Cooper Pairs and Their Influence on Superconductivity Theory

in completion of Drexel University's PHYS 517 Final

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June 7, 2013

Abstract

Cooper Pairs make up the backbone of the generally accepted BSC Theory of superconductivity. We attempt to look at the historical significance of the formulation of the pairs as well as go through their derivation. We start with the discovery of superfluidity in Helium. Using Landau's explanation of the formation of a Bose-Einstein Condensate of Helium atoms as a springboard, we attempt to apply the theory to superconductivity. We discuss how the Pauli Exclusion Principle prevents us from using the model on superconducting metals. We attempt to replicate Cooper's derivation of attracting electron interactions mediated by a phonon, called Cooper Pairs. Finally we discuss the application of Cooper Pairs to the more encompassing BCS Theory.

1 Background

Superconductivity in metals is not a new phenomenon. Far from it, the effect was first found in 1911 by H.Kamerling Onnes only three years after having first liquified Helium.

Superconductivity, the state in which certain metals lose all resistance allowing current to flow without an applied potential, requires incredibly low temperatures, something Onnes was able to reach with his newfound refrigeration techniques with Helium.[7] Thirty-six years later, Kapitsa discovered the superfluidity of liquid helium. Kapitsa observed that liquid helium, when moving through relatively thin channels with a speed below a critical velocity, experiences no viscosity, no resistance along within the channel.[4] Several years later Landau emerged with an explaination, stating that the helium atoms are bosons (particles with integer spin) and thus fall into a single wavefunction as a Bose-Einstein Condensate.

The enigma of the superconductive mechanism found in some, but not all, metals had baffled the scientific community for nearly half a century. Such emphasis was placed on the phenomenological explanation or description of these metals, yet the physics community failed to grasp the fundamental mechanism responsible for superconductivity. Feynman put it eloquently, saying,

> "... we do not understand, more or less, how superconductivity works and I would like to address my attention to the is problem of understanding it *more or less*, not of understanding the details of a lot of special phenomena. In other words, I would like to concentrate here on the problem of interpretation from first principles. We would like to connect the Schrödinger equation directly to some experimental facts." [3]

What's interesting is not even a year after stating that criticism of emphasis of describing phenomena rather than understanding the underlying mechanism, Cooper published his landmark paper on electron-pairs coupled by a phonon exchange/interaction.[2]

It is not coincidental that superfluidity and superconductivity share the same prefix. They both seem to experience the same phenomenon of resistanceless movement. The metals act like there is a superfluid of electrons moving without resistance within them. Describing superconductivity using superfluid theories, while attractive was seemingly impossible as electrons were obviously not bosons, but rather fermions. Fermions could not achieve the Bose-Einstein Condensate state needed for superfluidity. Unless physicists could find a way to have these repulsive fermions even remotely attract one another, or behave as one particle, there was no way to extend the theory of superfluidity to superconductivity.

Electrons follow Fermi-Dirac statistics as they are fermions. The reason the Bose-Einstein Condensate models were failing was because fermions must obey the Pauli Exclusion Principle. The principle states that two identical fermions cannot occupy the same state. A Bose-Einstein Condensate essential has a single, grand wavefunction describing the entire system of particles that it encompasses.[4] The problem with trying to apply that with fermions, specifically electrons, is that two of the electrons cannot share the exact same wavefunction, let alone all of them. Yet, the systems behaved almost exactly like the analogous superfluids. There must be some fundamental mechanism that we were missing. Something that would couple pairs of electrons together allowing them to act like bosons.

