The Quantum Hall Effect in Graphene

Daniel G. Flynn*
Department of Physics, Drexel University, Philadelphia, PA
(Dated: December 7, 2010)

Graphene is a single layer of carbon atoms arranged in a hexagonal lattice. The two-dimensional nature of graphene leads to many interesting electronic, thermal, and elastic properties. One particularly interesting property of graphene is that it exhibits the quantum Hall effect at room temperature.

I. INTRODUCTION

Classically, when an external magnetic field is applied perpendicularly to a current carrying conductor, the charges experience the Lorenz force and are deflected to one side of the conductor. Then, equal but opposite charges accumulate on the opposite side. The result is an asymmetric distribution of charge carriers on the conductor's surface. This separation of charges establishes an electric field that opposes further charge build-up. As long as charges flow, a steady electric potential exists called the Hall voltage and the resistivity of the conductor depends linearly on the magnetic field strength. This is known as the classical Hall effect.

In 1980, Klaus von Klitzing discovered that at low temperatures and high magnetic field strength, the plot of resistivity vs. applied magnetic field strength becomes an increasing series of plateaus. This implied that in quantum mechanics, resistance is quantized in units of $\frac{h}{e^2}$. The plateaus corresponded to the cases where the resistivity was related to the magnetic field by integer and some fractional values of a quantity known as the filling factor. These integer and fractional values led to the theory of the integer quantum Hall effect and the fractional quantum Hall effect. Both of these effects have since been observed in graphene, a single layer of carbon atoms in a hexagonal lattice, at room temperature. [1][2][3][4]

II. BACKGROUND

A. Particle Exchange and Fractional Statistics

The quantum Hall effect is only observed in two-dimensional systems. Thus the quantum Hall effect must be explained using the properties of two-dimensional systems. In three dimensions, particles with integer spin (bosons) follow Bose-Einstein statistics while particles with half-integer spin (fermions) follow Fermi-Dirac statistics. However, in two dimensions, particles can follow statistics that range continuously from fermionic to bosonic statistics. Such statistics are called parastatistics. To understand how these statistics work, consider the particle exchange operator, P, acting on n identical particles constrained along a chain. Let x_i be the position of the *ith* particle. Then the composite wavefunction for the particles in the chain is

$$\Psi := \Psi(x_1, x_2, ..., x_n) \tag{1}$$

For identical bosons, the particle exchange operator, P, flips two of the coordinates and the resulting wavefunction has eigenvalue +1.

$$P_{ij}\Psi(x_1, .., x_i, .., x_j, .., x_n) = \Psi(x_1, .., x_j, .., x_i, .., x_n) \tag{2}$$

For identical fermions, the particle exchange operator flips two of the coordinates and the resulting wavefunction has eigenvalue -1.

$$P_{ij}\Psi(x_1, .., x_i, .., x_j, .., x_n) = -\Psi(x_1, .., x_j, .., x_i, ..., x_n) \tag{3}$$

For particles in two dimensions obeying parastatistics

$$P_{ij}\Psi(x_1, .., x_i, .., x_j, .., x_n) = exp(i\alpha)\Psi(x_1, .., x_j, .., x_i, .., x_n) \tag{4}$$

In this case, applying the particle exchange operator swaps two of the coordinates and the resulting wavefunction has an eigenvalue of arbitrary phase, α. Applying the exchange operator a second time will swap the two coordinates back to their original position returning the system to its initial wavefunction.

$$\begin{aligned} P_{ji}P_{ij}\Psi(x_1, .., x_i, .., x_j, .., x_n) &= exp(2i\alpha)\Psi(x_1, .., x_j, .., x_i, .., x_n \\ &= \Psi(x_1, .., x_i, .., x_j, .., x_n) \qquad (5 \end{aligned}$$

This requires that the phase follows

$$exp(2i\alpha) = 1 \rightarrow exp(i\alpha) = \pm 1 \tag{6}$$

For bosons the phase is given by $\alpha = 2n\pi$, and for fermions the phase is given by $\alpha = \pi(2n + 1)$ where n is an integer. In two dimensions, however, the phase can range between the fermionic and bosonic values, and the statistics corresponding to these intermediate phases are called fractional statistics.

B. Anyons

The quasi-particles that follow the fractional statistics described above were given the name "anyons" by Frank

*Electronic address: dgf26@drexel.edu

Wilczek in 1982. One model for anyons is a spinless particle orbiting around a thin solenoid that produces a magnetic field that defines the z-axis. The charge of the particle is proportional to the applied flux Φ

$$q = C\Phi \tag{7}$$

where C is a constant. Before the magnetic flux is applied to the system, the particle's angular momentum is quantized, $l_z = n\hbar$. Then when a changing magnetic field is applied, it creates a circular electric field according to Maxwell's equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8}$$

This circular electric field exerts a torque on the particle changing its angular momentum. This change in angular momentum is due to the change of magnetic field and can be calculated usings Stoke's theorem

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{S} = \int \vec{E} \cdot d\vec{s} = E2\pi r = -\int \vec{B} \cdot \vec{S} = -\dot{\Phi} \tag{9}$$

where $d\vec{S}$ and $d\vec{s}$ are the infitesimal elements of the surface and its enclosing curve, respectively. This leads to

$$\dot{l_z} = qEr = -\frac{q\dot{\Phi}}{2\pi} = -\frac{C\Phi\dot{\Phi}}{2\pi} \tag{10}$$

Thus, the change in the angular momentum associated with applying magnetic flux to the system is

$$\Delta l_z = -\frac{c\Phi^2}{4\pi} = \frac{q\Phi}{4\pi} \tag{11}$$

If the particle's angular momentum in the absence of flux is zero and the dimensions of the solenoid and the particle's orbit are limited toward zero, then the system may be treated as a single object with spin determined by the magnetic flux passing through the solenoid.

The statistical properties of this model for anyons can be described as an extension of the properties of a two-anyon system. Such a system follows the Hamiltonian

$$H = \frac{(\vec{p_1} - q\vec{A_1})^2}{2m} + \frac{(\vec{p_2} - q\vec{A_2})^2}{2m} \tag{12}$$

where $\vec{p_i}$ are the momenta of the anyons and $\vec{A_i}$ are the vector potentials at the position of each anyon given by

$$\vec{A_i} = \pm\frac{\Phi}{2\pi}\hat{z} \times \frac{\vec{r}}{r^2} \tag{13}$$

where $\vec{r} = \vec{r_1} - \vec{r_2}$. In terms of the center of mass coordinate $\vec{P} = \vec{p_1} + \vec{p_2}$ and relative coordinate $\vec{p} = (\vec{p_1} - \vec{p_2})/2$, the Hamiltonian is

$$H = \frac{P^2}{4m} + \frac{(\vec{p} - q\vec{A_{rel}})^2}{m} \tag{14}$$

The first term in the Hamiltonian describes the motion of the center of mass while the second term describes a system where a particle of mass m/2 is orbiting a flux Φ. The particle is spinless giving the boundary conditions

$$\Psi(r, \theta + \pi) = \Psi(r, \theta) \tag{15}$$

Making use of the gauge transformation

$$\vec{A'} \to \vec{A} - \nabla\Lambda \tag{16}$$

where $\Lambda = \frac{\phi\theta}{2\pi}$, allows the Hamiltonian to be written as

$$H = \frac{P^2}{4m} + \frac{p^2}{m} \tag{17}$$

in the gauge where $\vec{A'} = 0$. This Hamiltonian is equivalent to the Hamiltonian of two free particles. The boundary conditions in this choice of gauge are also transformed, picking up a phase factor

$$\Psi'(r, \theta + \pi) = exp(-iq\Lambda)\Psi(r, \theta) = exp(-\frac{iq\Phi\theta}{2\pi})\Psi(r, \theta) \tag{18}$$

Then using the original boundary conditions and calculating $\Psi(-\vec{r})$ gives

$$\Psi'(r, \theta + \pi) = exp(-\frac{iq\Phi}{2})\Psi'(r, \theta) \tag{19}$$

This proves that the wavefunction describing two anyons is multiplied by a phase factor when the particles are interchanged. The phase factor describing the resulting fractional statistics, also known as anyonic statistics is

$$\alpha = \frac{q\Phi}{2} \tag{20}$$

Classically, the amount of magnetic flux produced by the solenoid is an independent parameter in this system so the phase factor and the corresponding statistics are arbitrary. [5]

C. Landau Levels

Another important concept in the explanation of the quantum Hall effect is Landau levels. Consider an electron confined to the x-y plane in the presence of a uniform magnetic field in the z-direction. The Hamiltonian is given by

$$H = \frac{1}{2m}(\vec{p} + \frac{e\vec{A}}{c})^2 \tag{21}$$

where $\vec{A}$ is the vector potential related to the applied magnetic field by the Maxwell relation $\nabla \times \vec{A} = \vec{B}$. The electron will follow Schrodinger's equation $H\Psi = E\Psi$ which is invariant under the Landau gauge transformation:

$$\vec{A}(x, y, z) = B(-y, 0, 0) \tag{22}$$

In the Landau gauge, the Hamiltonian can be written in terms of dimensionless quantities

$$\begin{aligned} y' &= \frac{y}{l} - lk_x \\ p'_y &= \frac{lp_y}{\hbar} \end{aligned} \tag{23}$$

where k_x is the x-component of the particle's wavevector and l is the magnetic length $l = \sqrt{\frac{\hbar c}{eB}}$. The Hamiltonian is then

$$H = \hbar\omega_c[\frac{1}{2}y'^2 + \frac{1}{2}(p'_y)^2] \tag{24}$$

where ω_c is the cyclotron frequency $\omega_c = \frac{eB}{mc}$. This is the Hamiltonian for the familiar one-dimensional harmonic oscillator with energy eigenvalues

$$E_n = (n + \frac{1}{2})\hbar\omega_c \tag{25}$$

where n=0,1,2,... The levels corresponding to each n are called Landau levels, and the correspondeing eigenvectors η are given by

$$\eta_{n,k_x}(\vec{r}) = [\pi 2^{2n}(n!)^2]^{\frac{1}{4}} exp[ik_x x - \frac{1}{2}(\frac{y}{l} - lk_x)^2] H_n(\frac{y}{l} - lk_x) \tag{26}$$

where H_n are Hermite polynomials. Some interesting properties of Lanau levels are that the energy of the level does not depend on k_x while the y position does depend on k_x. Because the energy is not dependent on k_x, the eigenstates with different values of k_x in a given Landau level are degenerate.

D. Filling Factor

Another important feature in describing the quantum Hall effect is the filling factor. In the Landau gauge described above, the electron orbital corresponding to a given n value is localized at $y = k_x l^2$ according to [23]. If the electron is confined to a space of length L_x in the x-direction with the periodic boudary conditions

$$e^{ik_x(x+L_x)} = e^{ik_x x} \tag{27}$$

then the allowed values of k_x and n_x are

$$k_x = 2\pi\frac{n_x}{L_x} \rightarrow n_x = \frac{L_x k_x}{2\pi} \tag{28}$$

where n_x is the corresponding Landau level quantum number. Counting the number of states in a region of $L_x L_y$ defined by $y = 0$ and $y = L_y$ (For simplicity, the state at $y = 0$ has $n_x = 0$ and the state $y = L_y$ corresponds to the wavevector $k_x = L_y/l^2$.) gives the total number of states in this region, N_x.

$$N_x = \frac{L_x L_y}{2\pi l^2} \tag{29}$$

The degeneracy of the Landau states, G, is then

$$G = \frac{N_x}{L_x L_y} = \frac{1}{2\pi l^2} = \frac{B}{\phi_0} \tag{30}$$

where is ϕ_0 is the flux quantum $\phi_0 = \frac{hc}{e}$. Therefore, there is one state per flux quantum in each Landau level. The filling factor, ν, is the number of occupied Landau levels for electrons in a given magnetic field. It is defined as

$$\nu = \frac{\rho}{G} = 2\pi l^2 \rho = \frac{\rho}{B/\phi_0} \tag{31}$$

where ρ is the two dimensional density of electrons. Hence, the number of electrons that can exist in a given Landau level increases proportionally with the magnetic field strength so that as the magnetic field strength increases fewer and fewer Landau levels are occupied. Thus, the filling factor is a measure of both the applied magnetic field strength and the number of Landau levels that are occupied in a system.

III. THE HALL EFFECT

A. Classical Hall Effect

The Hall effect describes how current flowing through a sample will be effected by the application of a magnetic field. Consider a two-dimensional conducting plate in an applied electric field, $\vec{E}$. According to Ohm's law, the current flowing through the plate, I, is proportional to the applied voltage, V, and is inversely proportional to the resistance of the plate, R. This is equivalent to

$$\vec{J} = \sigma\vec{E} \tag{32}$$

where σ is the conductivity of the plate, and $\vec{J} = q\rho\vec{v}$ is the current density for particles of charge q and density ρ moving with a velocity $\vec{v}$. Ohm's law states that current will flow in the same direction as the applied electric field.

In 1879, Edwin Hall discovered that in the presence of an applied magnetic field, the current in the plate wil actuallyl flow in a direction perpendicular to the applied electric field. Consider the conducting plate in the presence of an applied magnetic field perpendicular to the x-y plane, $\vec{B} = B\hat{z}$. Then the charges flowing through the plate are subject to the Lorenz force:

$$F_{Lorenz} = q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}) \tag{33}$$

If the applied electric field is in the y-direction, $\vec{E} = E\hat{y}$, then the particle will move with velocity $\vec{v}$ given by

$$q\vec{E} = -q\vec{v} \times \vec{B} \rightarrow \vec{v} = c\frac{E}{B}\hat{x} \tag{34}$$

Then from Ohm's law, the conductivity of the plate, or Hall conductivity, is given by

$$\sigma_H = \frac{\vec{J}}{\vec{E}} = \frac{q\rho\vec{v}}{E_y} = \frac{\rho qc}{B} \tag{35}$$

Similarly, the Hall resistance $R_H = 1/\sigma$ is defined as:

$$R_H = \frac{B}{\rho qc} \tag{36}$$

The Hall effect has been confirmed by countless experiments and is used routinely in solid-state physics to determine the density of charge carriers ρ through measurements of a system's Hall resistance.

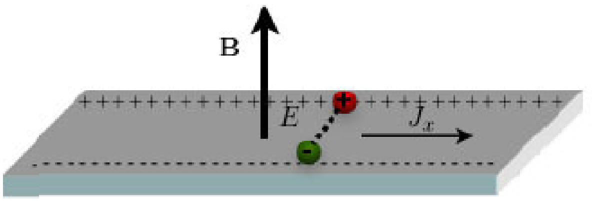

Figure 1: Physical picture of classical quantum Hall effect

B. Quantum Hall Effect - Integer and Fractional

The integral quantum Hall effect (IQHE) was discovered by Klaus von Klitzing in 1980. Von Klitzing was studying the Hall effect of two-dimensional electrons in silicon MOSFET(metal oxide-semiconductor field-effect transistor). He found that at low temperatures and high magnetic field, the Hall resistance of the system did not vary linearly with the strength of the magnetic field as predicted by the classical Hall effect. The plot of the resistivity as a function of magnetic field strength exhibited many plateaus which indicated that the Hall resistance is quantized.

In a two-dimensional system, the density of electrons,ρ, can be written as:

$$\rho = \nu\frac{B}{\phi_0} \tag{37}$$

Then, the classical Hall resistance with q=e is

$$R_H = \frac{B}{\rho ec} = \frac{B}{\frac{\nu Bec}{\phi_0}} = \frac{h}{\nu e^2} \tag{38}$$

Therefore, the Hall resistance is quantized in units of $\frac{h}{e^2}$ and is inversely proportional to the filling factor ν.

