The Topology of Chaos

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Abstract

Data generated by a low-dimensional dynamical system operating in a chaotic regime can be analyzed using topological methods. The process is (almost) straightforward. On a scalar time series, the following steps are taken:

1. Unstable periodic orbits are identified;
2. An embedding is constructed; $\star$ $\star$
3. The topological organization of these periodic orbits is determined;
4. Some orbits are used to identify an underlying branched manifold;
5. The branched manifold is used as a tool to predict the remaining topological invariants.

This algorithm has its own built in rejection criterion.

Abstract - Key Point

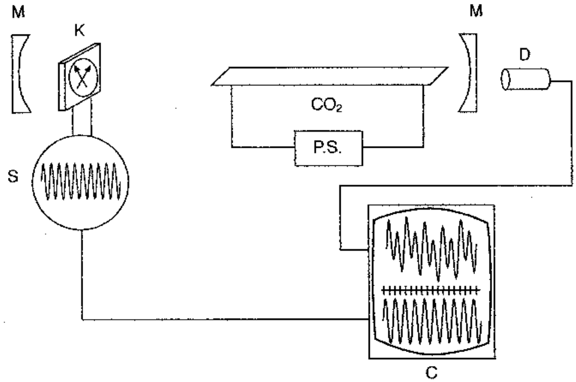
One soft spot in this analysis program is the embedding step. Different embeddings can yield different topological results. This makes the following question exciting:

> *When you analyze embedded data: How much of what you learn is about the embedding and how much is about the underlying dynamics?*

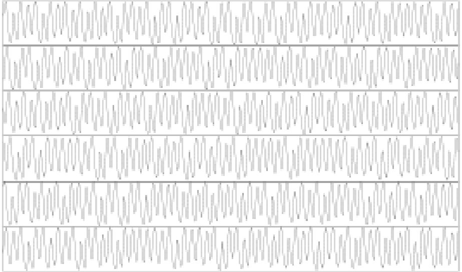
This question has been answered by creating a representation theory of low dimensional strange attractors. It is now possible to totally disentangle the mechanism generating the underlying dynamics from topological structure induced by the embedding step.

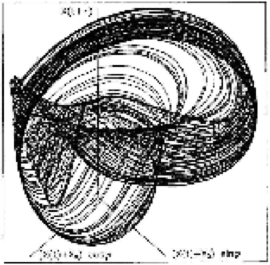
Outline

Laser Experimental Arrangement



Experimental Data: LSA





Lefranc - Cargese

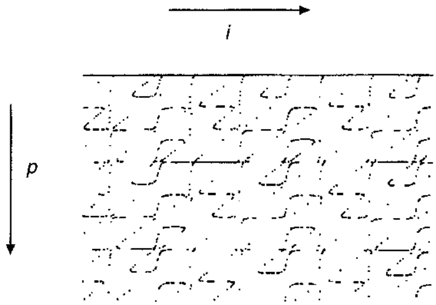
Periodic Orbits are the Key

Joseph Fourier
Linear Systems

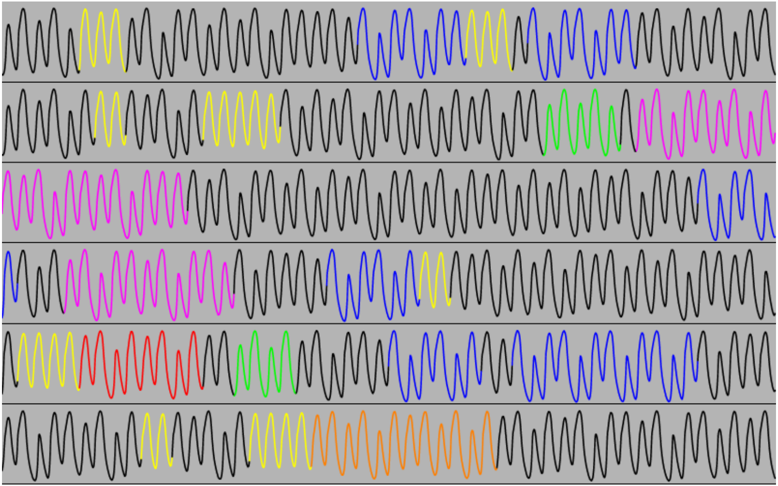
Henri Poincare
Nonlinear Systems

Searching for Periodic Orbits

$$\Theta(i, i+p) = |x(i) - x(i+p)|$$



"Periodic Orbits" in Real Data



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Exp'tal-01
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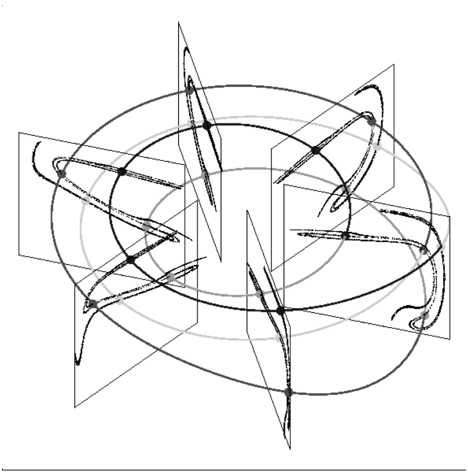
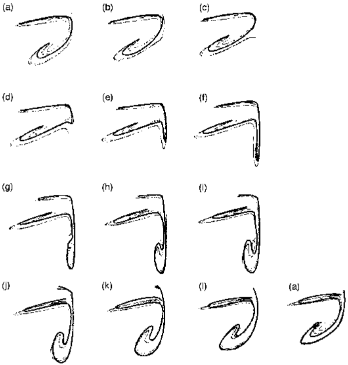
Exp'tal-04
Exp'tal-05
Exp'tal-06
Exp'tal-07
Exp'tal-08
Embed-01
Embed-02
Embed-03

