Black Holes, Thermodynamics, and Cosmology

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Abstract

There are deep connections between thermodynamics and Einstein's theory of General Relativity (GR). The most important example in astrophysics is the fact that we can define appropriate quantities to write down three laws of Black Hole (BH) mechanics which are analogous to the three laws of thermodynamics. I will here discuss the significance and origin of these laws as well as the implications of the relationship between thermodynamics and GR for cosmology.

1 Introduction

It was found in the 1970s by Hawking and Bekenstein that we can associate with a black hole an entropy which is proportional to the surface area of its event horizon. Following this, three laws of BH mechanics, directly analogous to the three laws of classical statistical mechanics, were found. It seems then that there may be further connections between GR and thermodynamics and that this may pave the way for a quantum theory of gravity.

This paper is organized as follows: In section 2, I state and discuss the three laws. In 3, I discuss equilibrium properties on BH from thermodynamic considerations. In 4, I discuss some of the quantum theory behind BH and problems associated with it. In 5, I consider possible ways of interpreting BH entropy, and how it relates to the more familiar entropy from thermodynamics. In 6, I very briefly note the relative importance of BH entropy compared to the rest of the entropy in the universe and its role in the formation of cosmic structure. Finally, in 7 I discuss some further implications of the GR-thermodynamics relationship, including the idea that Einstein's field equations may be derivable from thermodynamics.

2 The Three Laws

Noting the apparent requirement that BH area always increase, Bekenstein supposed an analogy between this and entropy [1]. Further evidence for this proposal is the increase of irreducible mass (see the section on Interpretation) and the observation that, just as one can obtain work by putting two systems which are independently in equilibrium together, this irreducible mass can decrease in BH mergers.

2.1 The First Law

Using the result that the "rationalized area" of a BH can be written

$$\alpha = A/(4\pi) = (M + (M^2 - Q^2 - L^2/M^2)^{1/2})^2 + L^2/M^2 \tag{1}$$

Taking the derivative, Bekenstein finds the relation

$$dM = (r_+ - r_-)/(4\alpha)d\alpha + \vec{L}/(M\alpha) \cdot d\vec{L} + Qr_+/\alpha dQ \tag{2}$$

where $r_\pm = M \pm (M^2 - Q^2 - L^2/M^2)^{1/2}$. This is similar to the first law of thermodynamics!

$$dE = TdS - PdV \tag{3}$$

The first term, as has been said, is the analogous entropy. The second is work done on the BH which increases its angular momentum and the third is work done that increases its charge.

2.2 The Second Law

Bekenstein proposes a "Generalized Second Law" of thermodynamics:

$$\Delta(S_{BH} + S_{thermo}) \geq 0 \tag{4}$$

It makes sense to have this generalization, because when a particle falls into the event horizon, its information is apparently lost to the outside universe, and so that is a decrease in entropy. A corresponding (greater or equal) increase in the BH entropy thus saves the second law.

2.3 The Third Law

In writing down the total mass inside and outside an event horizon, Bardeen et al find a term [4]:

$$1/(4\pi) \int \kappa dA \tag{5}$$

where $\kappa = -\nabla_b l_a n^a l^b$, n is a null vector normal to the surface being integrated over, and l is a null vector tangent to the generators of the horizon. This quantity is interpreted as follows: a particle outside the event horizon which rotates with the BH has an angular speed Ω and four-velocity $v^a = v^t(K^a + \Omega\widetilde{K}^a)$, where K_a is a time translational Killing vector. The particle then has acceleration $1/v^t \nabla_b (v^a) v^b$, which approaches κ as the particle is brought infinitesimally close to the horizon. Therefore, κ is interpreted as the BH surface gravity, and it is found to be constant over the horizon [4].

The first law then has the term $\kappa/(8\pi)\delta A$, and so κ is interpreted as being the BH temperature (Note that we then also have a "Zeroth Law" of BH mechanics because this quantity has been found to be constant). A Third Law, analogous to that of thermodynamics, could then be established if the surface gravity cannot be brought to zero in a finite number of steps. Bardeen et al does not prove this, but argues in favor of it by considering the following method of decreasing κ:

We can throw particles into a BH to increase angular momentum and decrease surface gravity. For a Kerr BH, it was found that

$$\kappa = (M^4 - J^2)^{1/2}/(2M(M^2 + (M^4 - J^2)^{1/2})) \tag{6}$$

where J is the angular momentum. So, $\kappa = 0$ when $J/M^2 = 1$. This decrease of κ becomes smaller with each particle as this ratio approaches 1 and it would take an infinitely long time to achieve this.

