$ is the electron density, $n_i$ is the density of atoms in the $i^{th}$ state of ionization, $g_i$ is the degeneracy of states for the $i$-ions, $\Lambda$ is the thermal de Broglie wavelength[1] of an electron, and $\epsilon_i$ is the energy needed to create an $i$-level ion.

We are assuming all the hydrogen atoms are either in the ground state or completely ionized[2]. Then the Saha equation can be simplified by taking note of the fact that $n_1 = n_e$.

$$\frac{n_e^2}{n_0} = \frac{2g_1}{\Lambda^3 g_0} e^{-\left(\frac{\epsilon}{kT}\right)} \tag{3}$$

Another simplification can be made using the total number density $n = n_0 + n_1$. Also, the degeneracy of states for atomic hydrogen is 2 and for ionized hydrogen it is 1. The equation then becomes

$$\frac{n_e^2}{n - n_e} = \frac{1}{\Lambda^3} e^{-\left(\frac{\epsilon}{kT}\right)} \tag{4}$$

It is useful to introduce a dimensionless quantity for the degree of ionization

$$\xi \equiv \frac{n_1}{n} = \frac{n_e}{n} \tag{5}$$

[1] $\Lambda = h/\left(2\pi m_e kT\right)^{1/2}$

[2] This is reasonable because of the separation of the hydrogen atom energy levels. If there is enough energy to excite the atom from the ground state to the next excited state, it only needs a 1/4 of that energy to be ionized.

so the equation is now written as

$$\begin{aligned}\frac{\xi^2}{1-\xi} &= \frac{1}{n\Lambda^3}e^{-\left(\frac{\epsilon}{kT}\right)} \\ &= \frac{4\times 10^{-6}}{\rho}T^{3/2}e^{-\left(\frac{1.57\times 10^5}{T}\right)}\end{aligned} \tag{6}$$

where $\rho$ is the density of the ionized hydrogen (protons).

It is important to note that as the density goes up, the degree of ionization goes down. This interesting fact can be understood as follows: the ionization rate per volume is obviously proportional to the hydrogen number density, but the recombination rate involves two particles so it is proportional to the product of their densities. This means that as the density goes up, the recombination rate grows faster than the ionization rate and the fraction of ionized particles goes down.

## Saturated States

It is impossible for a single proton and a single electron to form a bound state by colliding with each other - it would violate conservation of kinetic energy. However in the ionized gas there will be three body collisions that after some complicated interaction, could form a bound hydrogen state and send the third particle off in some direction.

This is how the protons and electrons recombine to form atomic hydrogen. But the most probable outcome of this type of collision is to leave the bound electron in a highly excited state.

The excited electron will then decay to a lower state via electric dipole transitions. The selection rules for this transition are

$$\begin{aligned}\Delta n &= anything \\ \Delta l &= \pm 1 \\ \Delta m_l &= \pm 1, 0\end{aligned} \tag{7}$$

The rate of transition can be found with Fermi's golden rule.

$$T_{i\to f} = \frac{2\pi}{\hbar}\left|\langle f|H'\,|i\rangle\right|^2 n(E) \tag{8}$$

It will be shown in the next section that this rate is proportional to the cube of the change in energy, so it takes the following form.

$$\begin{aligned}T_{i\to f} &\sim (E_{n'}-E_n)^3 \\ &= \left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^3 \\ &= \frac{(n+n')^3\,\Delta n^3}{n^6 n'^6}\end{aligned} \tag{9}$$

So the rate of transition increases like $\Delta n^2$. This means that while $l$ and $m_l$ can only change by one unit, a large change in $n$ is favored. The result is that the electron progresses towards a saturated state with $l = n-1$ and $m_l = \pm l$.

After proton-electron recombination creates states with saturated quantum numbers these states can only decay to the next lowest saturated state. This happens under electric dipole transitions because it gives rise to the selection rules (7) that lead to a possible transition[3].

## Lifetime and Laser Action

The state

$$|n, l, m\rangle = |n, n-1, \pm l\rangle \tag{10}$$

decays to lower energy levels at a rate governed by (8).

The matrix element in Fermi's golden rule can be written as follows

$$\langle f |H'| i\rangle = \left\langle F' \right| \otimes \left\langle n', l', m' \left| -\frac{q}{2mc}\left(\hat{p}\cdot\hat{A} + \hat{A}\cdot\hat{p}\right)\right| n, l, m\right\rangle \otimes \left| F \right\rangle \tag{11}$$

where $F$ is the field state. Because the momentum commutes with the vector potential

$$\left[\hat{p}, \hat{A}\right] = \frac{\hbar}{i}\nabla\cdot\hat{A} = 0 \tag{12}$$

the matrix element can be rewritten as

$$\langle f |H'| i\rangle = -\frac{2q}{2mc}\left\langle F' \left|\hat{A}\right| F\right\rangle \langle n', l', m' |\hat{p}| n, l, m\rangle \tag{13}$$