C. Integer Quantum Hall Effect

The Integer Quantum Hall Effect (IQHE) refers to the scenario where the filling factor, ν, has an integer value. Integer values of the filling factor describes a system of non-interacting electrons where the highest Landau level is completely filled. Once the Landau level is completely filled, a gap exists requiring a finite amount of energy to reach the next degenerate Landau level. However, impurities in the sample create localized potentials that can trap electrons in localized states. Therefore, if the filling is changed slightly, the extra electrons fill the localized states and do not contribute to the current. Thus, in regions where the filling factor has an integer value, there is a plateau in the plot of resistivity vs. magnetic field strength where the longitudinal resistance of the sample disappears.

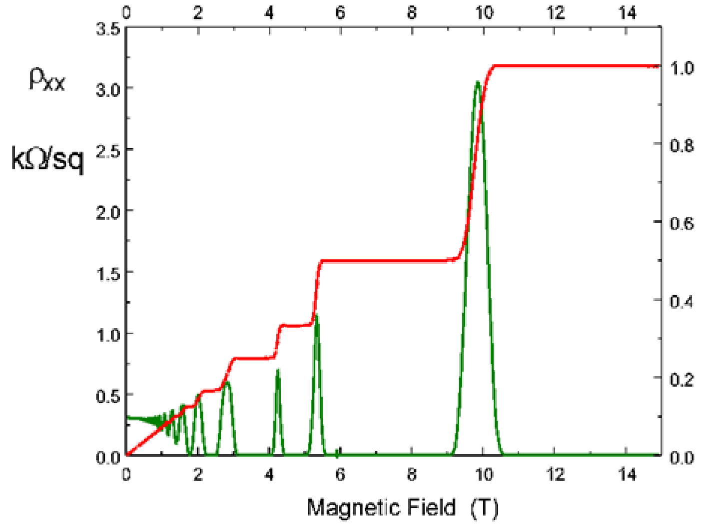

Figure 2: Plot of Hall resistivity and longitudinal resistivity vs. magnetic field strength exhibiting the integer quantum Hall effect

D. Fractional Quantum Hall Effect

In 1982, the fractional quantum Hall effect (FQHE) was discovered by Horst Stormer and Dan Tsui. By repeating von Klitzing's earlier experiments with cleaner samples in higher magnetic fields, they found that the plateaus in the plot of resistivity vs. magnetic field strength also occured at some fractional values of the filling factor. Because fractional values of the filling factor refers to partially filled Landau states, the plateaus could only be explained in terms of interacting particles. The Hamiltonian for N interacting electrons can be written as:

$$H = \Sigma_i^N \frac{(\vec{p_i} - eA(\vec{x_i}))^2}{2m} + \Sigma_{i<j}^N \frac{e^2}{\vec{x_i} - \vec{x_j}} \tag{39}$$

For the FQHE, the electron-electron interaction term is dominant and the system is strongly correlated. In 1983, Robert Laughlin introduced an ansatz wavefunction for filling factors, $\nu = 1/m$ where $m = (2j+1)$.

$$\Psi_m = \Pi_{i<j}(z_i - z_j)^m e^{-\Sigma_i |z_i|^2/4l^2} \tag{40}$$

where z_j are the complex electron coordinates in the plane, $z_j = x_j + iy_j$. This wavefunction is known as the Laughlin wavefunction. The Laughlin wavefunction gives a very uniform distribution of the electrons and minimizes the Coloumb energy of the system. The wavefunction is the lowest energy state for the system and is not degenerate. The Laughlin wavefunction describes a stable electronic ground state with high correlation for fractional values of the filling factor. Small deviations from these filling factors result in the excitation of quasi-particles that carry fractional charge and obey fractional statistics, i.e. anyons. As in the IQHE, these quasi-particles get trapped by impurities in real samples and do not contribute to the current. Therefore, the resistivity of the sample does not change until the filling factor reaches the next stable value.

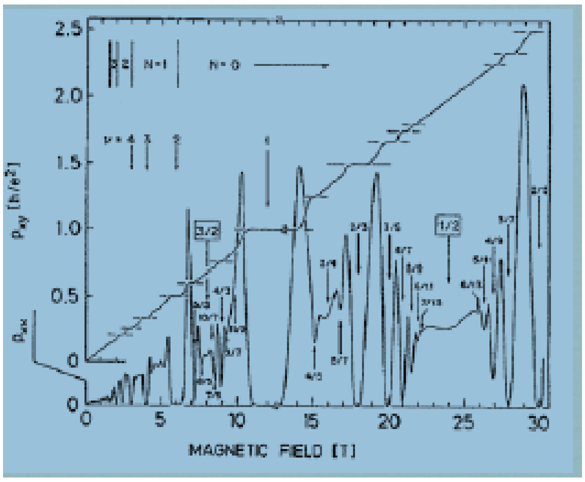

Figure 3: Plot of Hall resistivity vs. magnetic field strength exhibiting fractional quantum Hall effect

IV. THE QUANTUM HALL EFFECT IN GRAPHENE

Graphene is a single layer of carbon atoms in a two-dimensional hexagonal lattice. The carbon atoms bond to one another via covalent bonds leaving one 2p electron per carbon atom unbonded. The result is that the Fermi surface of graphene is characterized by six double cones. In the absence of applied fields, the Fermi level is situated at the connection points of these cones. Since the density of electrons is zero at the Fermi level, the electrical conductivity of graphene is very low. However, the application of an external electric field can change the Fermi level causing graphene to behave as a semi-conductor. In this case, near the Fermi level the dispersion relation for electrons is linear and the electrons behave as though they have zero effective mass (Dirac fermions). Because graphene exhibits this behavior even at room temperature, it is observed to exhibit both the integer and fractional quantum Hall effects.

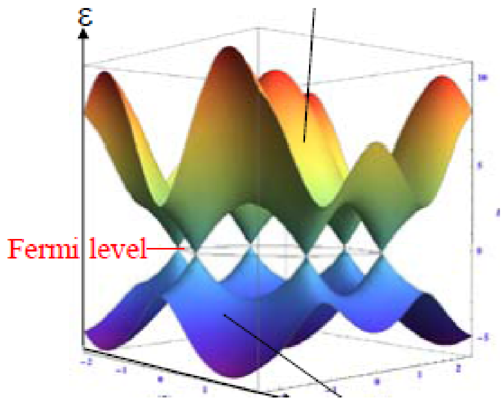

Figure 4: Energy level diagram for graphene showing the Fermi level in the absence of any applied fields

The existence of the QHE at such high temperatures in graphene is due to the large energy gaps characteristic of Dirac fermions. These energies are given by:

$$E_n = v_f \sqrt{|2ne\hbar B|} \tag{41}$$

where v_f is the Fermi velocity ($10^6 m/s$) and n is the Landau level quantum number. In a strong magnetic field (B=45T), the energy level spacing is 2800K. Graphene has a large concentration of charge carriers which keeps the lowest Landau level completely populated at high magnetic fields. Therefore, any carriers above the lowest Landau level will not be able to overcome the energy gap, and the quantum Hall effect is observed. The quantum Hall effect has been observed for the integer values of the filling factor $\nu = 0, \pm 1, \pm 2, \pm 6, \pm 10, ...$ as well as the fractional filling factors $\nu = \frac{1}{3}, \frac{2}{3}, and \frac{2}{5}$. Because graphene exhibits both the integer and fractional quantum Hall effects, it is ideal for studying QHE, and may be proven useful in the development of quantum computers. [6][7][8][9]

[1] K. von Klitzing, et al. Phys. Rev. Lett. **45**, 494 (1980).
[2] G. Yang. Class Lectures, December 2010.
[3] J. K. Jain. Composite Fermions. Cambridge University Press, Cambridge, NY, 2007.
[4] K.I. Bolotin, et al. Nature **462**, 196-199 (2009).
[5] F. Wilczek. Phys. Rev. Lett. **48**, 1144 (1982).
[6] K.S. Novoselov, et al. Natue **438**, 201-204 (2005).
[7] C. Toke, et al. Phys. Rev. B. **74**, 235417 (2006).
[8] K.S. Novoselov, et al. Science **315**, 1379 (2007).
[9] Z. Jiang, et al. Pys. Rev. Lett. **99**, 106802 (2007).