3 Black Hole Stability

An interesting application of BH horizon thermodynamics is in studying the stability of BH. In this framework, there are five thermodynamic variables E, P, V, T, S. From these, we can derive other thermodynamic quantities like specific heat, Gibbs free energy, compressibility, and an expansion coefficient to study the stability of a BH.

(Padmanabhan's horizon thermo gives first law from GR) For a static, spherically symmetric spacetime, we can write

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{7}$$

,

where $f(r) = 1 - 2m(r)/r$.

Plugging this into the Einstein equation gives us for both the 00 and 11 component

$$dm/dr = -4\pi r^2 T_0^0 \tag{8}$$

.

The proportionality of temperature and surface gravity

$$T = \kappa/(2\pi) = -1/(4\pi)\frac{\partial_r g_{tt}}{\sqrt{-g_{tt}g_{rr}}}|_{r=r_+} \tag{9}$$

then gives us an equation of state

$$1/2 = 2\pi r_+ T - 4\pi r_+^2 T_0^0 \tag{10}$$

Multiplying by dr and integrating, we also find the first law of horizon thermodynamics for a black hole

$$\boxed{dE = TdS - PdV} \tag{11}$$

Now, by integrating eq. (8), we can get the mass at the horizon and vary it to obtain

$$\delta M = (1/(4\pi r_+) + 2r_+ T_0^0)\delta A/4 - (4\pi \int_{r_+}^{\infty} r^2 \frac{\partial T_0^0}{\partial Q} dr)\delta Q \qquad (12)$$

,

which gives the more familiar first law of BH thermodynamics

$$\boxed{\delta M = T\delta S + \phi \delta Q} \qquad (13)$$

Since we derived both of the boxed equations above from the same field equation, they can be derived from each other. However, they yield very different stability properties. To see this, we can rewrite the equation of state (10)

$$P = T/(2r_+) - 1/(8\pi r_+^2) \qquad (14)$$

and use $V = 4\pi/3r_+^3$. For stable equilibrium, we require $\frac{\partial P}{\partial V}|_T \leq 0$. The constraint we then get for temperature tells us that $P \geq 1/(8\pi r_+^2) > 0$.

Another criterion is that the specific heats obey

$$C_P \geq C_V \geq 0 \qquad (15)$$

.

The definition is

$$C_P = T\frac{\partial S}{\partial T}|_P = \frac{2\pi r_+^2(8\pi P r_+^2 + 1)}{8\pi P r_+^2 - 1} \qquad (16)$$

Zero pressure gives negative C_P. Therefore, we must have $P > 0$, and we see that the larger a BH is, the more stable it will be when the horizon radius exceeds the critical radius.

However, after examining some special cases, it turns out nearly all static, spherically symmetric BH are unstable, according to this form of the first law! For instance, in the case of a Reissner-Nordstrom BH, the stress-energy tensor tells us that the pressure is negative, so this BH apparently cannot be stable. This is not the case when beginning from the more familiar first law of BH thermodynamics, because each form uses different thermodynamic variables, but it is not obvious which set of variables is the better choice.

3.1 Space of Equilibrium States

Quevedo and Vazquez show that it is also possible to define a metric tensor in the space of equilibrium states of a thermodynamic system. Writing the first law as

$$dM = TdS + \Omega dJ + \phi dQ \tag{17}$$

,

we can write a fundamental equation $M = M(S, J, Q)$ for a rotating, charged BH in the form

$$M = (\pi J^2/S + S/(4\pi)(1 + \pi Q^2/S)^2)^{1/2} \tag{18}$$

The space of equilibrium states (rather than spacetime) is thus a 3D, with coordinates corresponding to the BH's three variables. For $J = 0$, this gives the "Reissner-Nordstrom" metric [3]

$$g_{RN} = 1/(2S^2)(\pi Q^2 + S)(1/(8\pi S)(3\pi Q^2 - S)dS^2 - QdSdQ + SdQ^2) \tag{19}$$

