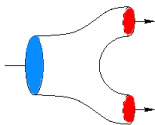
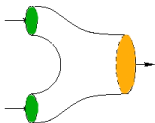


# The Topology of Chaos



**Robert Gilmore**

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**Physics & Topology Workshop**

**Drexel University, Philadelphia, PA 19104**

**Sept. 9, 2008**

## Outline

- 1 Overview
- 2 Experimental Challenge
- 3 Topology of Orbits
- 4 Topological Analysis Program
- 5 Basis Sets of Orbits
- 6 Bounding Tori
- 7 Covers and Images
- 8 Quantizing Chaos
- 9 Representation Theory of Strange Attractors
- 10 Summary

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**J. R. Tredicce**

**Can you explain my data?**

**I dare you to explain my data!**

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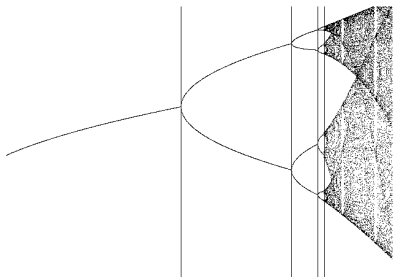
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## Where is Tredicce coming from?



**Feigenbaum:**

$$\alpha = 4.66920\ 16091\ \dots$$

$$\delta = -2.50290\ 78750\ \dots$$

# Experiment

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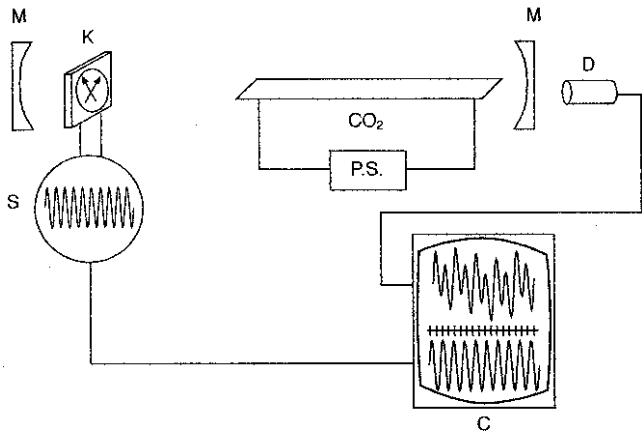
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## LSA Experimental Arrangement



## Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

# Result

**There is now a classification theory.**

- ① It is topological
- ② It has a hierarchy of 4 levels
- ③ Each is discrete
- ④ There is rigidity and degrees of freedom
- ⑤ It is applicable to  $R^3$  only — for now

## The 4 Levels of Structure

- **Basis Sets of Orbits**
- **Branched Manifolds**
- **Bounding Tori**
- **Extrinsic Embeddings**



# Organization

**LINKS OF PERIODIC ORBITS**

**organize**

**BOUNDING TORI**

**organize**

**BRANCHED MANIFOLDS**

**organize**

**LINKS OF PERIODIC ORBITS**

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# Experimental Schematic

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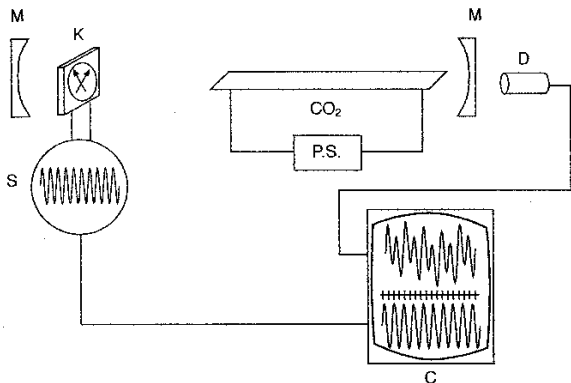
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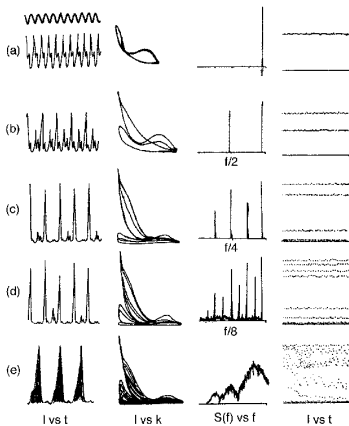
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## Laser Experimental Arrangement



# Experimental Motivation

## Oscilloscope Traces



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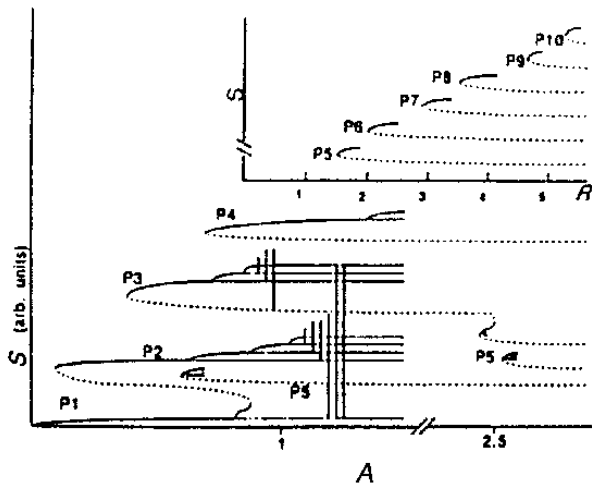
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# Results, Single Experiment

## Bifurcation Schematics



# Some Attractors

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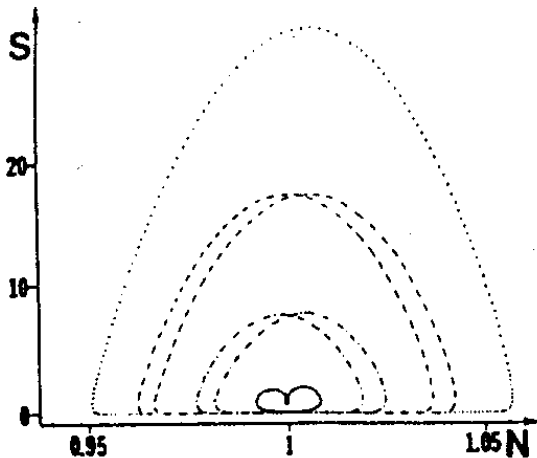
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## Coexisting Basins of Attraction



# Many Experiments

## Bifurcation Perestroikas

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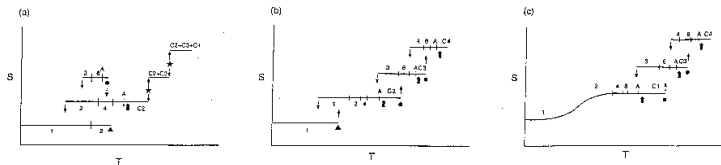
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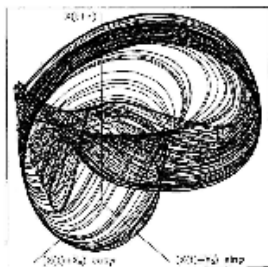
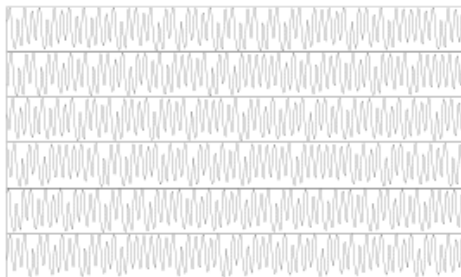
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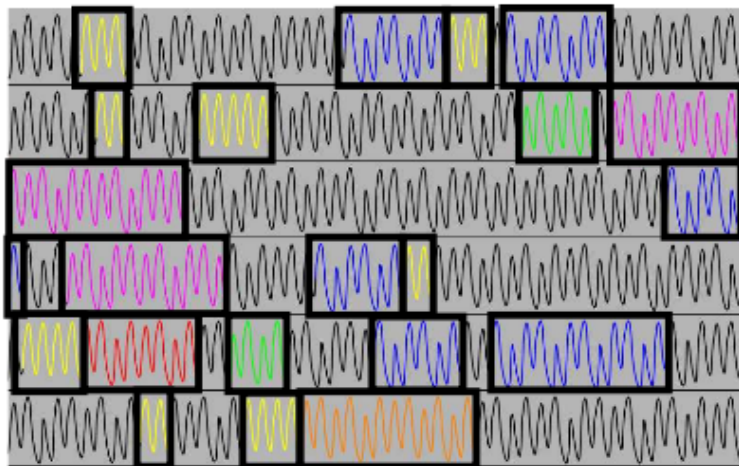


