

Neutron Degeneracy Pressure

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Introduction:

Within the core of a star we find a direct application of a quantum mechanical phenomenon, the degeneracy pressure. The purpose of the degeneracy pressure to combat the collapse of the star by its own gravity. Using an expression for the Fermi energy, we will construct a mathematical expression for the degeneracy pressure. We will then find the gravitational pressure due to the binding energy of a star. Next, we will examine an equilibrium state in a neutron star by balancing the pressures. We will examine the properties, such as the radius and density, of a particular neutron star of mass $M=2M_{\text{sun}}$ (solar masses). We will finally discuss the accuracy of our results based on observed values.

Background:

To begin our examination, we must discuss one of the most fundamental principles in quantum mechanics. The Pauli Exclusion Principle, was formulated by Wolfgang Pauli in 1925 while he was considering Bohr's atomic model, and in order to attempt to explain the results of his experiments on the Zeeman Effect in atomic spectroscopy. This principle states that no two identical fermions (spin $\frac{1}{2}$ particles) can occupy the same quantum state at the same time. The application of this principle to atoms, in the case of Neils Bohr, has allowed physicists to better explain how electrons fill orbitals around a nucleus of an atom, satisfying the different selection rules that atom. This paved the way for physicists to delve deeper into the study of the atom and examine other interactions such as the spin-spin interaction of two electrons or the spin-orbit

interaction. Solving these problems gave atomic physicists a closer approximation of the energy, for example, of non-Hydrogen like atoms. This idea of energy state filling becomes crucial when discussing degeneracy pressure.

The Pauli Exclusion Principle can also be applied to further our understanding of stars, black holes, supernovae, and many other celestial objects and phenomena studied in astrophysics. A vital application is the degeneracy pressure found in the interior of a white dwarf or neutron star. The degeneracy pressure is greatly needed in the interior of these stars to counteract the pressure due to gravity. The balance of the gravitational pressure and degeneracy pressure is what keeps these star stable. Without the degeneracy pressure in neutron stars and white dwarf stars, the pull of gravity would become too great for the thermal pressure to counteract, forcing the star to implode, which could possibly result in the formation of a black hole. Let us briefly look at some properties of white dwarf and neutron stars.

The two types of stars mentioned above are directly dependent on the degeneracy pressure because they are stars that no longer fuse elements, and thus do not burn as hot as larger, younger stars. This results in cooler temperatures at the core, causing the thermal pressure to decrease. The white dwarf star is a small, dense star with a mass less than the Chandrasekhar mass limit of 1.44 solar masses. Many white dwarf stars are born out of red giant stars that have burned their fuel, and lost a large fraction of their mass. These stars have fused all of their hydrogen and helium, leaving an abundance of carbon and, in many cases, oxygen in the core. At this point, the star no longer has enough energy to fuse carbon into a heavier element so the white dwarf star contracts. This contraction compresses the electrons in the core into degenerate energy levels, forming the electron degeneracy pressure. A typical white dwarf with a mass less

than 1.44 solar masses will become stable after balancing the gravitational pressure due to the contraction, and the electron degeneracy pressure.

More massive stars, typically greater than eight times the mass of our sun, however, are energetic enough to fuse elements up to iron. Fusion can no longer naturally occur in stars past iron, however, because it becomes an endothermic process, thus requiring energy. When iron is formed, it is deposited in the core, causing the density to rapidly increase and the core to begin to contract inward. This causes temperatures to rise in the core to help resist collapse. The rise in temperature and density allows for electron capture in the core by the reaction,



Both neutrinos and neutron rich matter are produced at the core of these large stars. Eventually, the core of these larger stars will become too massive, causing a gravitational core collapse supernova which, in many cases, leave behind a neutron star. These neutron stars are neutron rich due to reaction (1), and can weigh up to three solar masses. Neutron stars are much denser than white dwarf stars, which, once again, causes the core of the stars to collapse. The compression of neutrons in the contracting core, however, creates a neutron degeneracy pressure. This pressure, analogous to the electron degeneracy pressure in white dwarf stars, combats the gravitational collapse of the star. If, however, the neutron star is too massive (more than three solar masses), the neutron degeneracy pressure fails and the neutron star collapses into a black hole. We now see that the role of both the neutron degeneracy pressure, and the electron degeneracy pressure is crucial to the maintained stability of a star. We must now ask what neutron or electron degeneracy is, and how it forms a pressure in a star. Since these pressures are analogous, we will examine neutron degeneracy pressure in a neutron star's core.