The limit of thermodynamic stability is at the singularities of this metric [3]. The curvature scalar of the metric is

$$R_{RN} = -8\pi^2 Q^2 S^2(\pi Q^2 - 3S)/((\pi Q^2 + S)^3 * (\pi Q^2 - S)^2) \tag{20}$$

which has two critical points $S = \pi Q^2$ and $S = \pi Q^2/3$ which correspond, respectively, to the cases $M = Q$ and $M = 2Q/3^{1/2}$. Near these points, the curvature does indeed correspond to the thermodynamic behavior of the BH. This does not, however, appear to work for the $Q = 0$ case, for which the curvature is found to be zero, which cannot be correct. Quevedo and Vazquez state that this is an apparent failure of geometrothermodynamics, but that the situation ought to be salvageable with a more careful choice of metric tensor.

4 Quantum BH Thermodynamics

4.1 Problems with the Classical Theory

There are some problems with the theory so far.

1) The temperature of a BH seems to vanish.

2) In statistical mechanics, entropy is dimensionless, but S_{BH} has units of length squared

3) The area of a BH does not decrease, but only the total entropy need not decrease in thermodynamics.

4.2 Quantum

A theory of quantum gravity is necessary for a full treatment of BH thermodynamics because we need to incorporate quantum physics. Until we have this, a semiclassical treatment will suffice.

Essential to the field of BH thermodynamics essentially began with Hawking's discovery that a BH will emit radiation [2]. In quantum field theory, it is known that even vacuum undergoes fluctuations, producing particle-antiparticle pairs for very short periods of time before they annihilate. When a pair forms near enough a BH, one will fall into the horizon and the other can escape.

Hawking thus associated the radiation from a non rotating BH a temperature $T_H = \hbar\kappa/(2\pi)$, where κ is the horizon's surface gravity. The first law then gives

$$S_{BH} = A/(4\hbar\pi) \tag{21}$$

This resolves problems 1) and 2). Additionally, since a BH can evaporate via Hawking radiation, its area can decrease, so long as the total entropy is still nondecreasing (the Generalized Second Law).

4.3 The Unruh Effect

An accelerated observer in a Minkowski space vacuum will say there is radiation with temperature proportional to their acceleration. Although this was discovered later, it is more fundamental than the Hawking effect and is due to quantum fluctuations of the vacuum [7].

5 Interpreting BH Entropy

As in thermodynamics, BH entropy is related to the "degradation of energy" in that its increase means more energy is no longer "useful" [1]. Increasing the familiar form of entropy means the energy of the system cannot be converted into work; increasing a BH's entropy means that the irreducible mass (in the case of a rotating BH)

$$M_{ir} = (A/(16\pi))^{1/2} \tag{22}$$

increases. The irreducible mass is mass that cannot be extracted from the BH via a "Penrose process." I will not go into the details of this theoretical process, but the point is clear; there is more evidence yet that the BH area-entropy analogy is a deep one.

Introducing an entropy for BH is still troublesome because we expect, as in statistical mechanics, to be able to express it as a logarithm of the number of possible states it may be in. The first problem is that this number ought to be infinite. A more familiar thermodynamic system, a box full of radiation at constant energy and volume, has a finite number of states because the particles can only be in some range of wavelengths due to the two constraints. But since the horizon is a surface of infinite redshift, there is no high-wavelength cutoff in the case of a BH!

A proposed solution is that it should be possible to renormalize BH entropy, as is done for quantities in quantum field theory. But this is problematic because we would expect entropy to count dimensions in Hilbert space, and so we should not have to deal with infinities.

If the Generalized Second Law is true, then the most entropy that can be contained within an area A is the corresponding BH entropy. It has been proposed that we should then talk about a finite dimensional space of quantum states on a given region's boundary rather than its volume. This is called the Holographic Hypothesis. To gain some intuition here, we can consider the classical interpretation to be that we are really talking about the phase space of a gravitating system. This may be an indication that the Holographic Hypothesis will be a consequence of a theory of quantum gravity.

Another attempt at a reasonable definition of BH entropy (by Bekenstein) is that it is proportional to the log of the number of ways it could have formed. This is obviously similar to the classical $S = k_B log(\Omega)$. Hawking noted and proposed a solution to a possible issue with this: If there are more fundamental fields of nature than we know about, then there would be more ways a BH could have formed, but the entropy would be unchanged. He then supposed that, in this situation, we could save this definition of entropy by saying that the BH will radiate faster due to the extra fields, so each particle takes up less phase space than before, and perhaps this will cancel what appeared to give us extra entropy. This is again a highly speculative issue that requires a quantum theory of gravity to be fixed.

Two other possibilities are that BH entropy is the thermal entropy of Unruh radiation just outside the horizon and that it is "entanglement entropy," i.e. a measure of the information in correlated degrees of freedom inside and outside the horizon. The point is that it is unclear exactly what BH entropy is in relation to classical thermodynamic entropy, but the study of the topic may bring us closer to a theory of quantum gravity.

6 Cosmic Structure Formation

Cosmic entropy and cosmic structure are found to co-evolve with each other, with entropy increase being driven by structure formation (e.g. of galaxies and clusters). Simulations have found that BH entropy dominates the overall entropy of the universe by about 20 orders of magnitude.

7 The Einstein Equation as an Equation of State

One intriguing possible implication of the relationship between thermodynamics and general relativistic systems is that Einstein's field equation is in fact a consequence of the thermodynamics of spacetime. The argument is as follows [5]:

We start with the fact that BH entropy is proportional to its surface area and write the first law, $\delta Q = TdS$. Each term must be interpreted for the case of spacetime dynamics if we are to make the desired connection.

Just as heat in classical thermodynamics is energy flowing between the system's degrees of freedom, we can define heat in our case as energy flowing across a causal horizon; even if we cannot observe it from outside the horizon, it still affects us with its gravitational field. Entropy can be defined for a causal horizon as well, as it "hides" information from outside observers. It is proportional to the area of the horizon, as in BH thermodynamics. Finally, for the temperature we use the Unruh temperature seen by a uniformly accelerating observer just inside the horizon.

For any local Rindler horizon through a point in spacetime, the heat flux can be written

$$\delta Q = \int T_{ab}\chi^a d\Sigma^b \tag{23}$$

This can be rewritten using an affine parameter λ, area element A and using $\chi^a = -\kappa\lambda k^a$:

$$\delta Q = -\kappa \int \lambda T_{ab} k^a k^b d\lambda dA \tag{24}$$

Moving on, entropy of a piece of the horizon satisfies $dS = \eta\delta A$ for some constant η, and we can write

$$\delta A = \int \theta d\lambda dA \tag{25}$$

,

where θ is the expansion of the horizon generators. Geodesic deviation gives the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -(1/2)\theta^2 - \sigma^2 - R_{ab}k^a k^b \tag{26}$$

,

where σ_{ab} is the shear. Finally, the first law then implies the Einstein equation

$$R_{ab} - (1/2)Rg_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta}T_{ab} \tag{27}$$

To summarize, by interpreting the thermodynamic variables this way, the first law then requires that spacetime bend under the influence of mass/energy according to the Einstein Equation [5].

We can expand on this relationship between thermodynamics and GR by introducing the pair of variables (f^{ab}, N^c_{ab}), where the first is related to the metric by $f^{ab} = \sqrt{-g}g^{ab}$ and the second is the canonical momentum. There are analogous to the pair (S, T) [6] and we can use them to study gravity by taking the dot product of the Noether current and integrating in 3 dimensions to find

$$\int_R d^3x\sqrt{h}u_a g^{ij}(L_\xi N^a_{ij}) = \int_{\partial R} d^2x\sqrt{\sigma}r_\alpha(2Na^\alpha) - \int_R d^3x\sqrt{h}(2Nu^a u^b R_{ab}) \tag{28}$$

where L is a Lie derivative and r_α is the normal to the boundary of the 3D region R. Following this, [6] also defines the bulk and surface degrees of freedom

$$N_{surf} = \int_{\partial R} \frac{\sqrt{\sigma}d^2x}{L_P^2}, N_{Bulk} = \frac{|E|}{(1/2)k_B T_{avg}} \tag{29}$$

The above equation then tells us that coming observers in a static spacetime will say $N_{surf} = N_{bulk}$, which means that equipartition is holographic in these spacetimes! Further, we can find from our previous equation that

$$\int \frac{d^3x}{8\pi L_P^2}\sqrt{h}u_a g^{ij} L_\xi N^a_{ij} = \epsilon/2k_B T_{avg}(N_{surf} - N_{bulk}) \tag{30}$$

which means that the dynamics of the spacetime are driven by the departure from holographic equilibrium.

8 References

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