2 Derivation

We seek to derive an expression for the electron-phonon interaction needed to bind a Cooper Pair in a metal. More specifically we strive to show that the interaction term is negative, thus indicating that the interaction creates a bound state for the two electrons. Let us begin by looking at two electrons hovering just above the Fermi sphere in a metal. All of the energy states below our two electrons have been filled and do not allow any other electron to occupy the same state via the Pauli Exclusion Principle. This is illustrated in Figure 1. Considering only the two electrons then, a set of wavefunctions can be built to satisfy plane-wave energies, which Cooper writes as,

$$\varphi(\mathbf{k_1}, \mathbf{k_2}, \mathbf{r_1}, \mathbf{r_2}) = \frac{1}{V} \exp[i(\mathbf{k_1} \cdot \mathbf{r_1} + \mathbf{k_2} \cdot \mathbf{r_2})] \tag{1}$$

while maintaining periodic boundary conditions in a box with volume V.[2] As Tinkham points out, simply constructing the groundstate wavefunction requires $\mathbf{k_1}$ and $\mathbf{k_2}$ to be of the same magnitude.[7] Replacing them with a single $\mathbf{k}$, we must now take into account the anti-symmetry of the total wavefunction. This means φ is either a sum of products of $\cos[\mathbf{k}\cdot(\mathbf{r_1}-\mathbf{r_2})]$ having an antisymmetric singlet spin state of $(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)$, or a sum of products of $\sin[\mathbf{k}\cdot(\mathbf{r_1}-\mathbf{r_2})]$ having one of the symmetric triplet spin states of $(\uparrow_1\uparrow_2, \uparrow_1\downarrow_2 + \downarrow_1\uparrow_2, \downarrow_1\downarrow_2)$. As we have stated at the very beginning of this derivation, we expect the interaction between the electrons to be attractive. The singlet coupling will give us a stronger interaction between the electrons the closer they are relative to each other, given by $\cos[\mathbf{k}\cdot(\mathbf{r_1}-\mathbf{r_2})]$. Using this we can construct our complete wave equation which looks like,

$$\Psi(\mathbf{R}) = [\sum_{\mathbf{k}>\mathbf{k_F}} a_{\mathbf{k}} \cos(\mathbf{k}\cdot\mathbf{R})](\uparrow_1\downarrow_2 - \downarrow_1\uparrow 2) \tag{2}$$

where $\mathbf{R} = (\mathbf{r_1}-\mathbf{r_2})$, $a_{\mathbf{k}}$ is a weighting factor, and $\mathbf{k_F}$ is the wave number of the highest occupied state in the Fermi sphere. We can now write the Schrödinger equation as,

$$\sum_{\mathbf{k'}>\mathbf{k_F}} a_{\mathbf{k'}} H_{\mathbf{kk'}} = (E - \epsilon_{\mathbf{k}}) a_{\mathbf{k}} \tag{3}$$

where $\epsilon_{\mathbf{k}}$ are the combined energies of the electron pair above the Fermi surface, and $H_{\mathbf{kk'}}$ are the matrix elements of the Hamiltonian in Cooper's original derivation, or the "interaction potential" as Tinkham explains it. Cooper elaborates writing the matrix elements as,

$$H_{\mathbf{kk'}} = \frac{1}{V}\int e^{-i(\mathbf{k}\cdot\mathbf{R})} H(\mathbf{R}) e^{i(\mathbf{k'}\cdot\mathbf{R})} \tag{4}$$

but goes on to say that the actual $H(\mathbf{R})$ can be approximated by

$$H_{\mathbf{kk'}} = \begin{cases} -|F| & \text{for } k_F \leq k, k' \leq k_m \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where F is a constant, and $(\hbar^2/m)(k_m^2 - k_0^2) \simeq 2\hbar\omega_c \simeq 0.2\text{eV}$. Substituting this interaction matrix element into equation 3, we can reduce, rearrange, and turn the sum into an integral to obtain

$$1 = -|F| \int_{\epsilon_F}^{\epsilon_m} \frac{N(k,\epsilon)d\epsilon}{E - \epsilon - \varepsilon_k}, \text{ with } \epsilon - \varepsilon_k = \epsilon_k \tag{6}$$

requiring that $N(k, \epsilon)$ is the density of states of electrons with momentum k. As Cooper points out, to a good approximation $N(k, \epsilon) \simeq N(k, \epsilon_0)$, allowing us to pull it out of the integral. In this expression ϵ can range from ϵ_F which is the highest filled energy state on the Fermi sphere, to ϵ_m which is the maximum energy away from the sphere the electrons can be, using the relations set out in defining the Hamiltonian matrix elements. ε_k is the energy above the Fermi sphere of the pair of electrons. Integrating we observe

$$\frac{1}{N(k.\epsilon)|F|} = \ln \frac{E - \epsilon_m - \varepsilon_k}{E - \epsilon_F - \varepsilon_k} \tag{7}$$