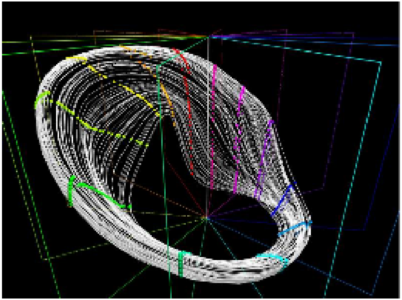
Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Rotating the Poincaré Section around the axis of the torus



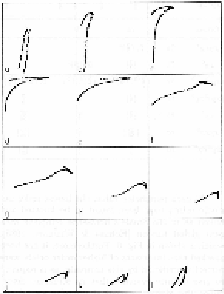


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

Lefranc - Cargese

Creating Something from "Nothing"

scalar time series $\longrightarrow$ vector time series

Embedding is an Art.

Perhaps more like Black Magic.

There are many ways to conjure an embedding.

Varieties of Embeddings

$$x(i) \rightarrow (y_1(i), y_2(i), y_3(i), \cdots)$$

Delay	$(x(i), x(i-\tau_1), x(i-\tau_2), x(i-\tau_3), \cdots)$
Delay	$y_j(i) = x(i - [j-1]\,\tau) \qquad \tau, N$
Differential	$y_1 = x, y_2 = dx/dt, y_3 = d^2x/dt^2, \cdots$
Int. – Diff.	$y_1 = \int_{-\infty}^{x} dx, y_2 = x, y_3 = dx/dt, \cdots$
SVD	EoM
Hilbert – Tsf.	
"Circular"	
Knotted	
Other	

Circular and Knotted Embeddings

If there is a "hole in the middle" then parameterize the scalar observable by an angle θ: $x(t) \rightarrow x(\theta)$

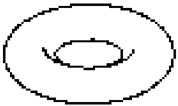
Introduce Knot coordinates ("harmonic knots")

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

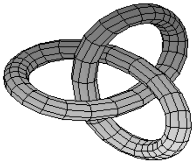
Repere Mobile: $\{\mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)\}$

$$x(t) \rightarrow x(\theta) \rightarrow \mathbf{K}(\theta) + y_1\mathbf{n}(\theta) + y_2\mathbf{b}(\theta)$$

Some Knotted Embeddings

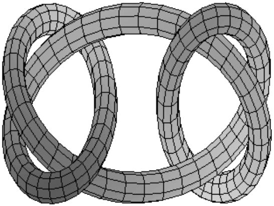


Simple "Unknot"

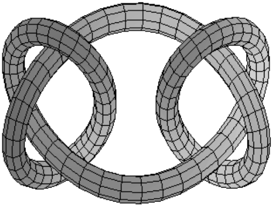


Trefoil Knot

More Knotted Embeddings



Granny Knot



Square Knot

Chaos

Motion that is

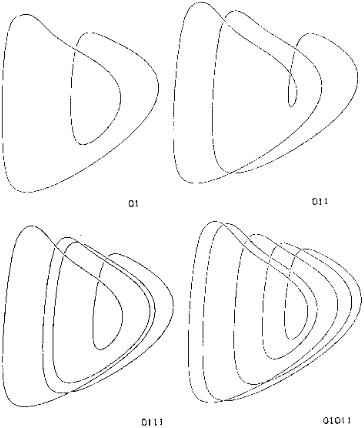
- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

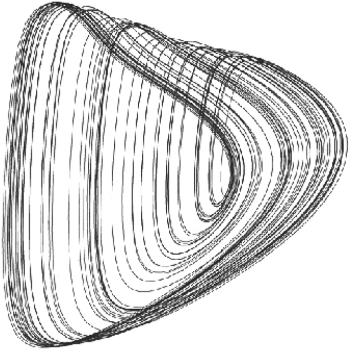
Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

UPOs Outline Strange Attractors





BZ reaction

UPOs Outline Strange attractors

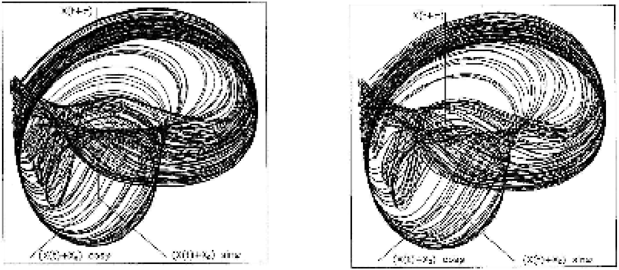


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

How Are Orbits Organized

Ask the Master:

Carl Friedrich Gauss

Organization of UPOs in R^3:

Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

Linking Number of Two UPOs

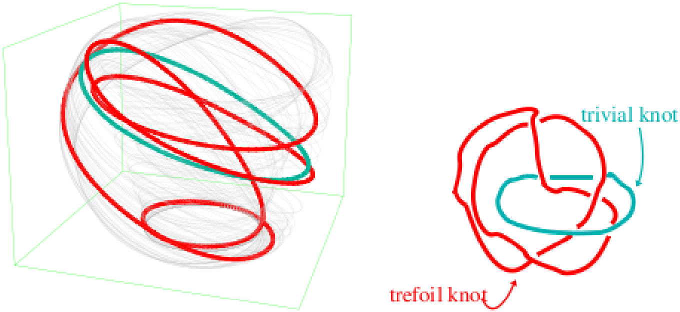
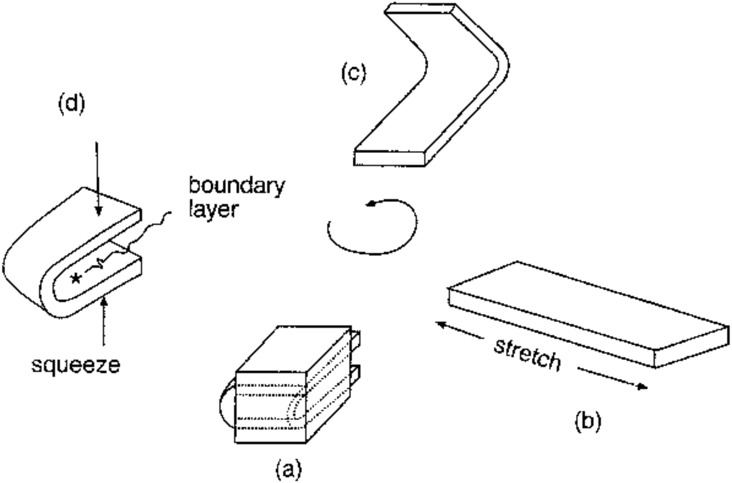


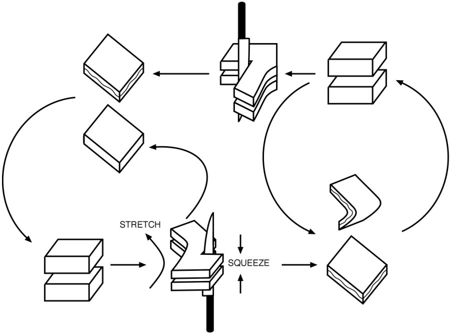
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

One Stretch-&-Squeeze Mechanism

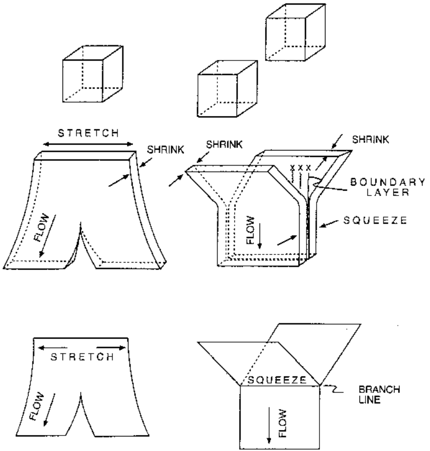


Another Stretch-&-Squeeze Mechanism



"Lorenz Mechanism"

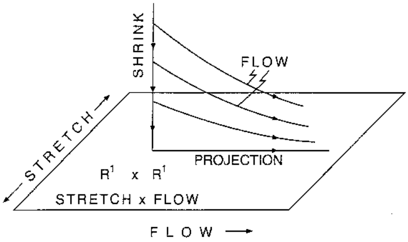
Stretching — Folding



Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$
- **Generates a** <u>**hyperbolic**</u> **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

- **The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.**

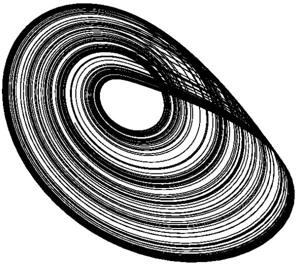
- **UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.**

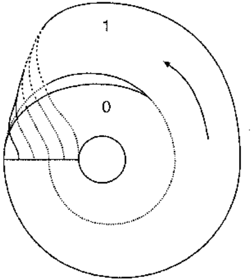
Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

Rössler:

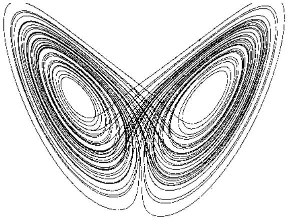
Attractor **Branched Manifold**

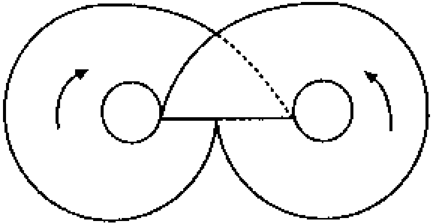




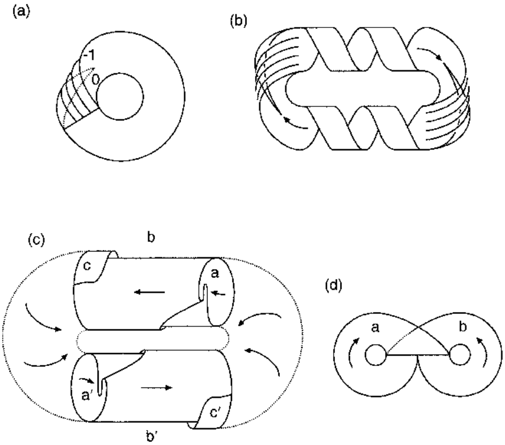
Lorenz:

Attractor **Branched Manifold**



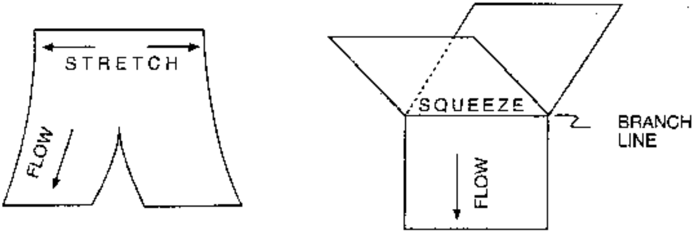


Inequivalent Branched Manifolds



Aufbau Princip for Branched Manifolds

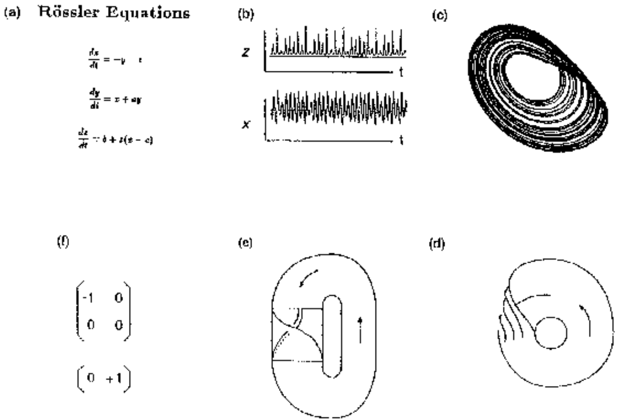
Any branched manifold can be built up from stretching and squeezing units



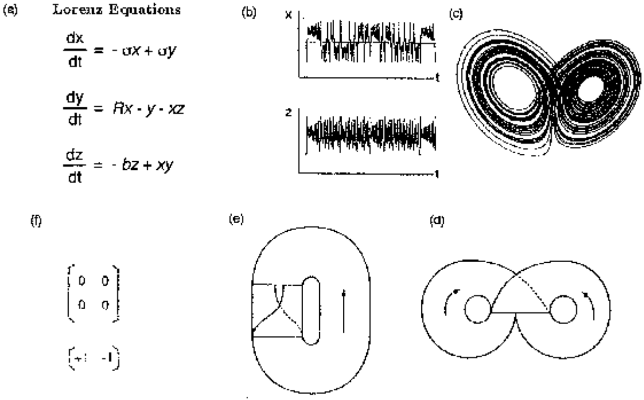
subject to the conditions:

- **Outputs to Inputs**
- **No Free Ends**

Rossler System



Lorenz System



Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

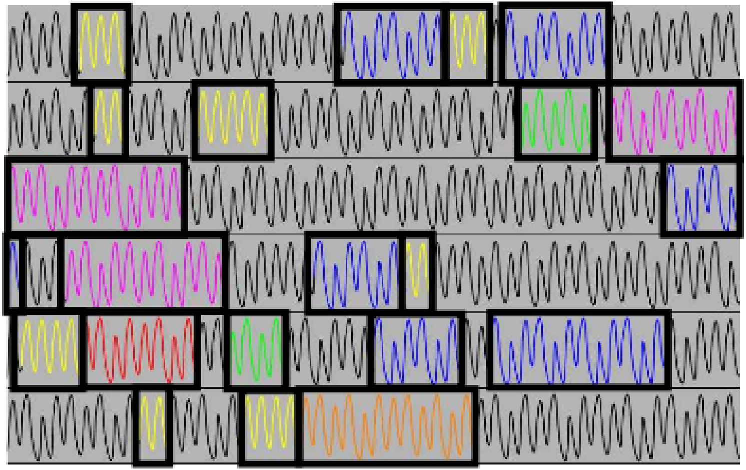
Identify a Branched Manifold

Verify the Branched Manifold

Model the Dynamics

Validate the Model

Method of Close Returns



Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	1s	1f	2_1	3f	3s	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Guess Branched Manifold

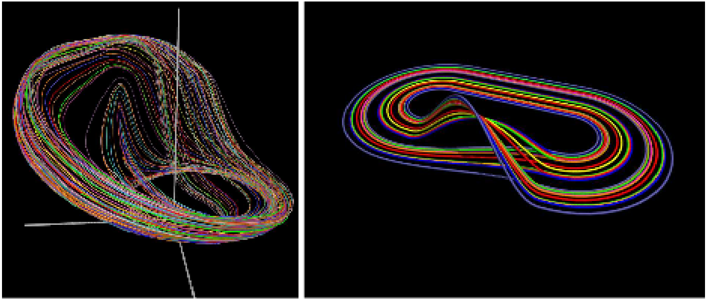


Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

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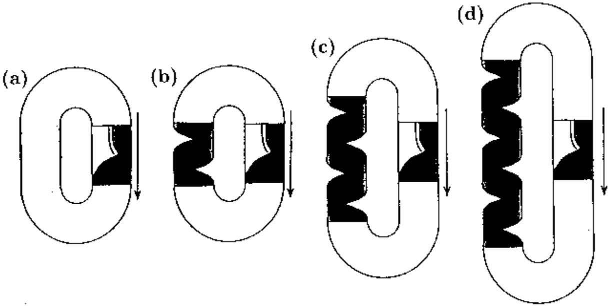
Identification & 'Confirmation'

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Reject or Fail to Reject)

