Neutron Degeneracy and Neutron Stars

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May 16, 2010

Abstract

Gravity never stops attracting matter to other massive objects; its pull can bring gas together from across space to form stars and is also responsible for the stars demise. Neutrons in a collapsing star will provide a quantum mechanically derived pressure to maintain the star in equilibrium. Degeneracy of energy states of the neutrons, explained by quantum mechanics as a byproduct of a predetermined way in which the particles can occupy the states, will prevent the collapse of stars with the requisite mass into black holes.

1. Introduction

Violent deaths on Earth, more often than not lost in the flow of history, have also been catalysts for the formation of new life as well as turning points in human history. Significance in such deaths during Earth's history, from the incineration of the dinosaurs to the crucifixion of Jesus, pales in comparison to that associated with the death of the first stars ever formed, whose demise populated the universe with the materials necessary for the emergence of organic life. Stars all begin to die as the self cannibalization of their bodies, necessary as a fuel source for the nuclear fusion processes at their core, becomes a fruitless search for fusible elements. Once a star can no longer fuse enough fuel to counteract the same gravitational force that powered its ignition, it will go through several stages of death dictated solely by how much mass the star contains. Total death for a star of sufficient mass involves a transformation into a black hole, a metamorphosis preceded by a stage during which the remnant core is composed entirely of neutrons. Extreme pressure caused by gravity's take over of the star's life is again the cause of a counter-insurgent pressure brought forth by the quantum mechanical effects that dominate the behavior of the neutrons.

Neutron stars represent a class of stars in a specific mass range that, through the degeneracy pressure exerted by the neutrons, will offer enough resistance to balance gravity's death grip. Neutron creation in stars that have collapsed beyond the white dwarf stage is due to the interaction of electrons and protons, modeled by the inverse beta decay reaction $p+e+1.36\text{MeV}\Leftrightarrow n+\bar{v}_e$. Although neutrons formed in this manner are usually unstable and decay in a matter of minutes, the white dwarf stage saw a degeneracy in the electron gas that resulted in all available electron states being filled. As no electrons can form with energy below or equal to 1.36Mev, the neutrons will not decay via a beta reaction to form an electron and a proton. Along with neutrons, the inverse beta decay reaction also forms neutrinos. Enough neutrons are formed so that the neutrinos' flux, coupled with the tightly packed matter in the surrounding star, is enough to begin counteracting the gravitational collapse of the star. As the neutrons become degenerate and occupy higher and higher energy states, they exert the degeneracy pressure that if produced in a small enough star will balance the effects of gravity. In stars whose mass prevents the equilibrium between degeneracy pressure and gravity, the continuous crushing of gravity leads inexorably to the next stage in star death, a black hole.

In order to quantify neutron degeneracy pressure, a quantum mechanical approach to a system of neutrons inside a three dimensional box in which the potential is zero, described in section 2, will be used to produce a model of the neutrons in a star. Section 3 will detail the gravitational collapse of a star and will contain the derivation of an expression of energy, which will be used to calculate the pressure due to gravity in the star. Section 4 will use the expression obtained from section 2 for the total energy contained in the degenerate system, whose relevance is predicated by the fact that the pressure exerted by a system on its surroundings can be calculated by $-\frac{dE}{dV}$. Of interest is the radius of a neutron star. Calculated in section 5, this radius will be the limit at which neutron degeneracy successfully balances gravity to produce a stable star.

2. Free Neutron Gas Degeneracy

Neutrons constrained to a box of length L with volume $V=L^3$ constitute a system that can be studied by making the simplifying assumptions that neutrons cannot exist outside the box, experience no neutron-neutron interactions and are free of any other potential. Such assumptions lead to the following form of the Schrödinger equation:

$$\frac{-\hbar^2}{2\mathrm{m}}\psi(x,y,z)=E\psi(x,y,z) \qquad (1)$$

The condition of restriction to inside the box lead to the boundary conditions:

$$\psi(0,y,z)=\psi(L,y,z)=0$$

$$\psi(x,0,z)=\psi(x,L,z)=0 \qquad (2)$$

$$\psi(x,y,0)=\psi(x,y,L)=0$$

Solutions of the Schrödinger equation under these conditions are very much the three dimensional generalization of the solutions for the case of a free particle in square well potential, with the eigenstates and corresponding energy eigenvalues given by:

$$\psi_{n_1n_2n_3}=\left(\frac{2}{L}\right)^{\frac{3}{2}}\sin\left(\frac{\pi n_1}{L}x\right)\sin\left(\frac{\pi n_2}{L}y\right)\sin\left(\frac{\pi n_3}{L}z\right) \qquad (3)$$

$$E_{n_1n_2n_3}=\frac{\pi^2\hbar^2}{2\mathrm{mL}^2}(n_1^{\ 2}+n_2^{\ 2}+n_3^{\ 2}) \qquad (4)$$

Neutrons with the highest allowed energy in this system will be at the Fermi Energy E_f , which can be obtained by noting that each energy eigenstate is associated with a set of three integers, namely n_1 , n_2 and n_3 . Each triplet of these integers (n_1,n_2,n_3) can represent a point corresponding to a specific energy in a three dimensional space in which it serves to specify a maximum radius $R^2=n_1^{\ 2}+n_2^{\ 2}+n_3^{\ 2}$ at the Fermi energy, which becomes:

$$E_f=\frac{\pi^2\hbar^2}{2\mathrm{mL}^2}R^2 \qquad (5)$$

Since n_1 , n_2 and n_3 are positive integers, all possible triplets will specify locations in the first octant of the three dimensional space. Each triplet specifies an energy level allowed to the neutrons, which, by virtue of being fermions have a spin of $\frac{1}{2}$, meaning two neutrons can occupy a single eigenstate at its corresponding energy, as long as their spins are opposite. Counting the total number of neutrons in the box then is a problem of counting the number of energy eigenvalue points in the first octant. Accounting for the spin gives:

$$N = 2\frac{1}{8}\frac{4}{3}\pi R^3 = \frac{1}{3}\pi R^3 \tag{6}$$

Equations 5 and 6 combine to give:

$$R = \left(\frac{2\mathrm{mL}^2}{\pi^2\hbar^2}E_f\right)^{\frac{1}{2}} \tag{7}$$

Equations 6 and 7 combine to give:

$$N = \frac{\pi}{3}L^3\left(\frac{2\mathrm{m}\,E_f}{\pi^2\hbar^2}\right)^{\frac{3}{2}} \tag{8}$$

Equation 8 leads to an expression of the Fermi energy:

$$E_f = \frac{\pi^2\hbar^2}{2\mathrm{m}}\left(\frac{3\mathrm{N}}{\pi L^3}\right)^{\frac{2}{3}} \tag{9}$$

The number of neutrons, given by N, divided by the volume of the box is the number density of the neutrons:

$$n_n = \frac{N}{L^3} = \frac{N}{V} \tag{10}$$

An equivalent expression of the Fermi energy that is independent of the physical dimensions of the box emerges when equations 9 and 10 are combined to give:

$$E_f = \frac{\pi^2\hbar^2}{2\mathrm{m}}\left(\frac{3}{\pi}n_n\right)^{\frac{2}{3}} \tag{11}$$

An integral can be used to compute the combined energy of all the states below the state that has the Fermi energy by integrating over the first octant in the three dimensional space:

$$E_{total} = \frac{\pi^2\hbar^2}{m L^2 8}\int_0^R n^4\,dn = \frac{\pi^3\hbar^2}{10\,m\,L^2}R^5 \tag{12}$$

Equations 6 says that $R = \left(\frac{3\mathrm{N}}{\pi}\right)^{\frac{1}{3}}$ which when combined with the fact that $L = V^{\frac{1}{3}}$ and equation 12 gives:

$$E_{total} = \frac{\pi^3\hbar^2}{10\mathrm{m}}\left(\frac{3\mathrm{N}}{\pi}\right)^{\frac{5}{3}}V^{\frac{-2}{3}} \tag{13}$$