## Experimental Data: LSA



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## Experimental Data: LSA



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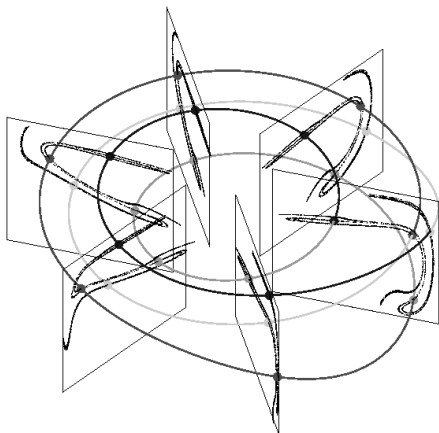
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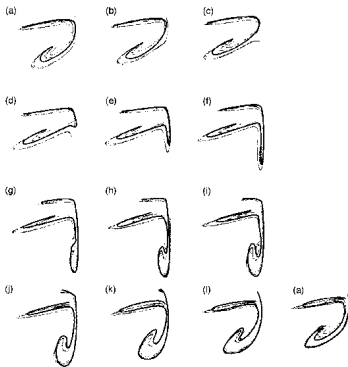
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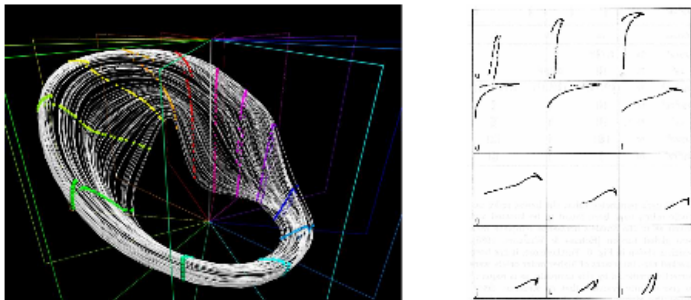
## Stretching & Squeezing in a Torus



## Rotating the Poincaré Section around the axis of the torus



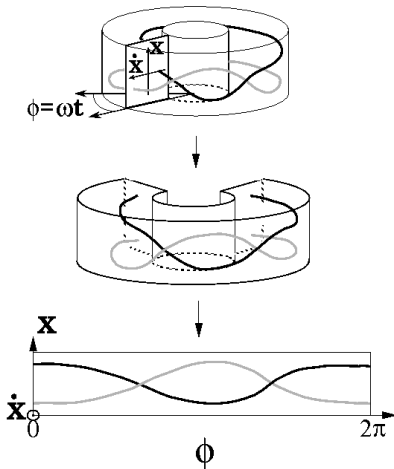
## Rotating the Poincaré Section around the axis of the torus



**Figure 2.** Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

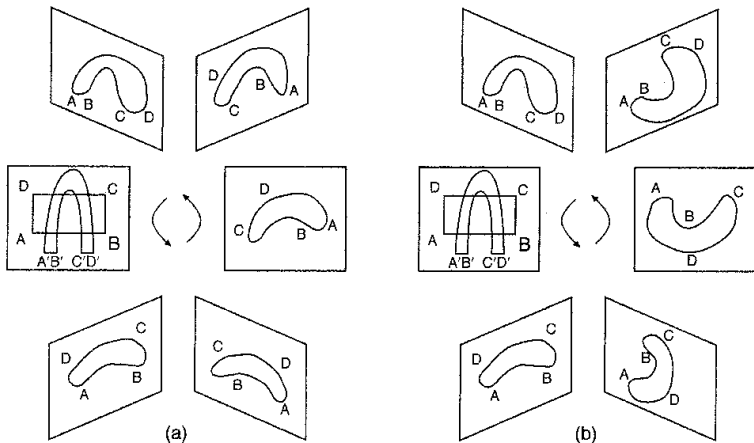
# Another Visualization

## Cutting Open a Torus



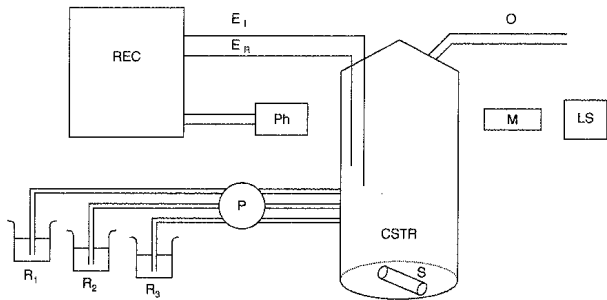
# Satisfying Boundary Conditions

## Global Torsion



## A Chemical Experiment

### The Belousov-Zhabotinskii Reaction



## Chaos

### Motion that is

- **Deterministic:**  $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

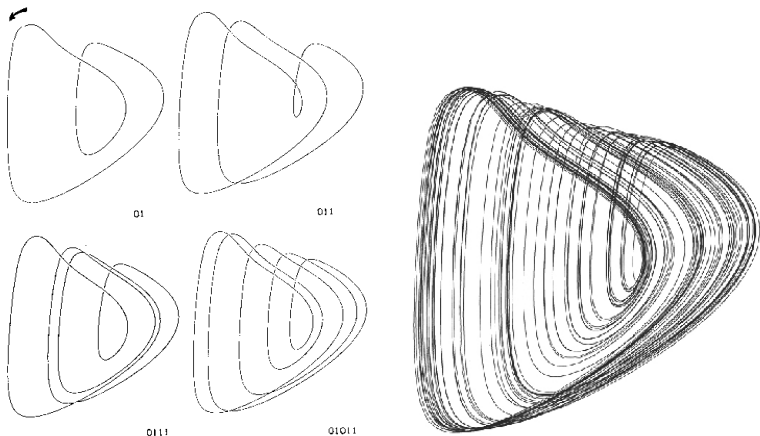
## Strange Attractor

The  $\Omega$  limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

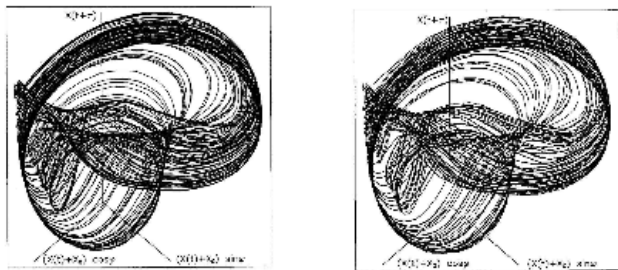


## UPOs Outline Strange attractors



BZ reaction

## UPOs Outline Strange attractors



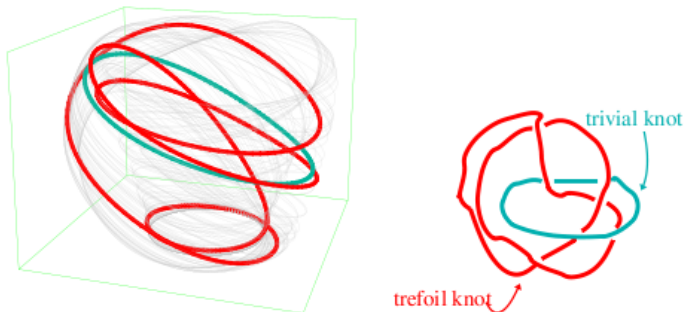
**Figure 5.** Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

## Organization of UPOs in $R^3$ : Gauss Linking Number

$$LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

# Interpretations of  $LN \simeq$  # Mathematicians in World

## Linking Number of Two UPOs



**Figure 6.** Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

# Evolution in Phase Space

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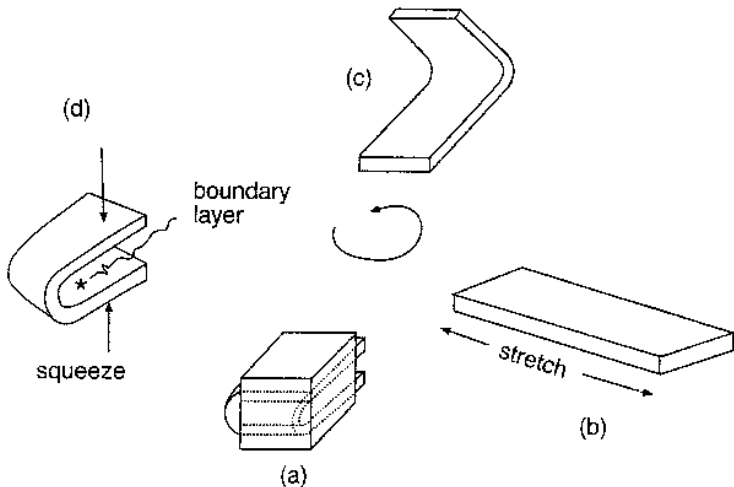
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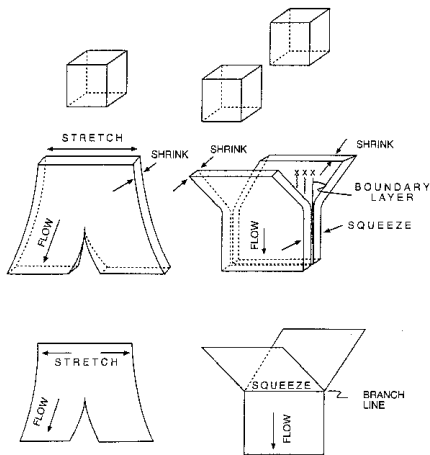
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## One Stretch-&-Squeeze Mechanism