Instead of working with momentum, it is easier if a switch is made to position using the commutator of the hamiltonian

$$\begin{aligned} [H, r] &= \left[\frac{p^2}{2m} - \frac{e^2}{r}, r\right] \\ &= -\frac{i\hbar}{m}\hat{p} \end{aligned} \tag{14}$$

Substituting this into the matrix element gives

$$\begin{aligned} \langle f |H'| i\rangle &= -\frac{q}{mc}\left\langle F' \left|\hat{A}\right| F\right\rangle \left\langle n', l', m' \left|\frac{im}{\hbar}[H, r]\right| n, l, m\right\rangle \\ &= -\frac{ie}{\hbar c}\left\langle F' \left|\hat{A}\right| F\right\rangle \langle n', l', m' |Hr - rH| n, l, m\rangle \\ &= -\frac{ie}{\hbar c}\left(E_{n'} - E_n\right)\left\langle F' \left|\hat{A}\right| F\right\rangle \langle n', l', m' |r| n, l, m\rangle \\ &= -\frac{ie\omega}{c}\left\langle F' \left|\hat{A}\right| F\right\rangle \langle n', l', m' |r| n, l, m\rangle \end{aligned} \tag{15}$$

Now calculate the field matrix element. The initial state is empty (no photons) and in the final state there is one photon leaving the transition.

$$\left\langle F' \left|\hat{A}\right| F\right\rangle = \left\langle 1 \left|\hat{A}\right| 0\right\rangle \tag{16}$$

The quantized field vector potential is

$$\hat{A} = \sum\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\epsilon_{k\sigma}\left(a_{k\sigma}e^{-i\left(\vec{k}\cdot\vec{x}-\omega t\right)} + a^\dagger_{k\sigma}e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\right) \tag{17}$$

[3]Note that in order to have a non-zero perturbation matrix element we must also have a parity change between states. This rules out the magnetic dipole and electric quadropole transitions. Although the electric octopole transition is still possible, it will occur at a much lower rate than the dipole transition.

which gives the following matrix element.

$$\begin{aligned}\left\langle F' \left| \hat{A} \right| F \right\rangle &= \left\langle 1 \left| \sum \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \epsilon_{k\sigma} a^{\dagger}_{k\sigma} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{x}-\omega t\right)} \right| 0 \right\rangle \\ &= \sum \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}} \epsilon_{k\sigma} e^{i\left(\overrightarrow{k}\cdot\overrightarrow{x}-\omega t\right)}\end{aligned} \tag{18}$$

The atomic matrix element can be written as follows.

$$\begin{aligned}\langle n', l', m' \left| r \right| n, l, m\rangle &= \left\langle n-1, n-2, l-1 \left| re^{-i\phi} \right| n, n-1, l \right\rangle \\ &= \langle R_{n-1,n-2} \left| r \right| R_{n,n-1}\rangle \left\langle l-1, l-1 \left| e^{i\phi} \right| l, l \right\rangle\end{aligned} \tag{19}$$

The radial part is

$$\begin{aligned}\langle R_{n-1,n-2} \left| r \right| R_{n,n-1}\rangle &= \frac{1}{N}\int_0^\infty \psi^* r \psi \\ &= \frac{1}{N}\int_0^\infty r^{n-2} e^{-\frac{r}{n-1}} (r) r^{n-1} e^{-\frac{r}{n}} r^2 dr \\ &= \frac{1}{N}\left(\frac{2n-1}{n(n-1)}\right)^{-(2n+1)} \Gamma(2n+1)\end{aligned} \tag{20}$$

where the normalization factor $N$ is

$$\begin{aligned}N &= \left(\int_0^\infty \left(r^{n-2} e^{-\frac{r}{n-1}}\right)^2 r^2 dr\right)^{1/2} \left(\int_0^\infty \left(r^{n-1} e^{-\frac{r}{n}}\right)^2 r^2 dr\right)^{1/2} \\ &= \sqrt{4^{-n} n^{2n+2}\Gamma(2n)}\sqrt{4^{1-n}(n-1)^{2n}\Gamma(2n-2)}\end{aligned} \tag{21}$$

so the final radial matrix element is

$$\langle R_{n-1,n-2} \left| r \right| R_{n,n-1}\rangle = \frac{\left(\frac{2n-1}{n(n-1)}\right)^{-(2n+1)} \Gamma(2n+1)}{\sqrt{4^{-n} n^{2n+2}\Gamma(2n)}\sqrt{4^{1-n}(n-1)^{2n}\Gamma(2n-2)}} \tag{22}$$

The angular matrix element can easily be computed by making use of Clebsch-Gordon coefficients.

$$\begin{aligned}\left\langle l-1, l-1 \left| e^{i\phi} \right| l, l \right\rangle &= \int Y_{l-1}^{l-1} e^{i\phi} Y_l^l d\Omega \\ &= \sqrt{\frac{3}{8\pi}} \int Y_{l-1}^{l-1} Y_1^{-1} Y_l^l d\Omega \\ &= \frac{3(-1)^{3l} l}{4\sqrt{2}\pi}\left(\frac{1}{l^3(2l+1)(2l-1)}\right)^{1/2}\end{aligned} \tag{23}$$