We have discussed earlier that the Pauli Exclusion Principle prevents two identical fermions from occupying the same state at the same time. Also, we know that when all of the lowest energy states of the neutrons in a neutron star are filled, the neutron star is at its lowest possible total energy. When a star contracts, the free neutrons get pushed closer together, and thus, by the Pauli Exclusion Principle, cannot all remain in their lowest energy states. This, therefore, forces the neutrons to occupy increasingly higher energy states, which, in turn, creates a pressure.

This pressure can be rationalized by considering a confining potential well and by applying the Heisenberg uncertainty principle. In one application, Heisenberg uncertainty principle asserts that complementary properties of a system, such as position and momentum, cannot be determined simultaneously, or mathematically, that $\Delta x \Delta p \geq \hbar/2$. Therefore, we see that there is a momentum, called the Fermi momentum, associated with the neutrons. As the core collapses and the energy states fill to higher levels, the Fermi momentum of the neutrons increases. The energy corresponding to this momentum, called the Fermi energy which we will discuss in the next section will also increase. So, with a decreasing volume and an increasing particle momentum, we can say that a pressure formed inside of the core of the star, and continues to increase as long as the volume continues to increase, and that there are degenerate neutron energy states. Now that we know where the pressure comes from, we can finally derive a mathematical expression for the neutron degeneracy pressure by non-relativistic neutrons inside of a neutron star.

Degeneracy Pressure Approximation:

To approximate the neutron degeneracy pressure in the core of a neutron star, we must first begin by finding the Fermi energy. The Fermi energy assumes a cold, and in our case, three dimensional space filled with N fermions (neutrons) where all of the lowest energy states are occupied. Then, we fill energy states up to, but not exceeding, a particular energy called the Fermi energy. This method is, of course, an approximation because we assume that the neutrons have zero temperature, which, at the center of a star, is not the case. We then assume, appropriately, that all states lower than the Fermi energy are filled and states higher than the Fermi energy are empty. We approximate that the Fermi energy in n -space is similar to the square well energy,

$$E_F = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2) \quad (2)$$

$$= \frac{\pi^2 \hbar^2}{2mL^2} r_n^2 \quad (3)$$

where r_n is the radius in n -space, and n_x , n_y , and n_z are the principle quantum numbers in three dimensions. We can find the total energy of every the fermions in the sphere (assuming a uniform spherical star) by integrating the Fermi energy over the volume of one eighth of the sphere. We know that the quantum numbers n_x , n_y , and n_z are positive integers, so we only need to use the positive axes, which corresponds to one eighth of the sphere. We find that the total energy is,

$$E_{tot} = 2 \frac{1}{8} \int_0^{r_n} \frac{\pi^2 \hbar^2}{2mL^2} r_n^2 (4\pi r_n^2) dr_n \quad (4)$$

$$E_{tot} = \frac{\pi^3 \hbar^2}{10mL^2} r_n^5 \quad (5)$$

Then two corresponds to two spin states for the fermion. Next we find the total number of states available to the fermions within the radius of the Fermi energy. If we consider a sphere of radius r_n , the number of available states in one eighth of that sphere would be,

$$N = (2) \left(\frac{1}{8}\right) \left(\frac{4}{3} \pi r_n^3\right) \quad (6)$$

Where we have two spin states, and the rest is the volume of the one eighth of the sphere. We can then solve for the radius in equation (6) and plug in to equation (5) to find the total energy and the Fermi energy as a function of the number of particles.

$$r_n = \left(\frac{3N}{\pi}\right)^{\frac{1}{3}} \quad (7)$$

$$E_{tot} = \frac{\pi^3 \hbar^2}{10mL^2} \left(\frac{3N}{\pi}\right)^{\frac{5}{3}} \quad (8)$$

$$E_F = \frac{\pi^2 \hbar^2}{2m} \left(\frac{3N}{\pi L^3}\right)^{\frac{2}{3}} \quad (9)$$