3. Gravitational Collapse

In order to stay a neutron star, the dying core needs to exert enough pressure through the neutron degeneracy to balance the gravitational force driving the collapse. Gravity acts to gather all material in a system to the center of mass. For a star of radius r, assumed uniform density ρ, and mass $m_{star}=4\frac{\pi}{3}\rho r^3$, the work dW gravity does to add mass $dm=4\pi\rho^3 dr$ to the star can written in terms of its radius, the number of neutrons and the mass in the following way:

$$dW=-G\frac{m_{star}}{r}dm \quad (14)$$

Solving and plugging in gives the total energy needed to collapse the star:

$$E=\frac{-16\text{G}\rho^2\pi^2}{3}\int_0^R r^4 dr=\frac{-16\text{G}\rho^2\pi^2}{15}R^5 \quad (15)$$

In this case, the density given by ρ is the total mass of the neutrons divided by the volume of the star:

$$\rho=NM\frac{1}{4\frac{\pi}{3}R^3} \quad (16)$$

Combining equations 15 and 16 gives the total energy needed to form the star of mass M, radius R and containing N neutrons:

$$E=\frac{-3\text{G}N^2M^2}{5\text{R}} \quad (17)$$

As noted, the pressure of a system can be given by $-\frac{dE}{dV}$. The gravitational pressure is then:

$$P_{grav}=\frac{-dE_{grav}}{dV}=-\frac{1}{5}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}GM^2V^{-\frac{4}{3}} \quad (18)$$

4. Degeneracy Pressure

The neutron degeneracy pressure is given by applying the derivative on equation 13 in the same way as the gravitational pressure:

$$P_{degeneracy} = \frac{-d\,E_{degeneracy}}{dV} = \frac{\hbar^2 \pi^3}{15\,m_n} \left(\frac{3N}{\pi}\right)^{\frac{5}{3}} V^{\frac{-5}{3}} = \frac{\hbar^2 \pi^3}{15\,m_n} \left(\frac{3\,n_n}{\pi}\right)^{\frac{5}{3}} \qquad (19)$$

5. Neutron Star

Neutrons produced by the joining of an electron and proton, when formed in energy levels necessary to produce a gravity balancing degeneracy pressure, will result in an equilibrium between the gravitational and the degeneracy pressure. To find the radius of the star for which the equilibrium will be reached, the pressures in equations 18 and 19 can be equalized and solved for the volume V:

$$P_{grav} = P_{degeneracy} = \frac{1}{5} \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} G\,M^2\,V^{-\frac{4}{3}} = \frac{\hbar^2 \pi^3}{15\,m_n} \left(\frac{3N}{\pi}\right)^{\frac{5}{3}} V^{\frac{-5}{3}} \qquad (20)$$

Plugging in the expression $R = \left(\frac{3V}{4\pi}\right)^{1/3}$, the radius of a neutron star in terms of the number of neutrons present and their mass is:

$$R = \left(\frac{81\pi^{20}}{16}\right)^{\frac{1}{3}} \frac{\hbar^2}{G\,m_n^3} N^{-1/3} \qquad (21)$$

6. Conclusion

Outlined in this paper is an example of the connection between quantum mechanics and the force of gravity. In the transformation of a star from a fusion reactor to its stint as a neutron star, gravity was opposed in its habitual attractive tendencies by a pressure whose origin rests solely in quantum mechanics. Ultimately dependent on the amount of matter present in the star, its path can lead to the point of equilibrium between neutron degeneracy pressure and gravitational pressure. As gravity crushes the stellar material with no resistance from fusion reactions, the electrons and protons are brought close enough to fuse into neutrons, whose collective energy fills every available state up to the Fermi energy $E_f = \frac{\pi^2 \hbar^2}{2\text{m}} (\frac{3}{\pi} n_n)^{\frac{2}{3}}$. Fermi energy is of the same source as that from which neutron degeneracy pressure arises, namely the fermion nature of the neutron that allows two neutrons of the same energy but with different spin to occupy the same state. Gravity continues in its attempt to compress the star, which serves to impart the energy on the neutrons, causing the population of higher energy states that push out against the collapse. A star that is in equilibrium with gravity will have the theoretical radius $R = (\frac{81\pi^{20}}{16})^{\frac{1}{3}} \frac{\hbar^2}{G m_n^3} N^{-1/3}$, which is determined by the amount of neutrons present and their mass.

7. References

Griffiths, David J. Introduction to Quantum Mechanics. Upper Saddle River, NJ; Prentice Hall, 2005.