# Motion of Blobs in Phase Space

## Stretching — Squeezing

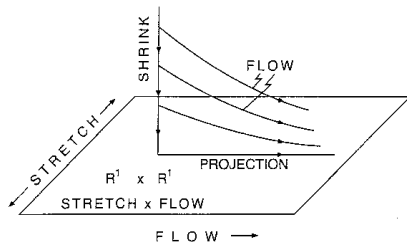


# Collapse Along the Stable Manifold

## Birman - Williams Projection

Identify  $x$  and  $y$  if

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| \rightarrow 0$$



## Birman - Williams Theorem

**If:**

**Then:**

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## Birman - Williams Theorem

**If:**                      **Certain Assumptions**

**Then:**

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## Birman - Williams Theorem

**If:**                      **Certain Assumptions**

**Then:**                    **Specific Conclusions**

## Assumptions, B-W Theorem

**A flow**  $\Phi_t(x)$

- on  $R^n$  is dissipative,  $n = 3$ , so that  $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$ .
- Generates a hyperbolic strange attractor  $SA$

IMPORTANT: The underlined assumptions can be relaxed.

## Conclusions, B-W Theorem

- **The projection maps the strange attractor  $\mathcal{SA}$  onto a 2-dimensional branched manifold  $\mathcal{BM}$  and the flow  $\Phi_t(x)$  on  $\mathcal{SA}$  to a semiflow  $\bar{\Phi}(x)_t$  on  $\mathcal{BM}$ .**
- **UPOs of  $\Phi_t(x)$  on  $\mathcal{SA}$  are in 1-1 correspondence with UPOs of  $\bar{\Phi}(x)_t$  on  $\mathcal{BM}$ . Moreover, every link of UPOs of  $(\Phi_t(x), \mathcal{SA})$  is isotopic to the correspond link of UPOs of  $(\bar{\Phi}(x)_t, \mathcal{BM})$ .**

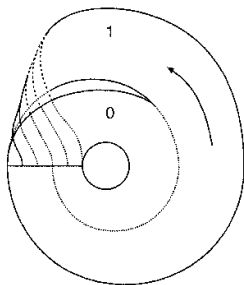
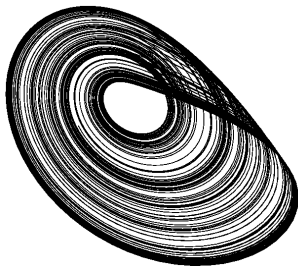
Remark: "One of the few theorems useful to experimentalists."

# A Very Common Mechanism

**Rössler:**

**Attractor**

**Branched Manifold**



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# A Mechanism with Symmetry

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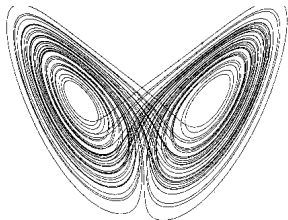
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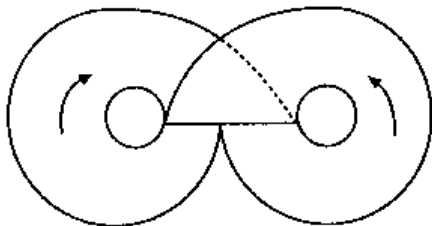
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## Lorenz:

### Attractor

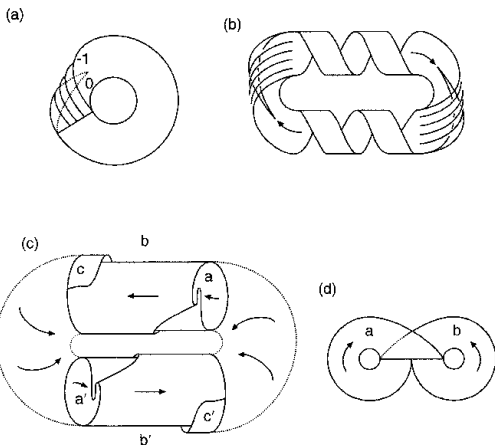


### Branched Manifold



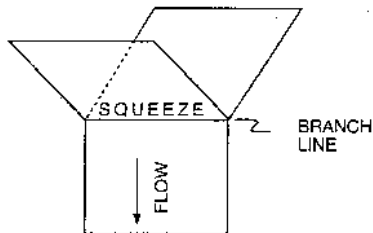
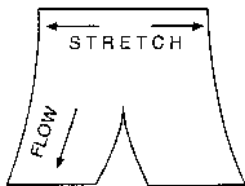
# Examples of Branched Manifolds

## Inequivalent Branched Manifolds



# Aufbau Princip for Branched Manifolds

Any branched manifold can be built up from stretching and squeezing units



subject to the conditions:

- Outputs to Inputs
- No Free Ends



## Rössler System

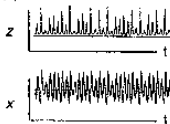
(a) Rössler Equations

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

(b)



(c)



(f)

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & +1 \end{pmatrix}$$

(e)



(f)



## Lorenz System

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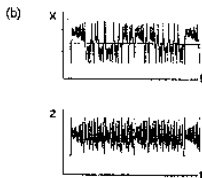
Experimental-

(a) Lorenz Equations

$$\frac{dx}{dt} = -\alpha x + \alpha y$$

$$\frac{dy}{dt} = \beta x - y - xz$$

$$\frac{dz}{dt} = -bz + xy$$

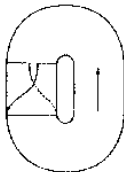


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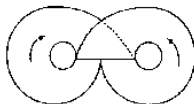
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

(e)



(d)



## Poincaré Smiles at $U$ s in $R^3$

- **Determine organization of UPOs**  $\Rightarrow$
- **Determine branched manifold**  $\Rightarrow$
- **Determine equivalence class of  $\mathcal{SA}$**

## Topological Analysis Program

**Locate Periodic Orbits**

**Create an Embedding**

**Determine Topological Invariants (LN)**

**Identify a Branched Manifold**

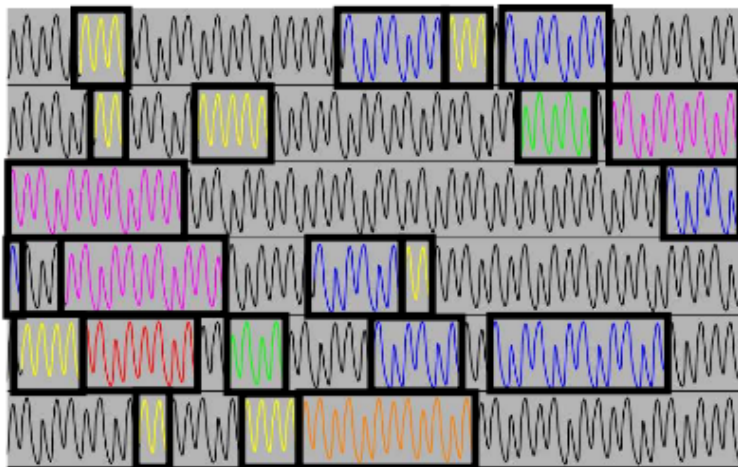
**Verify the Branched Manifold**

---

**Model the Dynamics**

**Validate the Model**

## Method of Close Returns



## Embeddings

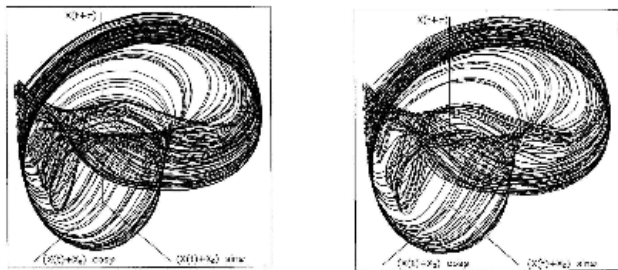
Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological<sup>†</sup>