We can then replace L^3 from the square well by the volume, V , using the equation $L^3=V$, so equations (8) and (9) become:

$$E_F = \frac{\pi^2 \hbar^2}{2m} \left(\frac{3N}{\pi V}\right)^{\frac{2}{3}} \quad (10)$$

$$E_{tot} = \frac{\pi^3 \hbar^2}{10mV^{2/3}} \left(\frac{3N}{\pi}\right)^{\frac{5}{3}} \quad (11)$$

Finally, we can use a thermodynamic relation to find the pressure from the total energy expression in equation (11). If we assume constant entropy and constant particle number in the

fundamental equation, we can approximate the pressure, P , by taking the derivative of the energy with respect to the volume. Therefore, we obtain:

$$P_{tot} = - \left(\frac{\partial E_{tot}}{\partial V} \right)_{S,N} \quad (12)$$

$$= \frac{\pi^3 \hbar^2}{15m} \left(\frac{3N}{\pi V} \right)^{\frac{5}{3}} \quad (13)$$

This is a general approximation for the degeneracy pressure for fermions in the center of a uniform, spherical star. For a white dwarf star, we stated above that the electron degeneracy pressure balances the gravitational collapse, so equation (13) becomes:

$$P_e = \frac{\pi^3 \hbar^2}{15m_e} \left(\frac{3N_e}{\pi V} \right)^{\frac{5}{3}} \quad (14)$$

This is a calculation of the electron degeneracy pressure, where m_e is the mass of an electron and N_e is the number of electrons. Similarly, for a neutron star, we can find the neutron degeneracy pressure by substituting into equation (13) the neutron mass and the number of neutrons in the star. We obtain:

$$P_n = \frac{\pi^3 \hbar^2}{15m_n} \left(\frac{3N_n}{\pi V} \right)^{\frac{5}{3}} \quad (15)$$

Gravitational Pressure approximation:

As stated earlier, a gravitational pressure is exerted on the star due to its own mass, which helps form the electron or neutron degeneracy pressure. To solve for the gravitational pressure, we must first find the gravitational binding energy. This can be found (for a uniform sphere) by considering two masses in the star. We must first consider a mass in the shape of a spherical

shell in the star with mass $m_s=4\pi r^2\rho$, where ρ is the density of the shell, and the rest is the surface area element. This mass shell has a gravitational attraction to all of the material it surrounds of mass of $m_i=4\pi r^3\rho/3$. We can then plug these two masses into the formula for gravitational potential energy, and, assuming constant ρ , we integrate over every shell in the star's radius. We obtain an equation of the form:

$$E_g = -G \int_0^R \frac{m_s m_i}{r} dr \quad (16)$$

$$= -G \int_0^R \frac{(4\pi r^2 \rho) \left(\frac{4}{3}\pi r^3 \rho\right)}{r} dr \quad (17)$$

$$= -\frac{16G\pi\rho^2 R^5}{15} = -\frac{3}{5} \frac{GM^2}{R} \quad (18)$$

Where we use $M=4\pi r^3\rho/3$ for the mass of the star. This, again, is an approximation of the gravitational binding energy inside of a star because we assume that the density of the material inside of the star is constant. Using this approximation and equation (12), we can calculate the pressure due to this energy. We rewrite (18) in terms of the volume V and differentiate:

$$E_g = -\frac{3}{5} G(NM_N)^2 \left(\frac{4\pi}{3V}\right)^{\frac{1}{3}} \quad (19)$$

$$P_g = -\frac{1}{5} G(NM_N)^2 \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} V^{-\frac{4}{3}} \quad (20)$$

The mass of the star is mainly comprised of the number of nucleons in the star, N , multiplied by the masses of those nucleons, M_N . If, however we know the mass of the star, we can leave the mass as M . Thus, we have an expression for pressure due to gravity and now we must relate that to the degeneracy pressures in our neutron star.