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change

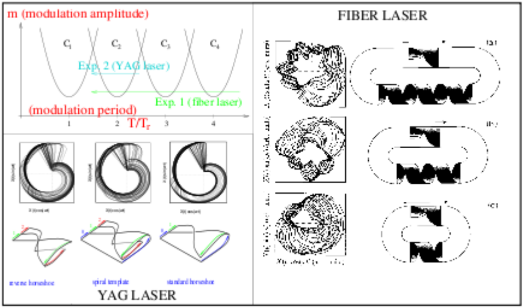
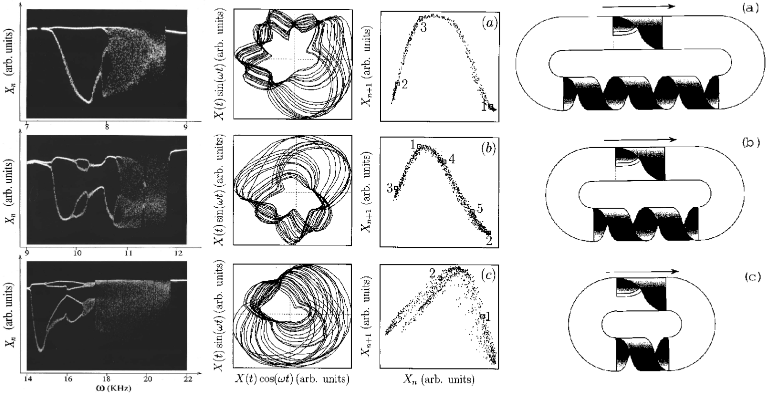


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Lefranc - Cargese

Evolution Under Parameter Change



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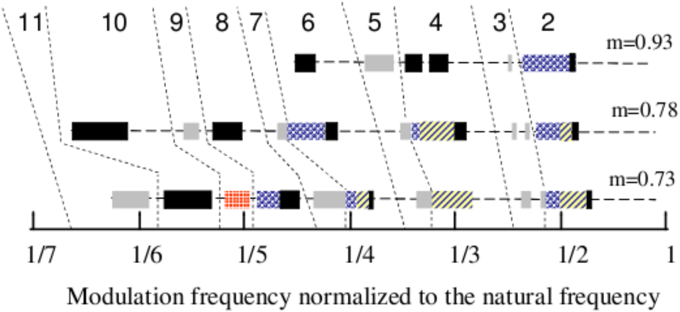
Evolution Under Parameter Change

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						

Used and Martin — Zaragosa

Evolution Under Parameter Change



Used and Martin — Zaragosa

Analysis of Nonstationary Data

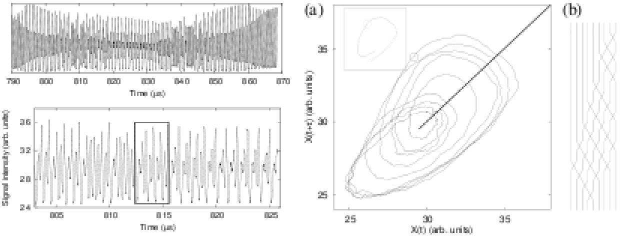


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.

Inequivalent Irreducible Representations

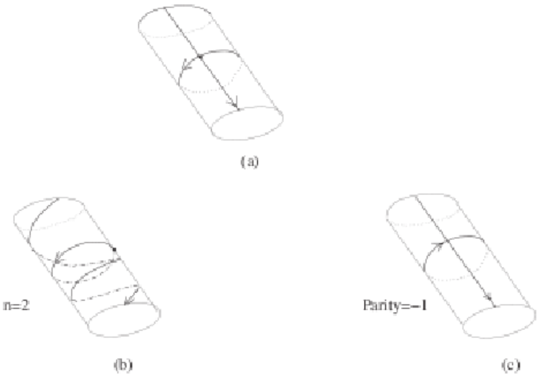
Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

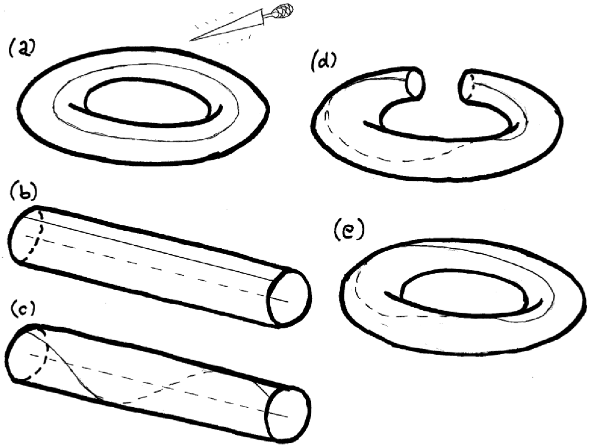
$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Global Torsion & Parity



Inequivalence in R^3



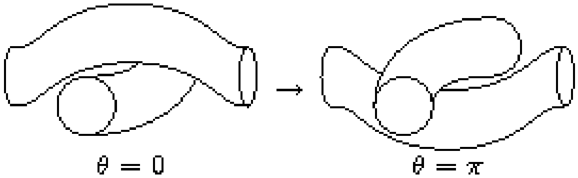
Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

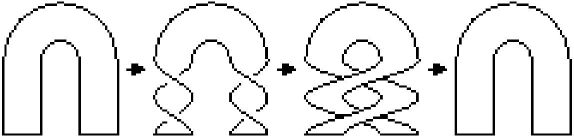
Systematically?

Crossing Exchange in R^4



Parity reversal is also possible in R^4 by isotopy.

2 Twists = 1 Writhe = Identity



$$Z \longrightarrow Z_2$$

Global Torsion $\longrightarrow$ Binary Op

Equivalences by Injection

Obstructions to Isotopy

R^3	$\rightarrow$	R^4	$\rightarrow$	R^5
Global Torsion		Global Torsion		
Parity				
Knot Type				

There is one *Universal* reducible representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

Can We Classify Strange Attractors?

Chemists have their classification.

Nuclear Physicists have their classification.

Particle Physicists have their classification.

Astronomers have their classification.

Mendeleev's Table of the Chemical Elements

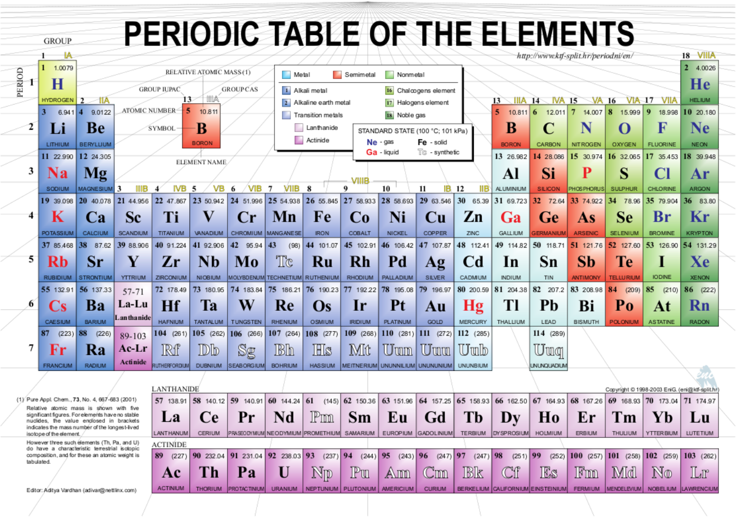
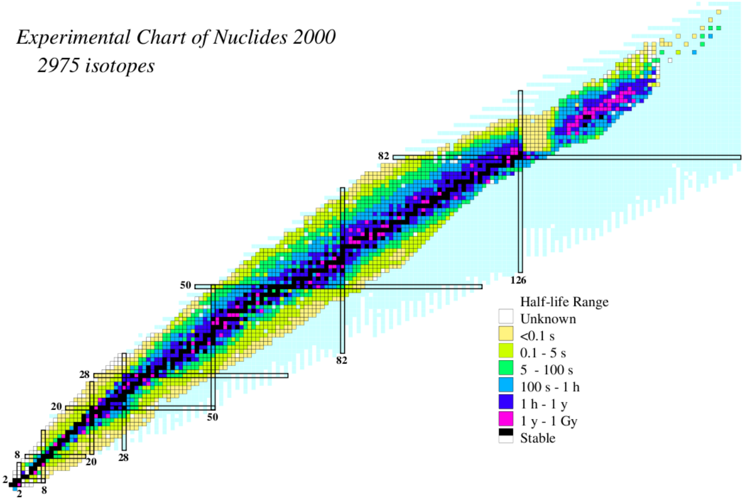
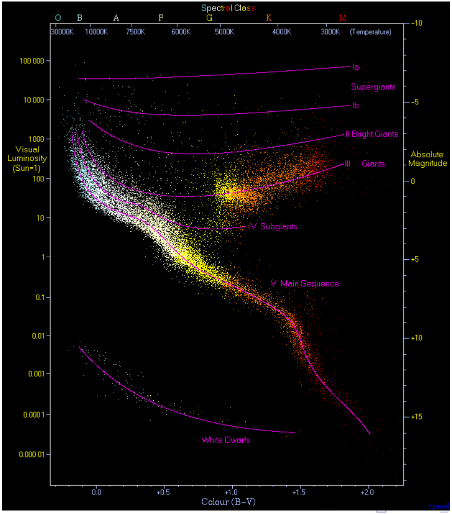


Table of Atomic Nuclei



Hertzsprung-Russell Diagram



How Does it Work Out?

Chemical Elements	1 Integer: N_P
Atomic Nuclei	2 Integers: N_P, N_N
Stars	1 Continuous variable M + exceptions
Strange Attractors	? ? ? ?

What We Did

We analyzed data from a Laser with Saturable Absorber (LSA).

3 Different absorbers were used.

For each absorber data were taken under 6 - 10 operating conditions.

There was a total of 25 different data sets.

We wanted to "prove experimentally" that changing the absorber/operating condition served merely to push to flow around on the same branched manifold.

What We Found

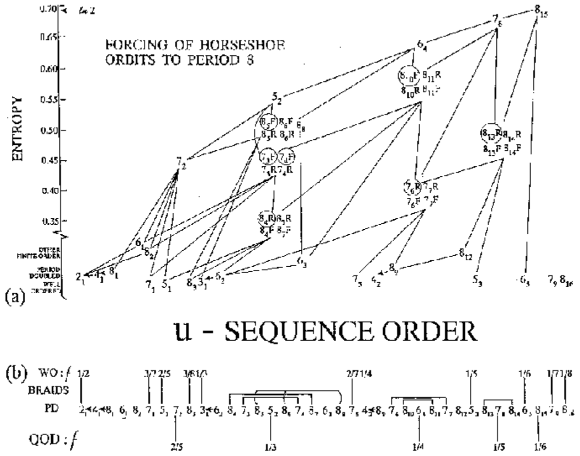
When certain orbits (UPOs) were present they were invariably accompanied by a specific set of other orbits.