None Good

We Demand a 3 Dimensional Embedding

## An Embedding and Periodic Orbits



**Figure 5.** Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

# Determine Topological Invariants

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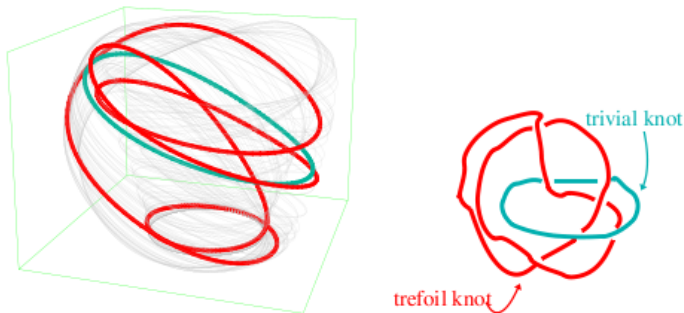
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## Linking Number of Orbit Pairs



**Figure 6.** Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

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# Determine Topological Invariants

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## Compute Table of Expt'l LN

**Table 7.2** Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data<sup>a</sup>

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

<sup>a</sup>All indices are negative.

# Determine Topological Invariants

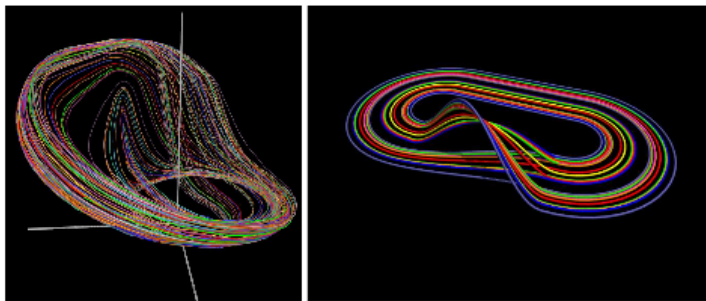
## Compare w. LN From Various $BM$

**Table 2.1** Linking numbers for orbits to period five in Smale horseshoe dynamics.

	$1^s$	$1^f$	$2_1$	$3^f$	$3^s$	$4_1$	$4_2^f$	$4_2^s$	$5_2^f$	$5_2^s$	$5_2^f$	$5_2^s$	$5_1^f$	$5_1^s$
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

# Determine Topological Invariants

## Guess Branched Manifold



**Figure 7.** “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

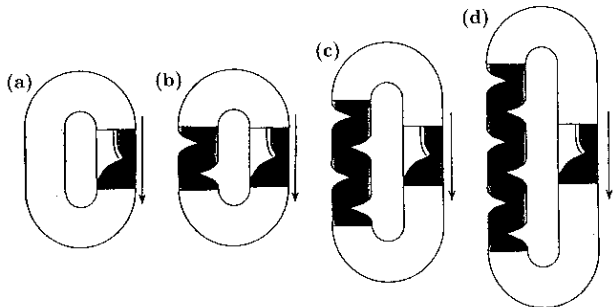
## Identification & ‘Confirmation’

- $\mathcal{BM}$  Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

# Determine Topological Invariants

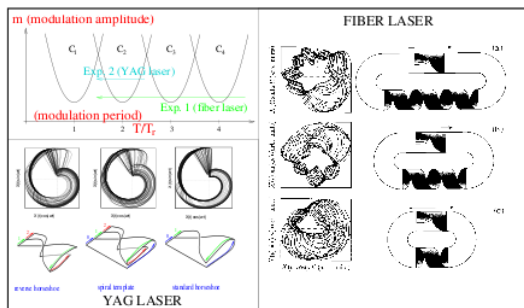
## What Do We Learn?

- $BM$  Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



# Perestroikas of Strange Attractors

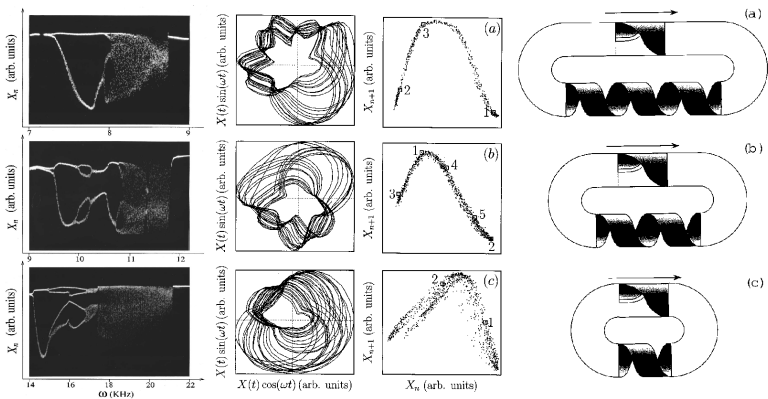
## Evolution Under Parameter Change



**Figure 11.** Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

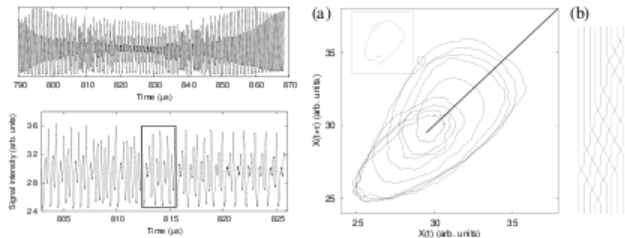
# Perestroikas of Strange Attractors

## Evolution Under Parameter Change



Lefranc - Cargese

## Analysis of Nonstationary Data



**Figure 16.** Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is  $h_T = 0.377$ , showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

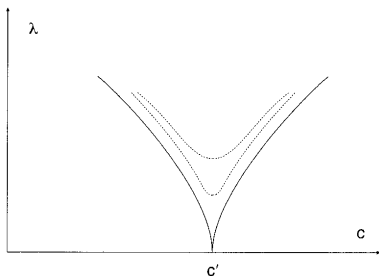


## Model the Dynamics

A hodgepodge of methods exist: # Methods  $\simeq$  # Physicists

## Validate the Model

Needed: Nonlinear analog of  $\chi^2$  test. OPPORTUNITY:  
Tests that depend on entrainment/synchronization.



# Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

# Orbits Can be “Pruned”

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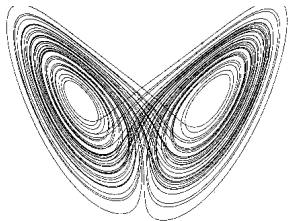
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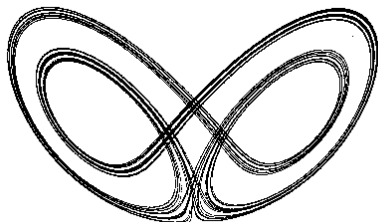
Experimental-  
02

Experimental-

## There Are Some Missing Orbits

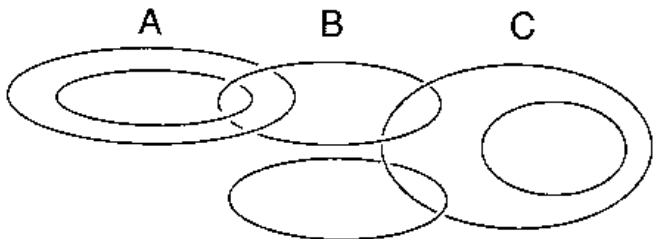


Lorenz



Shimizu-Morioka

## Orbit Forcing



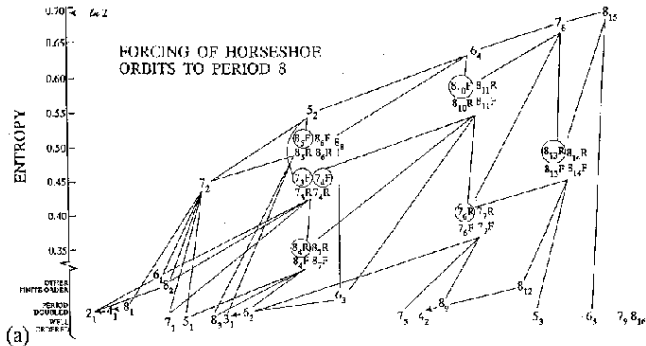
$$A \Rightarrow B$$

$$B \Rightarrow C$$

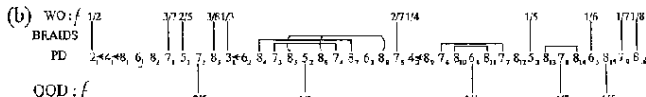
$$A \Rightarrow C$$

# An Ongoing Problem

## Forcing Diagram - Horseshoe



### U - SEQUENCE ORDER



## Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

## Constraints on Branched Manifolds

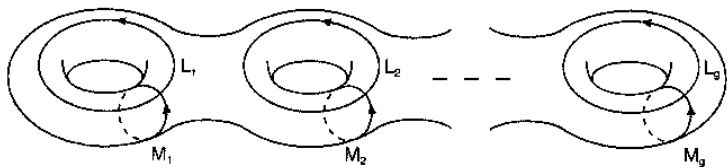
**“Inflate” a strange attractor**

**Union of  $\epsilon$  ball around each point**

**Boundary is surface of bounded 3D manifold**

**Torus that bounds strange attractor**

## Torus, Longitudes, Meridians





## Surface Singularities

**Flow field: three eigenvalues: +, 0, -**

**Vector field “perpendicular” to surface**

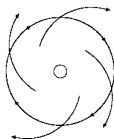
**Eigenvalues on surface at fixed point: +, -**