Neutron Star Balancing Act:

Now that we have obtained expressions for the gravitational pressure and the degeneracy pressure, we can examine the interaction between the neutron degeneracy pressure and the gravitational pressure in a neutron star. Under equilibrium conditions, we know that the two pressures are equal in magnitude, but act opposite each other, such that $P_n + P_g = 0$. Assuming our neutron star has a mass of $M = 2M_{\text{sun}}$, which is a reasonable mass for a neutron star, we can make an estimate on the radius of the star by solving $P_n + P_g = 0$ as follows:

$$\frac{\pi^3 \hbar^2}{15m_n} \left(\frac{3N_n}{\pi V} \right)^{\frac{5}{3}} = \frac{1}{5} GM^2 \left(\frac{4\pi}{3} \right)^{\frac{1}{3}} V^{-\frac{4}{3}} \quad (21)$$

$$\frac{\pi^3 \hbar^2}{m_n} \left(\frac{3N_n}{\pi} \right)^{\frac{5}{3}} = 3GM^2 \left(\frac{4\pi}{3} \right)^{\frac{2}{3}} R \quad (22)$$

Where $V = 4/3\pi R^3$ for our approximation, and thus solving for R:

$$\frac{\pi^3 \hbar^2}{3GM^2 m_n} \left(\frac{3}{4\pi} \right)^{\frac{2}{3}} \left(\frac{3N_n}{\pi} \right)^{\frac{5}{3}} = R \quad (23)$$

$$R = \frac{\hbar^2}{\left(Gm_n^{\frac{8}{3}} \right)} \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} M^{-\frac{1}{3}} \quad (24)$$

Equation (24) comes from the fact that the neutron star mass, M , comes from the number of neutrons present multiplied by the mass of a single neutron. For this reason, we can replace N_n with the total mass of the star divided by the mass of a neutron, in other words $N_n = M/m_n$.

Calculations and Conclusions:

Equation (24) says that the radius of a neutron star is inversely proportional to the cube root of the mass. Using the mass estimate of $M=2M_{\text{sun}}$ and equation 24, we find that the radius $R\sim 9.8\text{km}$. Comparatively, this radius gives a circular area of approximately 485 mi^2 which is roughly three times larger than the city of Philadelphia. We can also calculate the density of the star using $\rho=M/V$, where the volume is $(4/3)\pi R^3$, we obtain that the density of a neutron star to be approximately $9.8*10^{17}\text{ kg/m}^3$. Finally, we calculate the neutron degeneracy pressure present in the neutron star by rearranging equation (15) and plugging these values. When we plug in $N_n=M/m_n$ and $V=(4/3)\pi R^3$ into equation (15) we obtain:

$$P_n = \frac{3^{\frac{10}{3}} \hbar^2}{\left(15\pi^{\frac{1}{3}} m_n^{\frac{8}{3}} R^5\right)} \left(\frac{M}{4}\right)^{\frac{5}{3}} \quad (25)$$

After substituting the values $m_n=1.67*10^{-27}\text{kg}$ for neutron mass, $M=2M_{\text{sun}}$ for the mass of the neutron star, $\hbar=1.054*10^{-34}$, and $R=9800\text{m}$ calculated above, we calculate that the neutron degeneracy pressure is $P_n\sim 5.24*10^{33}\text{kg/ms}^2$. We noted above that expression (13) was simply an approximation for the pressure inside of the neutron star, because we assumed the neutrons inside of the star were cold for the Fermi energy calculation, that the neutron star had a uniform neutron density, and that the neutrons behaved classically. It turns out that documentation of different observations of neutron stars agrees with our calculation. The typical radius of a neutron star of mass $M=2M_{\text{sun}}$ is reported as 10-11 km, which is consistent with our results. Thus, using $\rho=M/V$ once again, we see that the densities match up very closely as well. For our calculations, we assumed that there were only two pressures in the neutron star, neutron degeneracy pressure and gravitational pressure. It turns out there is another that could account

for the slight difference, and that is thermal pressure. This pressure is considered to be an insignificant pressure compared to the neutron degeneracy pressure, which is why our values seem to be consistent with reported numbers from papers and textbooks.

Using quantum mechanical principles, we were able to derive an expression for both the electron degeneracy pressure and the neutron degeneracy pressure in a general form. We found an expression for the gravitational binding energy of a star, and the pressure in the star it creates. We were able to balance the neutron degeneracy pressure and the gravitational pressure to find a general expression for the radius of a neutron star. Finally, by assuming a mass of $2M_{\text{sun}}$ we were able to calculate the radius and the density of the star to very close approximations.

References:

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