Solving for the eigenvalue energies, we can see that

$$E = \epsilon_F + \varepsilon_k + \Delta \tag{8}$$

where

$$\Delta = (\epsilon_F - \epsilon_m)/(e^{\frac{1}{N(k,\epsilon)|F|}} - 1). \tag{9}$$

But we know $\epsilon_F < \epsilon_m$, so Δ will *never* be positive. This subtle result is exactly what we have striven to prove, that the binding energy of the electron pair interaction is negative (i.e. attractive) for pairs of electrons above the Fermi surface with equal and opposite k. And thus two electrons can, over a relatively large distance in a metal, become attracted to one another via phonon interaction.

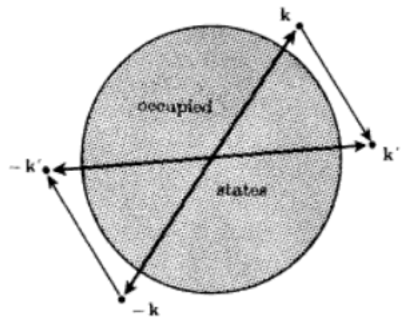

Figure 1: This is a visual representation of the two electrons with($k, -k$) or $(k', -k')$ just above the Fermi sphere.[6]

3 Conclusion

Now that we have successfully shown that two electrons *can* indeed be remotely attracted to each other by way of a phonon interaction, we should take a step back and ask ourselves what this means physically. What exactly is going on in the superconducting metals?

The first thing that we must realize is that all of the electrons in the metal are interacting with each other and with the lattice surround them. This is what is called the Landau quasiparticle or the Fermi Liquid of electrons.[5] The vibrations of the lattice can be quantized, and these quantizations are what we call phonons. As an electron moves by lattice points, it actively distorts the overall structure, attracting the nearby positive nuclei. This distortion, in a sense, pulls on other electrons, thus changing its original path. These distortions are exactly what we mean by phonons and that is how the phonon interaction can mediate momentum from one electron to another over relatively large distances. It is important to realize though that the momentum change in the electron pair mediated by a phonon happens in conjunction with all the other electrons above the Fermi sphere, all at the same time. This is truly a complicated quantum system Cooper attempts to describe.

Looking back at our derivation, Δ cannot ever be positive in our model. This is the binding energy associated with the Cooper Pair. The negative allows us to say without a doubt that they are attracted to each other, yet their totally energy is not actually less than zeros. It is also important to note that our derivation is only valid for a single pair of electrons above the Fermi surface, not for the entire sum of electrons at $T = 0$. That is what Bardeen, Cooper, and Schrieffer layout in their later letter to the editor in February of 1957.[1] In that paper, they explain the full nature of BSC Theory, as it has been come to call. BSC Theory takes into account the sum of the electrons in the metal of a superconductor, describing their eventual fall into the Bose-Einstein Condensate at $T = 0$.

We have successfully shown that two electrons, though both of negative charge and fermion nature, can become briefly bound together. This interaction is mediated over a relatively large distance by a scattering phonon. The electrons, we have found, must be hovering above the Fermi Sphere to be able to interact with each other. All other electrons for our outlined model are normal electrons occupying energy states less than the Fermi energy. When two particles are bound in this phonon-mediated attraction state, we call them a Cooper Pair. Cooper Pairs are the only way discovered so far, to explain the electron Bose-Einstein Condensate present in superconductive metals.

References

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