**All singularities are regular saddles**

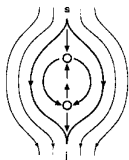
$$\sum_{s.p.} (-1)^{\text{index}} = \chi(S) = 2 - 2g$$

**# fixed points on surface = index =  $2g - 2$**

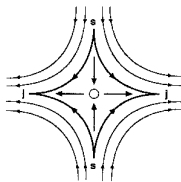
## Flow Near a Singularity



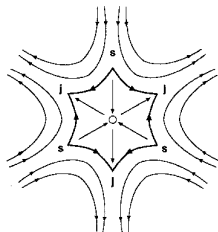
(a)



(b)



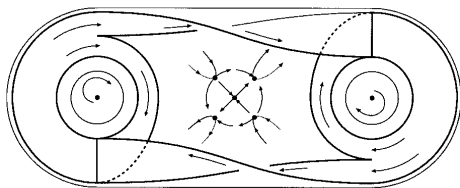
(c)



(d)

# Some Bounding Tori

## Torus Bounding Lorenz-like Flows



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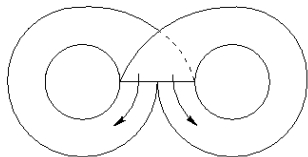
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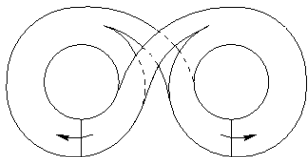
Experimental-  
02

Experimental-

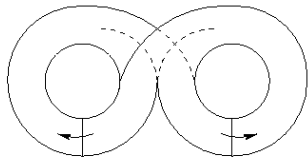
## Twisting the Lorenz Attractor



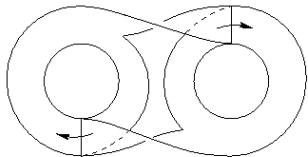
(a)



(c)



(b)



(d)

# Constraints Provided by Bounding Tori

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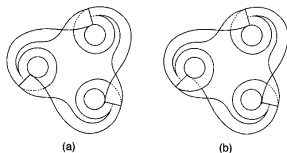
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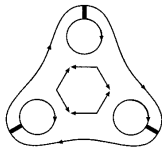
Experimental-

**Two possible branched manifolds  
in the torus with  $g=4$ .**



(a)

(b)



(c)

# Bounding Tori contain all known Strange Attractors

Tab.1. All known strange attractors of dimension  $d_L < 3$  are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g - 1$ Orbit
Rössler, Duffing, Burke and Shaw	$A_1$	1
Various Lasers, Gateau Roule	$A_1$	1
Neuron with Subthreshold Oscillations	$A_1$	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	$A_2$	$(12)^2$
Multispiral attractors	$A_n$	$(12^{n-1})^2$
$C_n$ Covers of Rössler	$C_n$	$1^n$
$C_2$ Cover of Lorenz <sup>(a)</sup>	$C_4$	$1^4$
$C_2$ Cover of Lorenz <sup>(b)</sup>	$A_8$	$(122)^2$
$C_n$ Cover of Lorenz <sup>(a)</sup>	$C_{2n}$	$1^{2n}$
$C_n$ Cover of Lorenz <sup>(b)</sup>	$P_{n+1}$	$(1n)^n$
$2 \rightarrow 1$ Image of Fig. 8 Branched Manifold	$A_8$	$(122)^2$
Fig. 8 Branched Manifold	$P_8$	$(14)^4$

<sup>(a)</sup> Rotation axis through origin.  
<sup>(b)</sup> Rotation axis through one focus.

## Labeling Bounding Tori

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**Poincaré section is disjoint union of  $g-1$  disks**

**Transition matrix sum of two  $g-1 \times g-1$  matrices**

**One is cyclic  $g-1 \times g-1$  matrix**

**Other represents union of cycles**

**Labeling via (permutation) group theory**

# Some Bounding Tori

## Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

$g$	$m$	$(p_1, p_2, \dots, p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11313133
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

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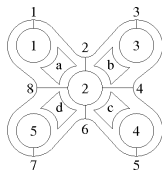
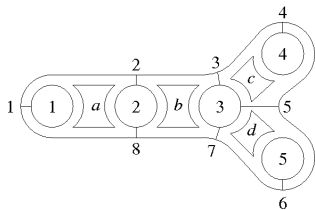
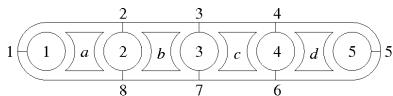
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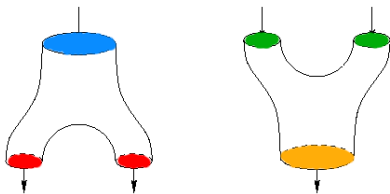


## Some Genus-9 Bounding Tori



# Aufbau Princip for Bounding Tori

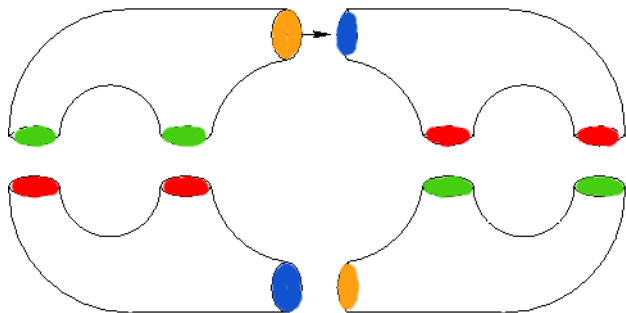
Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

# Aufbau Princip for Bounding Tori

## Application: Lorenz Dynamics, $g=3$



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## Construction of Poincaré Section

P. S. = Union 

# Components =  $g-1$

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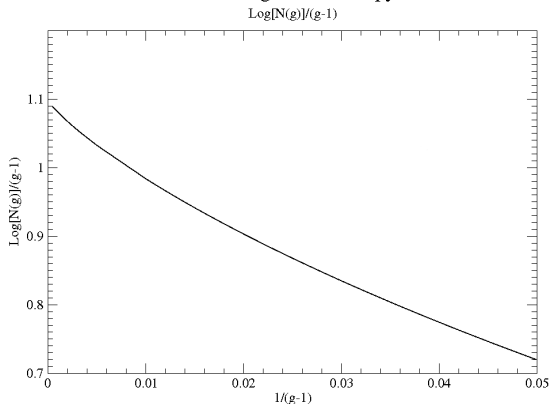
## The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus,  $g$ .

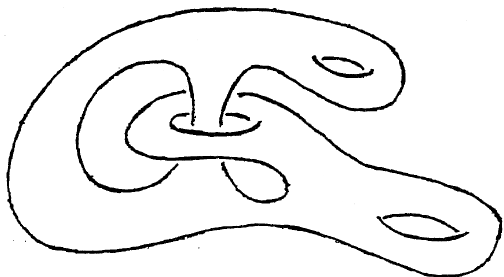
$g$	$N(g)$	$g$	$N(g)$	$g$	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

## The Growth is Exponential The Entropy is $\log 3$

Bounding Torus Entropy



## Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.  
Nightmare Numbers are Expected.