The last thing needed to calculate the transition rate is the density of states. The emitted photon can come out in any direction, so start with the wavevector for a 3-D box of length L.

$$\overrightarrow{k} = \frac{2\pi}{L}\left(n_x, n_y, n_z\right) \tag{24}$$

then the number of states is

$$\begin{aligned}n(E) &= \frac{4\pi}{3} n^3 = \frac{4\pi}{3}\left(\frac{kL}{2\pi}\right)^3 = \frac{Vk^3}{6\pi^2} \\ &= \frac{V\omega^2 d\omega}{2\pi^2 c^3}\end{aligned} \tag{25}$$

In computing the transition rate from one saturated state to the next, there is only one possible atomic state. However the emitted photon can be emitted in any direction so the field matrix element must be summed over all states. The rate is thus

$$T_{i\to f} = \frac{2\pi}{\hbar} \left|\langle n', l', m' \left| r \right| n, l, m\rangle\right|^2 \sum \left(\frac{e\omega}{c}\right)^2 \left|\left\langle F' \left|\hat{A}\right| F\right\rangle\right|^2 n(E) \tag{26}$$

The sum can be converted to an integral. Substituting in the previously computed matrix element and the density of states, the integral is

$$\int_0^\infty \left(\frac{e\omega}{c}\right)^2 \left(\frac{2\pi\hbar c^2}{V\omega}\epsilon\right) \left(\frac{V\omega^2 d\omega}{2\pi^2 c^3}\right) = \frac{e^2\epsilon^2\hbar}{\pi c^3} \int_0^\infty \omega^3 d\omega \tag{27}$$

The emitted energy will always be between a very small range for the transition so the integral can be written as a delta function and solved.

$$\frac{e^2\epsilon^2\hbar}{\pi c^3} \int_0^\infty \omega^3 \delta(\hbar\omega - \Delta E) d\omega = \frac{e^2\epsilon^2\hbar}{\pi c^3} \left(\frac{\Delta E}{\hbar}\right)^3 = \frac{e^2\epsilon^2}{\pi c^3 \hbar^2} \Delta E^3 \tag{28}$$

Putting everything together, the rate is

$$T_{i\to f} = \frac{2\pi}{\hbar} \left( \frac{9(-1)^{6l} l^2 16^{n-1} (n-1)^{1-2n} n^{-2n} \left(\frac{2n-1}{n(n-1)}\right)^{-4n-1} n(n-1)}{\pi^2 (2l-1) l^3 [2l+1]} \right) \frac{e^2\epsilon^2}{\pi c^3 \hbar^2} \Delta E^3 \tag{29}$$

Note that this is the same dependence on $\Delta E$ that was used without verification in equation (9).

If values are put in for $n$ and $l (= n - l)$, it is seen that the lifetime will be longer for higher values of $n$. This leads to a population inversion as particles continually make these transitions.

Once this population inversion occurs stimulated emission can take place creating a laser.

# Observations

The first observation of an interstellar maser was in 1965 by Weaver et al [1]. They detected a 1665 MHz microwave line from a hydroxyl cloud in Orion. Three years later in 1968 Wilson and Barret also detected an OH line but this time from four different stellar sources [2].

The first water maser was detected in 1969 when Cheung et al. detected an $H_2O$ emission line in three different interstellar sources [3]. That same year Knowles, in collaboration with Cheung and others detected water maser activity from a compact stellar source [4].

In 1970 a methyl alcohol maser was detected near the galactic center by Ball et al [5].

Silicon dioxide masers from molecular clouds and stellar sources were detected in 1974. Both were found by Snyder and Buhl and both sources were in Orion [6, 7].

The first so-called megamaser with a luminosity about $10^6$ times greater than normal was found in 1982 by Baan and Wood [8].

# References


[1] H Weaver et al. Observations of a strong unidentified microwave line and of emission from the oh molecule. *Nature*, 208:29, Jan 1965.

[2] William J Wilson et al. Discovery of hydroxyl radio emission from infrared stars. *Science*, 161:778, Aug 1968.

[3] A. C Cheung et al. Delection of water in interstellar regions by its microwave radiation. *Nature*, 221:626, Feb 1969. A&AA ID. AAA001.131.006.

[4] S. H Knowles et al. Spectra, variability, size, and polarization of h2o microwave emission sources in the galaxy. *Science*, 163:1055, Mar 1969.

[5] John A Ball et al. Detection of methyl alcohol in sagittarius. *Astrophysical Journall*, 162:L203, Dec 1970. A&AA ID. AAA004.141.202.

[6] L. E Snyder et al. Detection of possible maser emission near 3.48 millimeters from an unidentified molecular species in orion. *Astrophys J*, 189:L31, Apr 1974. A&AA ID. AAA011.131.060.

[7] David Buhl et al. Silicon monoxide: Detection of maser emission from the second vibrationally excited state. *Astrophysical Journal*, 192:L97, Sep 1974. A&AA ID. AAA012.132.027.

[8] W. A Baan et al. Broad hydroxyl emission in ic 4553. *Astrophys J*, 260:L49, Sep 1982. A&AA ID. AAA032.158.107.