This led us to propose that certain orbits 'force' other orbits.

Forcing is topological.

A discrete set of "basis orbits" serves to identify the complete collection of UPOs present in an attractor.

Forcing Diagram - Horseshoe



Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required
- Higher dimension - ???

Constraints

Branched manifolds largely constrain the 'perestroikas" that forcing diagrams can undergo.

Is there some mechanism/structure that constrains the types of perestroikas that branched manifolds can undergo?

Constraints on Branched Manifolds

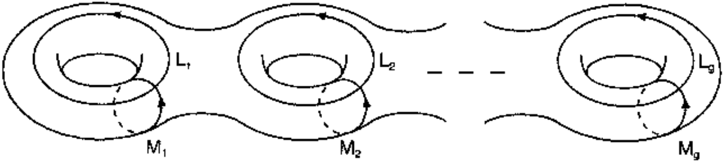
"Inflate" a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Tori are identified by genus g and dressed with a surface flow induced from that creating the strange attractor.

Surface Singularities

Flow field: three eigenvalues: +, 0, −

Vector field "perpendicular" to surface

Eigenvalues on surface at fixed point: +, −

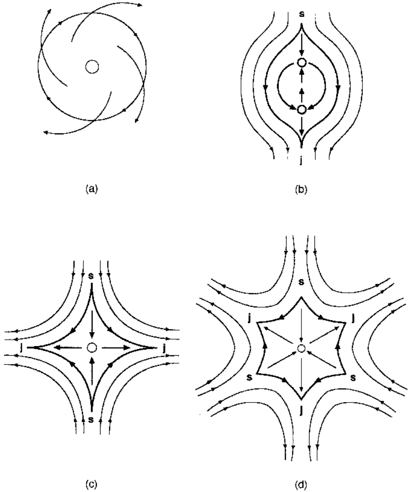
All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

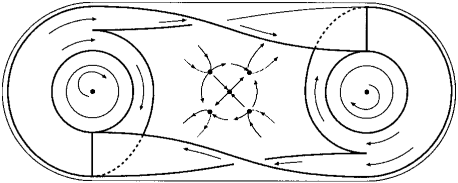
 # fixed points on surface = index = 2g - 2

Singularities organize the surface flow dressing the torus

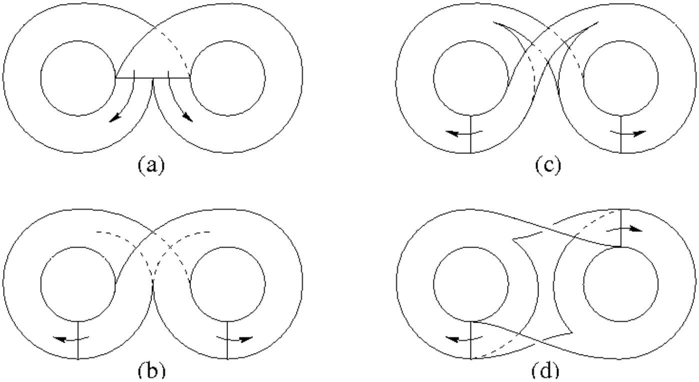
Flow Near a Singularity



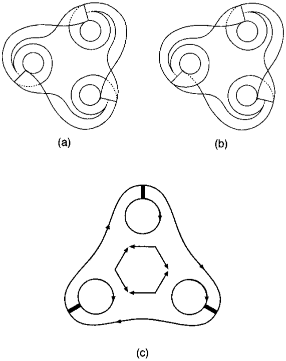
Torus Bounding Lorenz-like Flows



Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.



Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks.

Transition matrix sum of two g-1 $\times$ g-1 matrices.

Both are g-1 $\times$ g-1 permutation matrices.

They identify mappings of Poincaré sections to P'sections.

Bounding tori labeled by (permutation) group theory.

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

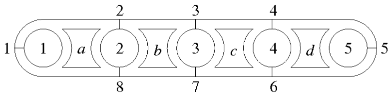
g	m	$(p_1, p_2, \dots p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

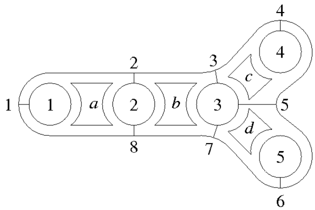
The Growth is Exponential

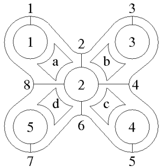
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

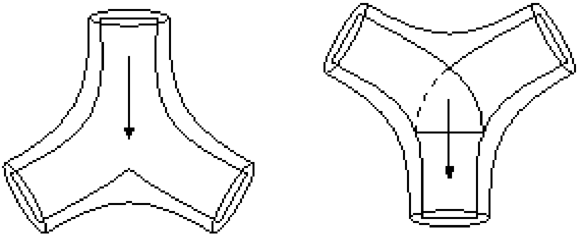
Some Genus-9 Bounding Tori







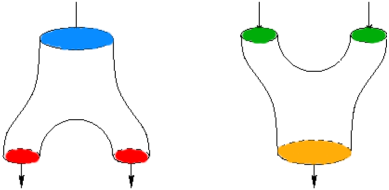
Aufbau Princip for Bounding Tori



These units ("pants, trinions") surround the stretching and squeezing units of branched manifolds.

Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



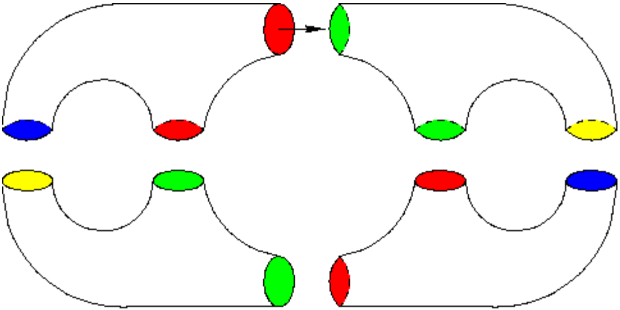
- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Construction of Poincaré Section

P. S. = Union 

Components = g-1

Application: Lorenz Dynamics, g=3



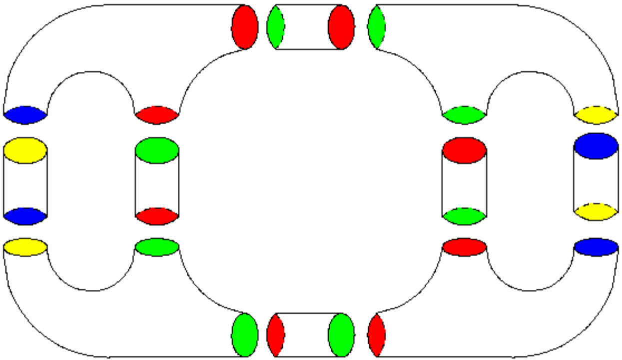
Representation Theory for $g > 1$

Can we extend the representation theory for strange attractors "with a hole in the middle" (i.e., genus $= 1$) to higher-genus attractors?

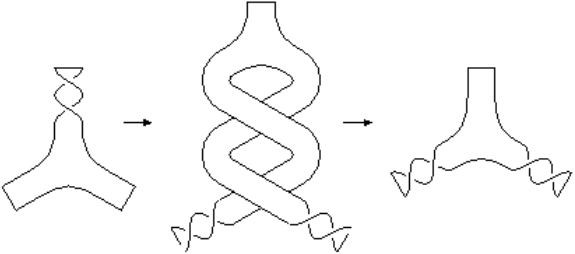
Yes. The results are similar.

Begin as follows:

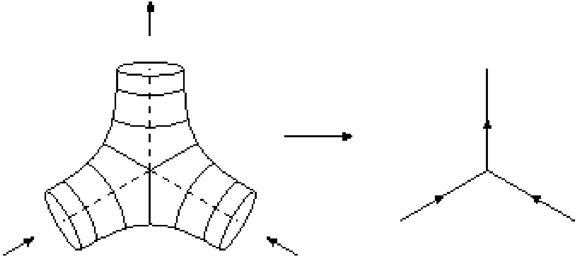
Application: Lorenz Dynamics, g=3



Embeddings

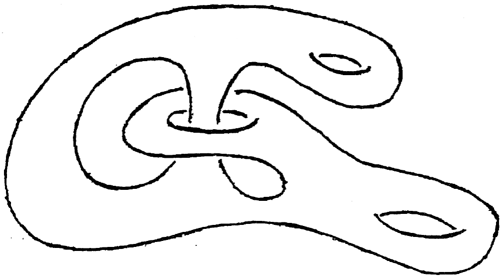


Preparations for Embedding tori into



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Intrinsic Tori



A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

Equivalences by Injection

Obstructions to Isotopy

Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus-g strange attractor become equivalent under isotopy.

Summary

1 Question Answered $\Rightarrow$

2 Questions Raised

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds

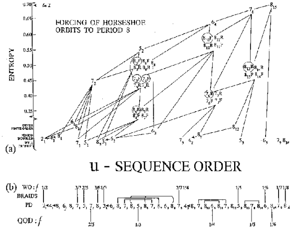
The Classification Theory has 4 Levels of Structure

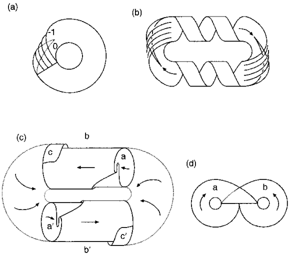
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

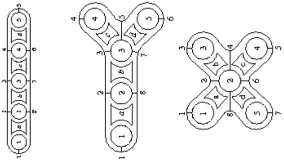
The Classification Theory has 4 Levels of Structure

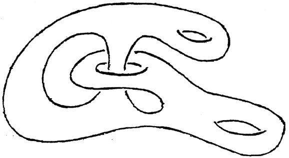
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

Answered Questions

There is a Representation Theory for Strange Attractors

There is a complete set of rerpesentation labels for strange attractors of any genus g.

The labels are complete and discrete.

Representations can become equivalent when immersed in higher dimension.

All representations (embeddings) of a 3-dimensional strange attractor become isotopic (equivalent) in R^5.

The *Universal Representation* of an attractor in R^5 identifies mechanism. No embedding artifacts are left.

The topological index in R^5 that identifies mechanism remains to be discovered.

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g - 1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuuation
- Cauchy-Riemann symmetries
- Quantizing Chaos

Unanswered Questions

We hope to find:

- Robust topological invariants for R^N, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \to D^2$, $n \times D^2 \to n \times D^2$ (e.g., Lorenz), $D^N \to D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy

Thanks

To my colleagues and friends:

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