# Modding Out a Rotation Symmetry

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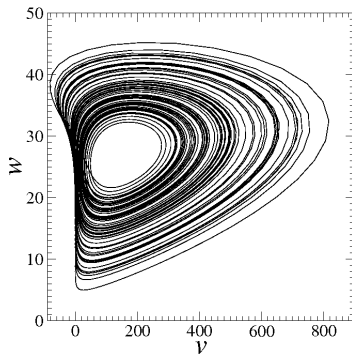
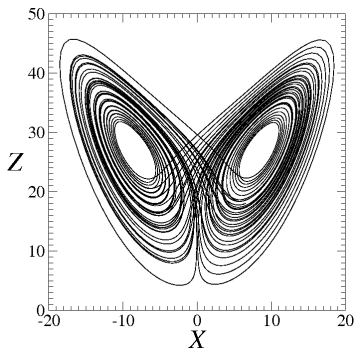
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# Modding Out a Rotation Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} (X + iY)^2 \\ \operatorname{Im} (X + iY)^2 \\ Z \end{pmatrix}$$





# Lorenz Attractor and Its Image

## The Topology of Chaos

Robert Gilmore

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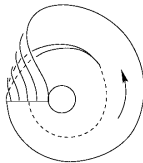
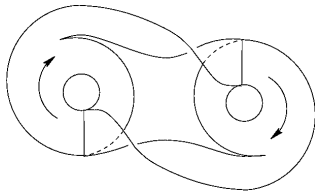
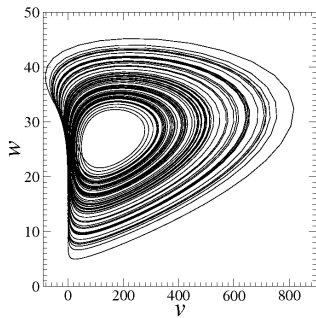
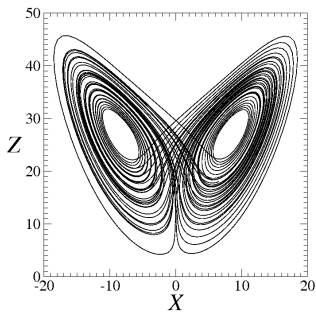
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# Lifting an Attractor: Cover-Image Relations

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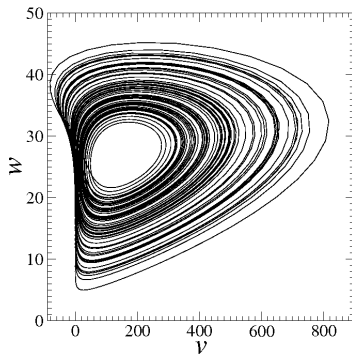
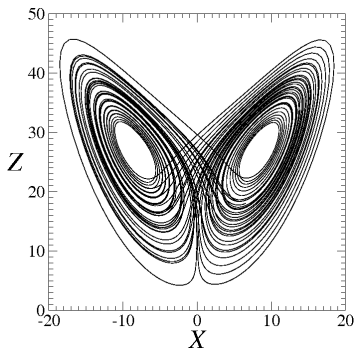
Experimental-  
01

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## Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \operatorname{Re} (X + iY)^2 \\ \operatorname{Im} (X + iY)^2 \\ Z \end{pmatrix}$$



# Cover-Image Related Branched Manifolds

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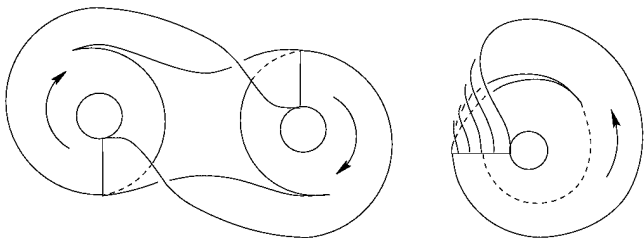
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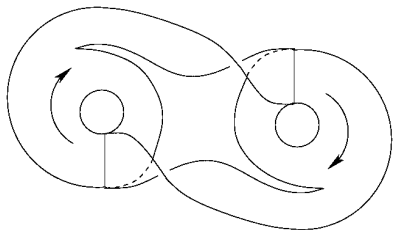
Experimental-  
02

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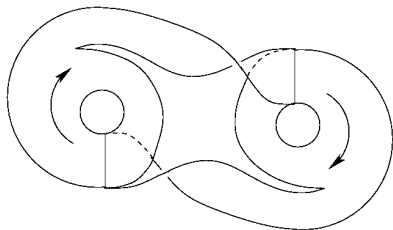
## Cover-Image Branched Manifolds



## Two Two-fold Lifts Different Symmetry

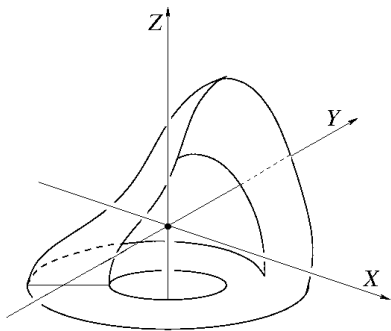


**Rotation  
Symmetry**

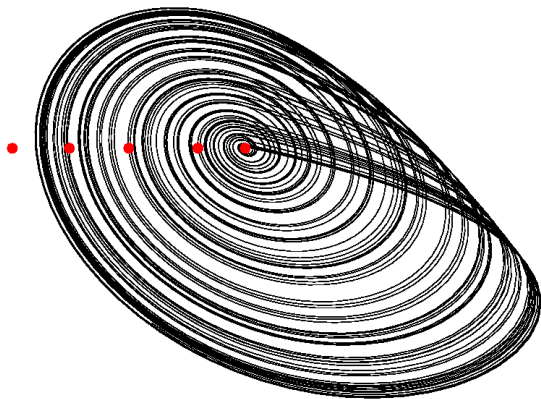


**Inversion  
Symmetry**

## Topological Index: Choose Group Choose Rotation Axis (Singular Set)



## Different Rotation Axes Produce Different (Nonisotopic) Lifts



# Nonisotopic Locally Diffeomorphic Lifts

## The Topology of Chaos

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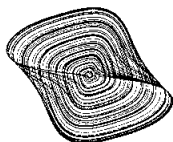
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(a)  $\mu = 0.0$



(c)  $\mu = -2.083$



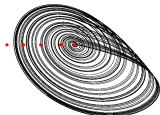
(e)  $\mu = -4.166$



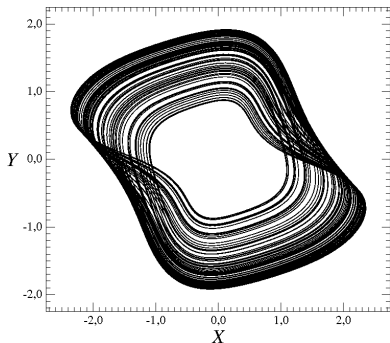
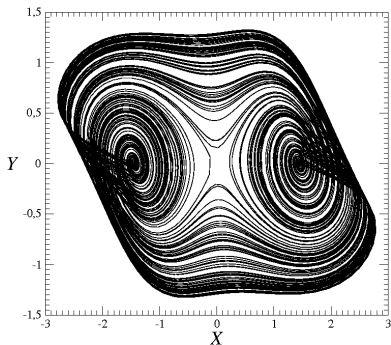
(b)  $\mu = -0.84548$



(d)  $\mu = -3.14674$

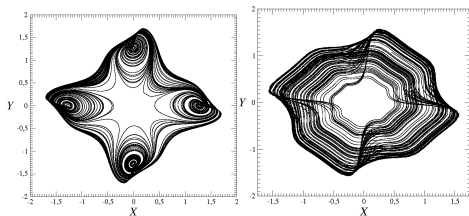
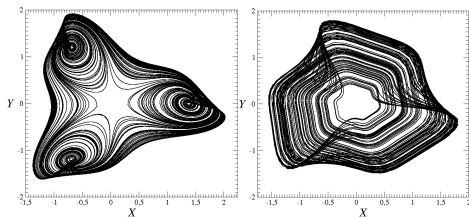


# Two Two-fold Covers Same Symmetry





# Three-fold, Four-fold Covers



# Two Inequivalent Lifts with $V_4$ Symmetry

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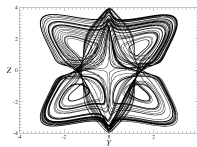
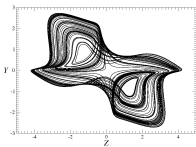
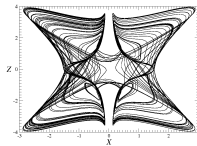
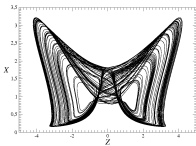
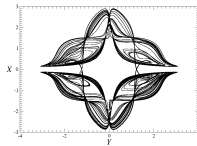
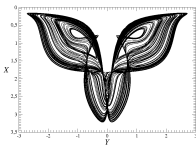
Overview-06

Overview-07

Experimental-01

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## Algorithm

- Construct Invariant Polynomials, Syzygies, Radicals
- Construct Singular Sets
- Determine Topological Indices
- Construct Spectrum of Structurally Stable Covers
- Structurally Unstable Covers Interpolate

## Symmetries Due to Symmetry

- Schur's Lemmas & Equivariant Dynamics
- Cauchy Riemann Symmetries
- Clebsch-Gordon Symmetries
- Continuations
  - Analytic Continuation
  - Topological Continuation
  - Group Continuation

# Covers of a Trefoil Torus

The Topology  
of Chaos

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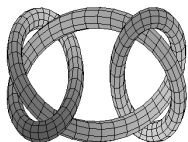
Overview-06

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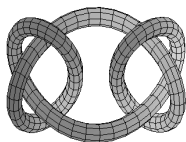
Experimental-  
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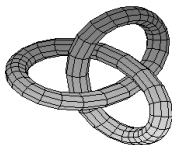
Experimental-



**Granny Knot**



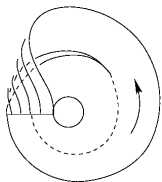
**Square Knot**



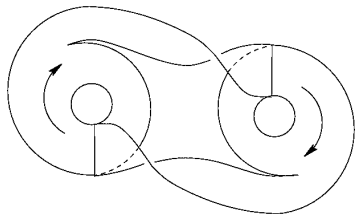
**Trefoil Knot**

# You Can Cover a Cover = Lift a Lift

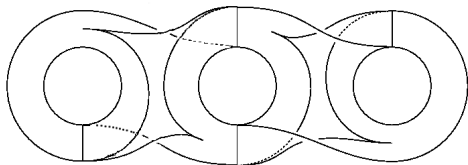
## Covers of Covers of Covers



Rossler

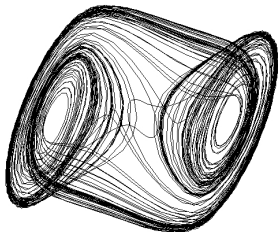
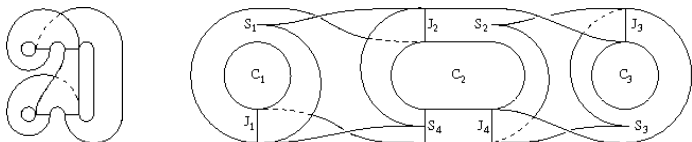


Lorenz



# Universal Branched Manifold

## Every Knot Lives Here



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## Local Stuff

**Groups:**

**Local Isomorphisms**

**Cartan's Theorem**

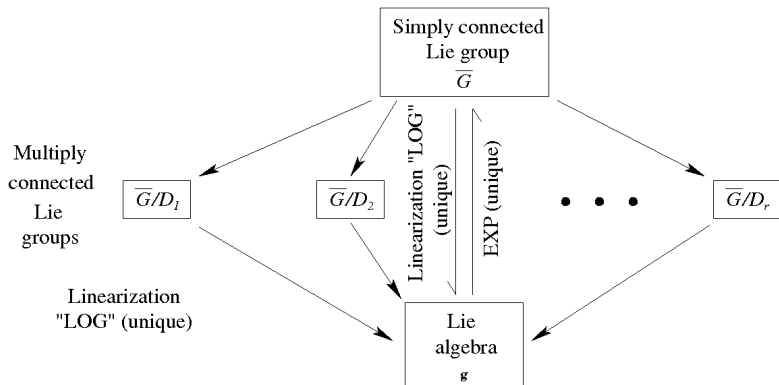
**Dynamical Systems:**

**Local Diffeomorphisms**

**??? Anything Useful ???**

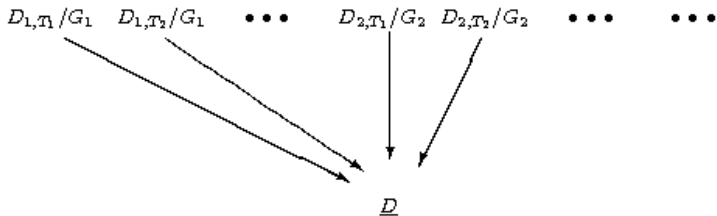


## Cartan's Theorem for Lie Groups



# Universal Image Dynamical System

## Locally Diffeomorphic Covers of $\underline{D}$



$\underline{D}$ : Universal Image Dynamical System

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## Local Isomorphisms & Diffeomorphisms

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## Local Isomorphisms & Diffeomorphisms

### Lie Groups

## Local Isomorphisms & Diffeomorphisms

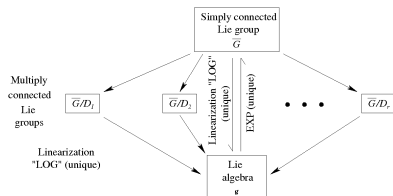
### Lie Groups

### Local Isomorphisms

## Local Isomorphisms & Diffeomorphisms

### Lie Groups

### Local Isomorphisms

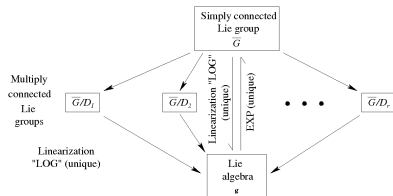


## Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms



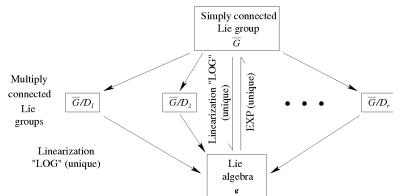
# Useful Analogs

## Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms    Local Diffeos





# Useful Analogs

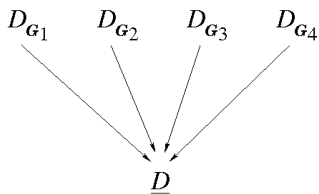
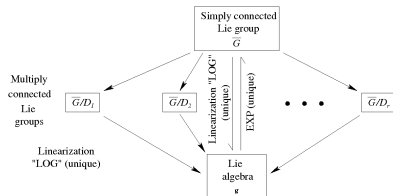
## Local Isomorphisms & Diffeomorphisms

Lie Groups

Dynamical Systems

Local Isomorphisms

Local Diffeos



## Rotating the Attractor

$$\frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F_1(X, Y) \\ F_2(X, Y) \end{bmatrix} + \begin{bmatrix} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{bmatrix}$$

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \begin{bmatrix} -v \\ +u \end{bmatrix}$$

$$\Omega = n \omega_d$$

$$q \Omega = p \omega_d$$

Global Diffeomorphisms

Local Diffeomorphisms

(p-fold covers)

# Two Phase Spaces: $R^3$ and $D^2 \times S^1$

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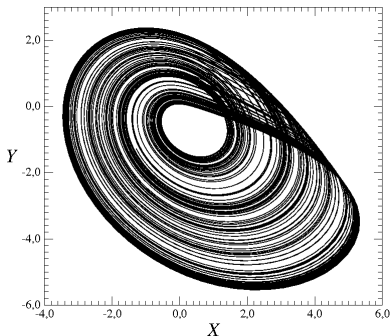
Experimental-  
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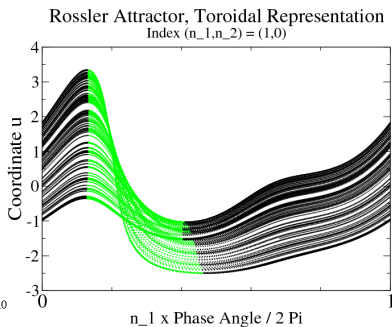
Experimental-

## Rosler Attractor: Two Representations

$R^3$



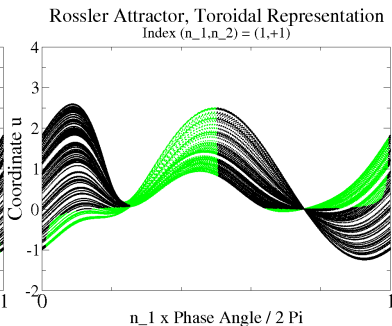
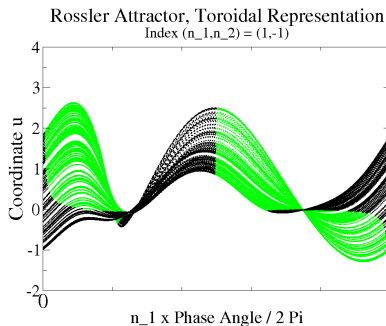
$D^2 \times S^1$



# Other Diffeomorphic Attractors

## Rossler Attractor:

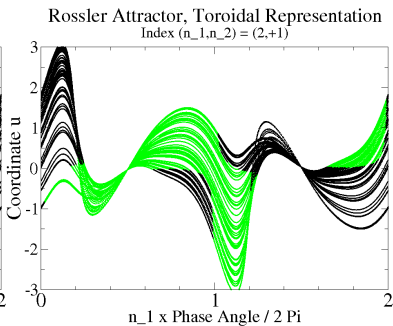
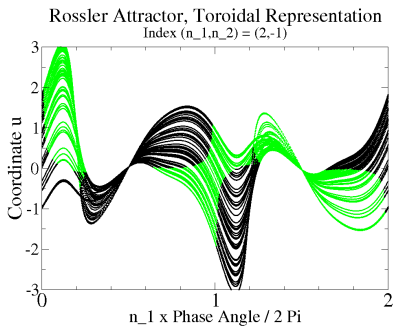
### Two More Representations with $n = \pm 1$



# Subharmonic, Locally Diffeomorphic Attractors

## Rossler Attractor:

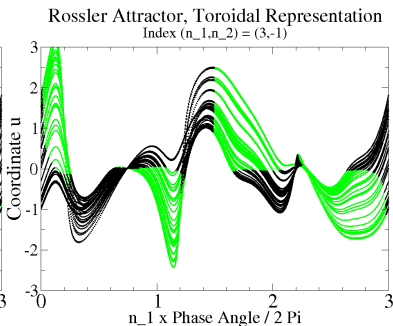
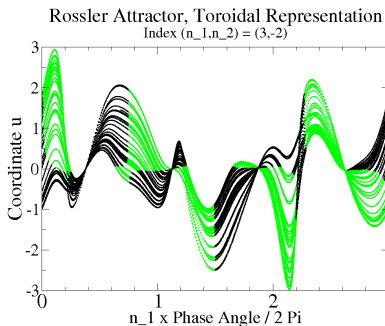
### Two Two-Fold Covers with $p/q = \pm 1/2$



# Subharmonic, Locally Diffeomorphic Attractors

## Rossler Attractor:

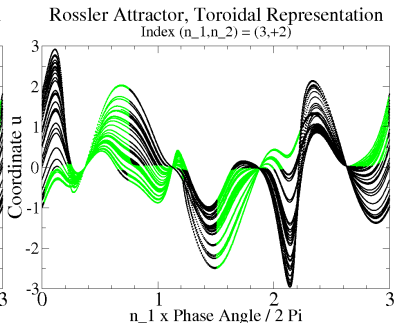
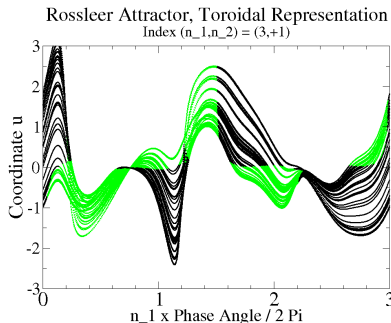
Two Three-Fold Covers with  $p/q = -2/3, -1/3$



# Subharmonic, Locally Diffeomorphic Attractors

## Rossler Attractor:

And Even More Covers (with  $p/q = +1/3, +2/3$ )



## Angular Momentum and Energy

$$L(0) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau X dY - Y dX \quad K(0) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) dt$$

$$L(\Omega) = \langle uv - vu \rangle \quad K(\Omega) = \left\langle \frac{1}{2} (\dot{u}^2 + \dot{v}^2) \right\rangle$$

$$= L(0) + \Omega \langle R^2 \rangle \quad = K(0) + \Omega L(0) + \frac{1}{2} \Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau (X^2 + Y^2) dt = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau (u^2 + v^2) dt$$



# New Measures, Diffeomorphic Attractors

## Energy and Angular Momentum

### Diffeomorphic, Quantum Number $n$

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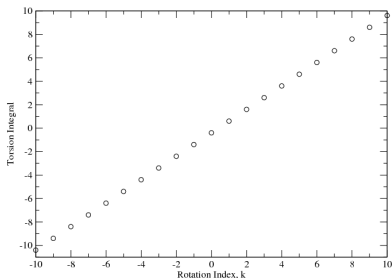
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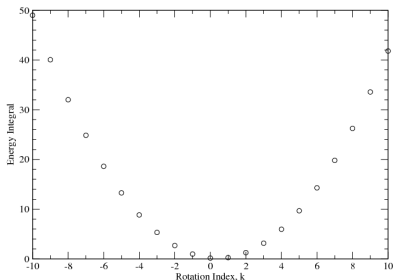
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Torsion Integral



Energy Integral



# New Measures, Subharmonic Covering Attractors

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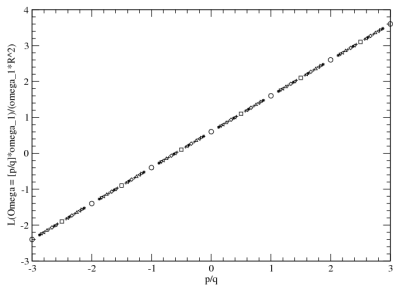
Experimental-  
01

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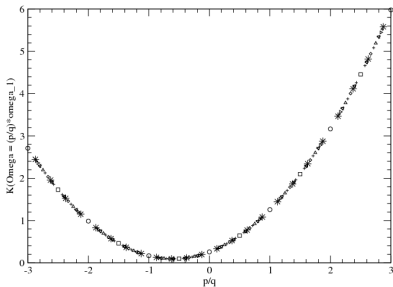
Experimental-

## Energy and Angular Momentum Subharmonics, Quantum Numbers $p/q$

Torsion Integral



Energy Integral



## Embeddings

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

## Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

*Mechanism* (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

## Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

## Equivalences by Injection

### Obstructions to Isotopy

$$R^3 \quad \rightarrow \quad R^4 \quad \rightarrow \quad R^5$$

Global Torsion

Global Torsion

Parity

Knot Type

There is one *Universal* reducible representation in  $R^N$ ,  $N \geq 5$ .  
In  $R^N$  the only topological invariant is *mechanism*.

## Summary

**1 Question Answered  $\Rightarrow$   
2 Questions Raised**

**We must be on the right track !**

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# Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.



## Result

**There is now a classification theory  
for low-dimensional strange attractors.**

- ① It is topological
- ② It has a hierarchy of 4 levels
- ③ Each is discrete
- ④ There is rigidity and degrees of freedom
- ⑤ It is applicable to  $R^3$  only — for now

# The Classification Theory has 4 Levels of Structure

# The Classification Theory has 4 Levels of Structure

## ① Basis Sets of Orbits

# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds

# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori

# The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori
- 4 Extrinsic Embeddings

# Four Levels of Structure

## The Topology of Chaos

Robert Gilmore

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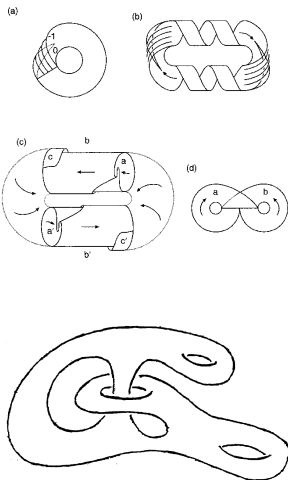
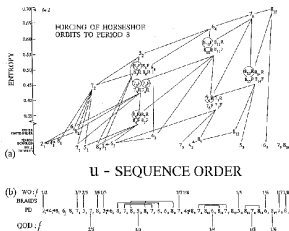
Overview-06

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# Poetic Organization

**LINKS OF PERIODIC ORBITS**

organize

**BOUNDING TORI**

organize

**BRANCHED MANIFOLDS**

organize

**LINKS OF PERIODIC ORBITS**



## Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of  $g - 1$  disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuation
- Cauchy-Riemann symmetries
- Quantizing Chaos
- Representation labels for inequivalent embeddings
- Representation Theory for Strange Attractors

## We hope to find:

- Robust topological invariants for  $R^N$ ,  $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of  $\chi^2$  test for NLD
- Better forcing results: Smale horseshoe,  $D^2 \rightarrow D^2$ ,  
 $n \times D^2 \rightarrow n \times D^2$  (e.g., Lorenz),  $D^N \rightarrow D^N$ ,  $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points  
(0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy