

The Topology of Chaos

Robert Gilmore

Physics Department
Drexel University
Philadelphia, PA 19104
robert.gilmore@drexel.edu

2011 Brazilian Conference on Dynamics, Control, and Applications
Águas de Lindóia, Sao Paulo, Brazil
August 29 — September 2, 2011

August 16, 2011

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Abstract

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Data generated by a low-dimensional dynamical system operating in a chaotic regime can be analyzed using topological methods. The process is (almost) straightforward. On a scalar time series, the following steps are taken:

- 1 Unstable periodic orbits are identified;
- 2 An embedding is constructed; $\star \star$
- 3 The topological organization of these periodic orbits is determined;
- 4 Some orbits are used to identify an underlying branched manifold;
- 5 The branched manifold is used as a tool to predict the remaining topological invariants.

This algorithm has its own built in rejection criterion.

Abstract - Key Point

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

One soft spot in this analysis program is the embedding step. Different embeddings can yield different topological results. This makes the following question exciting:

When you analyze embedded data: How much of what you learn is about the embedding and how much is about the underlying dynamics?

This question has been answered by creating a representation theory of low dimensional strange attractors. It is now possible to totally disentangle the mechanism generating the underlying dynamics from topological structure induced by the embedding step.

Outline

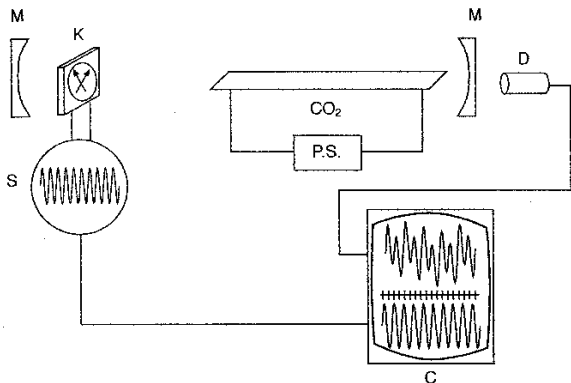
- 1 Overview
- 2 Experimental Challenge
- 3 Embedding Problems
- 4 Topological Analysis Program
- 5 Representation Theory of Strange Attractors
- 6 Classification of Strange Attractors
- 7 Basis Sets of Orbits
- 8 Bounding Tori
- 9 Summary

Experimental Schematic

The Topology
of Chaos

Robert
Gilmore

Laser Experimental Arrangement



Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

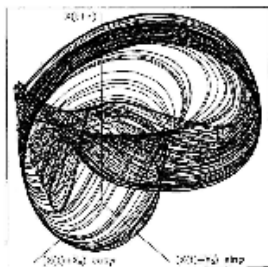
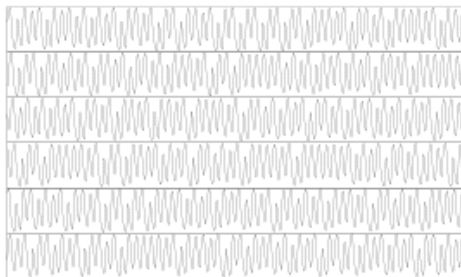
Exp'tal-08

Embed-01

Embed-02

Embed-03

Experimental Data: LSA



Lefranc - Cargese

Periodic Orbits are the Key



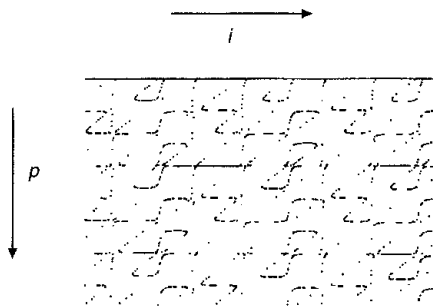
Joseph Fourier
Linear Systems



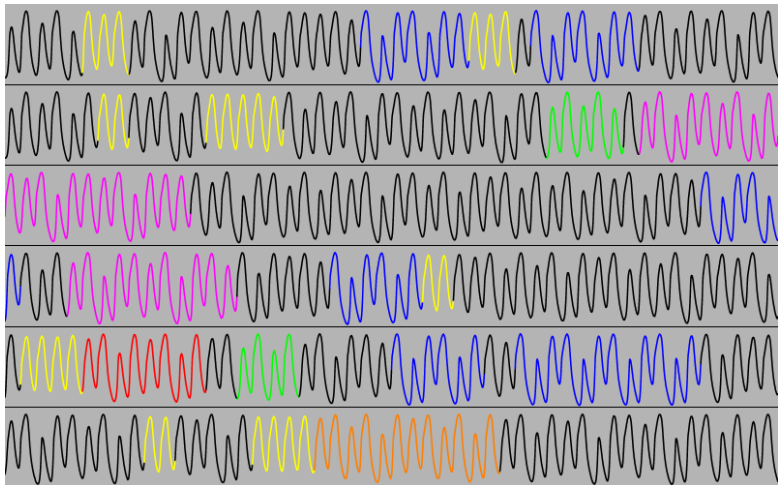
Henri Poincaré
Nonlinear Systems

Searching for Periodic Orbits

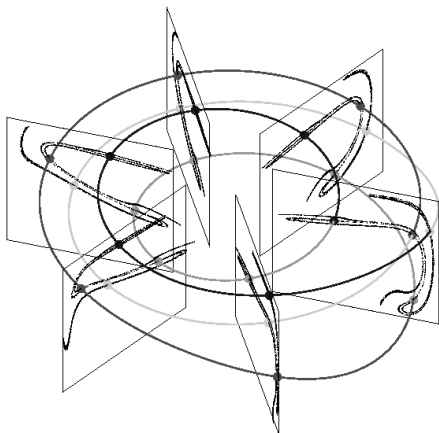
$$\Theta(i, i + p) = |x(i) - x(i + p)|$$



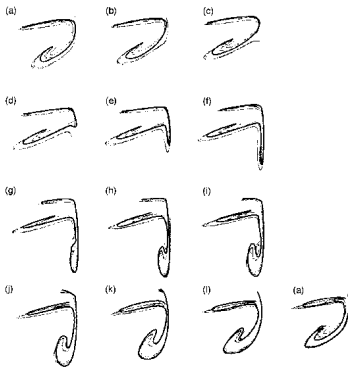
“Periodic Orbits” in Real Data



Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Rotating the Poincaré Section around the axis of the torus

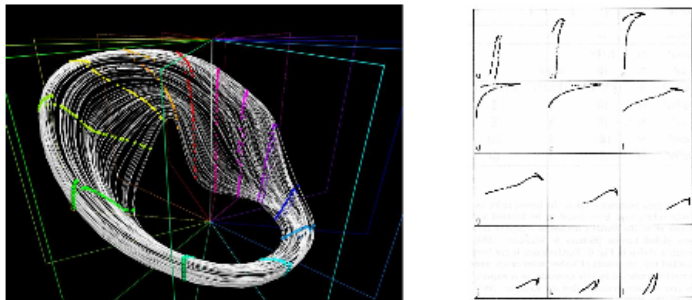


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

Creating Something from “Nothing”

scalar time series \longrightarrow vector time series

Embedding is an Art.

Perhaps more like Black Magic.

There are many ways to conjure an embedding.

Varieties of Embeddings

$$x(i) \rightarrow (y_1(i), y_2(i), y_3(i), \dots)$$

Delay	$(x(i), x(i - \tau_1), x(i - \tau_2), x(i - \tau_3), \dots)$
Delay	$y_j(i) = x(i - [j - 1] \tau) \quad \tau, N$
Differential	$y_1 = x, y_2 = dx/dt, y_3 = d^2x/dt^2, \dots$
Int. – Diff.	$y_1 = \int_{-\infty}^x dx, y_2 = x, y_3 = dx/dt, \dots$
SVD	EoM
Hilbert – Tsf.	
“Circular”	
Knotted	
Other	

Circular and Knotted Embeddings

If there is a “hole in the middle” then parameterize the scalar observable by an angle θ : $x(t) \rightarrow x(\theta)$

Introduce Knot coordinates (“harmonic knots”)

$$\mathbf{K}(\theta) = (\xi(\theta), \eta(\theta), \zeta(\theta)) = \mathbf{K}(\theta + 2\pi)$$

Repere Mobile: $\{\mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)\}$

$$x(t) \rightarrow x(\theta) \rightarrow \mathbf{K}(\theta) + y_1 \mathbf{n}(\theta) + y_2 \mathbf{b}(\theta)$$

Some Knotted Embeddings

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

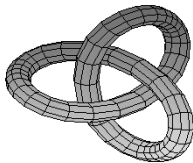
Embed-01

Embed-02

Embed-03



Simple “Unknot”



Trefoil Knot

More Knotted Embeddings

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

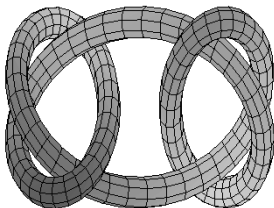
Exp'tal-07

Exp'tal-08

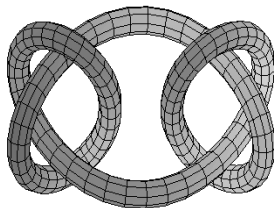
Embed-01

Embed-02

Embed-03



Granny Knot



Square Knot

Chaos

Motion that is

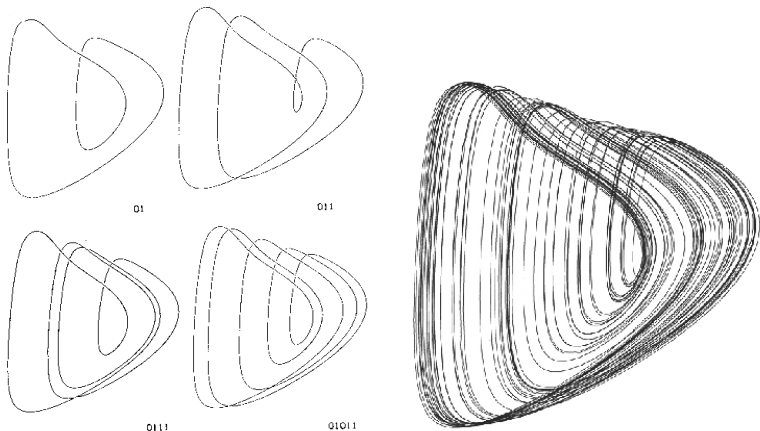
- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits “in” the strange attractor. They are

- “Abundant”
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

UPOs Outline Strange Attractors



BZ reaction

UPOs Outline Strange attractors

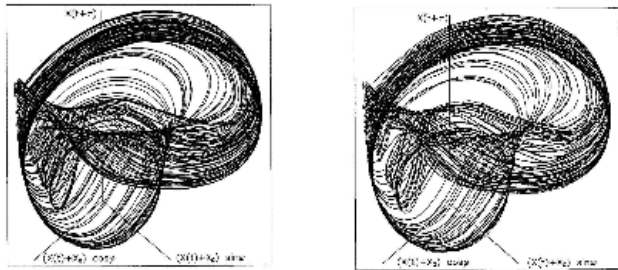


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

How Are Orbits Organized

Ask the Master:



Carl Friedrich Gauss

Organization of UPOs in R^3 : Gauss Linking Number

$$LN(A, B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of $LN \simeq$ # Mathematicians in World

Linking Number of Two UPOs

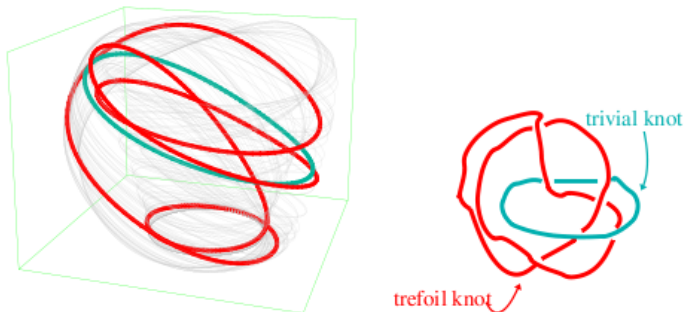


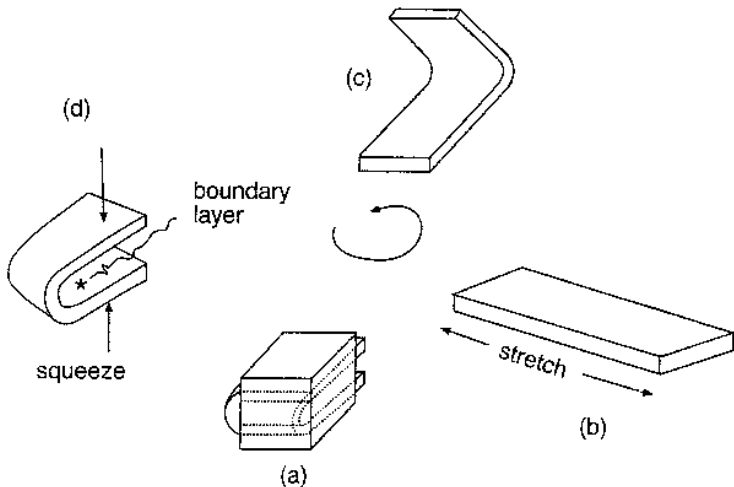
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Evolution in Phase Space

The Topology
of Chaos

Robert
Gilmore

One Stretch-&-Squeeze Mechanism



Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Motion of Blobs in Phase Space

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

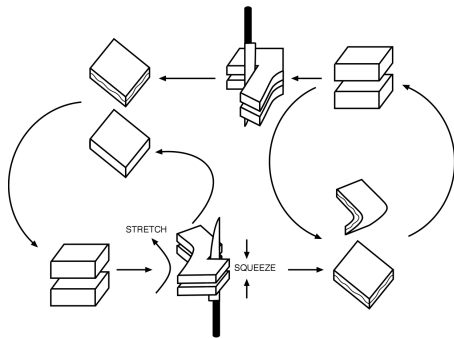
Exp'tal-08

Embed-01

Embed-02

Embed-03

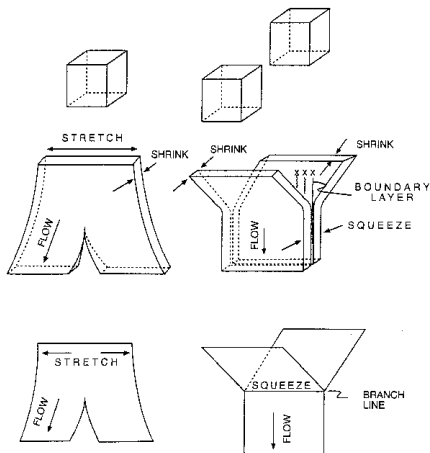
Another Stretch-&-Squeeze Mechanism



“Lorenz Mechanism”

Motion of Blobs in Phase Space

Stretching — Folding

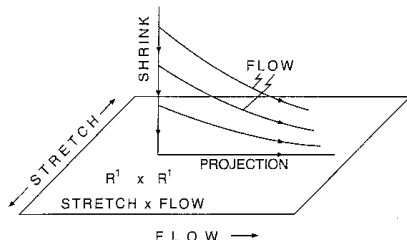


Collapse Along the Stable Manifold

Birman - Williams Projection

Identify x and y if

$$\lim_{t \rightarrow \infty} |x(t) - y(t)| \rightarrow 0$$



Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- on R^n is dissipative, $n = 3$, so that $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0$.
- Generates a hyperbolic strange attractor SA

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

- **The projection maps the strange attractor \mathcal{SA} onto a 2-dimensional branched manifold \mathcal{BM} and the flow $\Phi_t(x)$ on \mathcal{SA} to a semiflow $\bar{\Phi}(x)_t$ on \mathcal{BM} .**
- **UPOs of $\Phi_t(x)$ on \mathcal{SA} are in 1-1 correspondence with UPOs of $\bar{\Phi}(x)_t$ on \mathcal{BM} . Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\bar{\Phi}(x)_t, \mathcal{BM})$.**

Remark: “One of the few theorems useful to experimentalists.”

A Very Common Mechanism

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

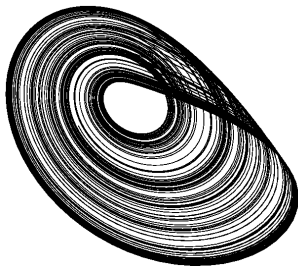
Embed-01

Embed-02

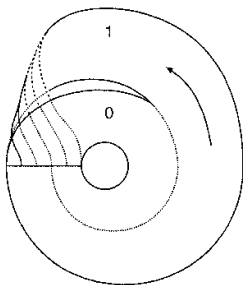
Embed-03

Rössler:

Attractor



Branched Manifold



A Mechanism with Symmetry

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

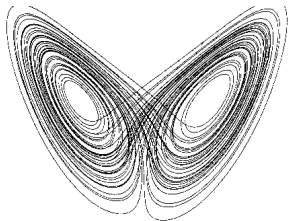
Embed-01

Embed-02

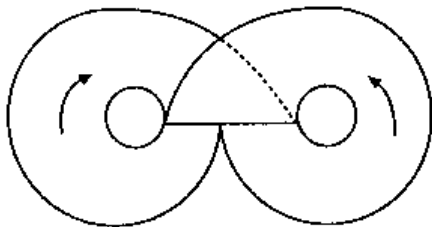
Embed-03

Lorenz:

Attractor



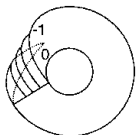
Branched Manifold



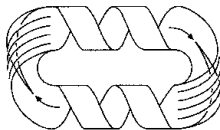
Examples of Branched Manifolds

Inequivalent Branched Manifolds

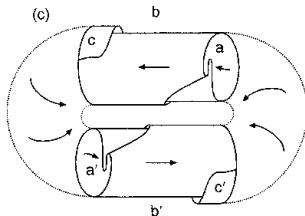
(a)



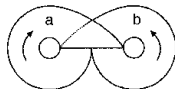
(b)



(c)



(d)



Aufbau Princip for Branched Manifolds

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

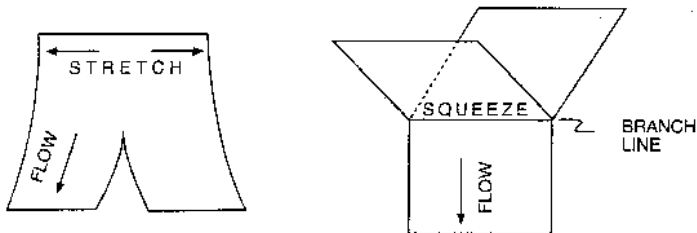
Exp'tal-08

Embed-01

Embed-02

Embed-03

Any branched manifold can be built up from stretching and squeezing units



subject to the conditions:

- **Outputs to Inputs**
- **No Free Ends**

Rössler System

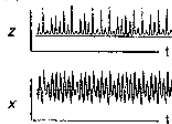
(a) Rössler Equations

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

(b)



(c)



(f)

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & +1 \end{pmatrix}$$

(e)



(f)



Lorenz System

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

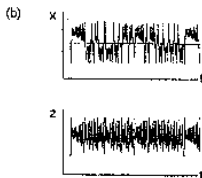
Embed-03

(a) Lorenz Equations

$$\frac{dx}{dt} = -\alpha x + \alpha y$$

$$\frac{dy}{dt} = \beta x - y - xz$$

$$\frac{dz}{dt} = -bz + xy$$

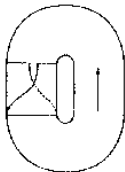


(f)

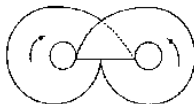
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} +i & -i \end{bmatrix}$$

(e)



(d)



Poincaré Smiles at U s in R^3

- Determine organization of UPOs \Rightarrow
- Determine branched manifold \Rightarrow
- Determine equivalence class of \mathcal{SA}

Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

Identify a Branched Manifold

Verify the Branched Manifold

Model the Dynamics

Validate the Model

Method of Close Returns

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

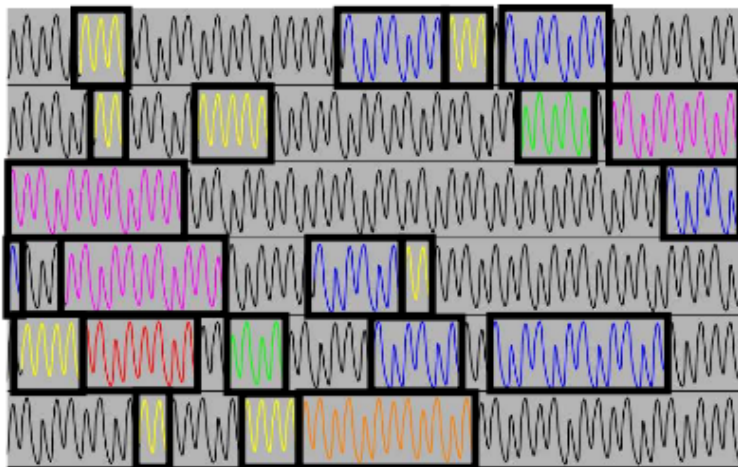
Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03



Embeddings

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological[†]

None Good

We Demand a 3 Dimensional Embedding

Determine Topological Invariants

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov-Zhabotinskii data^a

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

^aAll indices are negative.

Determine Topological Invariants

The Topology
of Chaos

Robert
Gilmore

Compare w. LN From Various BM

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	1^s	1^f	2_1	3^f	3^s	4_1	4_2^f	4_2^s	5_2^f	5_2^s	5_2^f	5_2^s	5_1^f	5_1^s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Determine Topological Invariants

Guess Branched Manifold

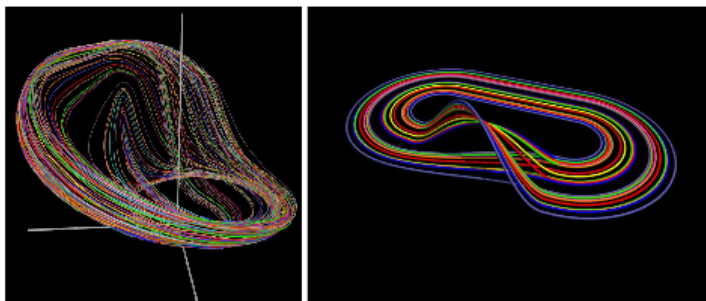


Figure 7. “Combing” the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

Lefranc - Cargese

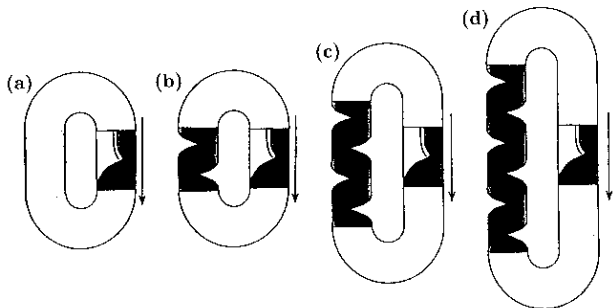
Identification & 'Confirmation'

- \mathcal{BM} Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion (Reject or Fail to Reject)

Determine Topological Invariants

What Do We Learn?

- BM Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Perestroikas of Strange Attractors

Evolution Under Parameter Change

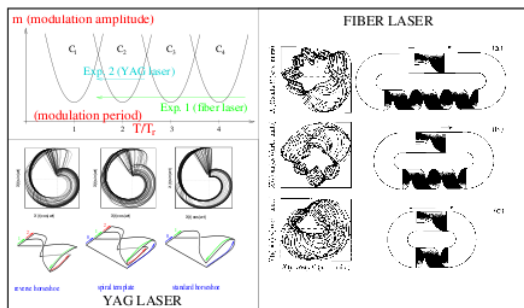
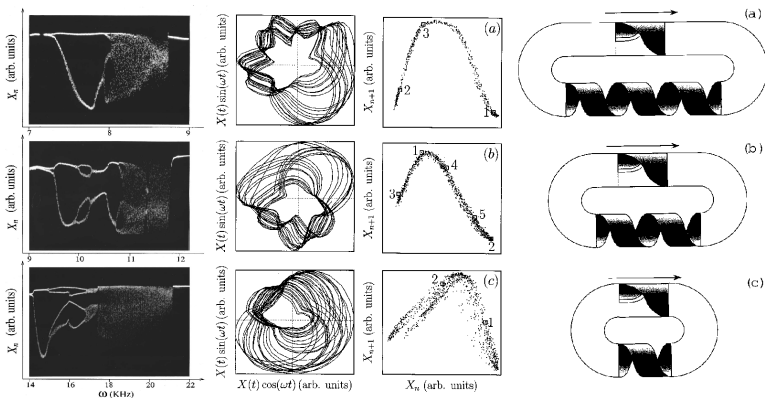


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Perestroikas of Strange Attractors

Evolution Under Parameter Change







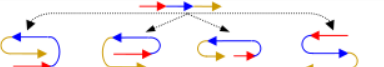


Lefranc - Cargese

Perestroikas of Strange Attractors

Evolution Under Parameter Change

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						

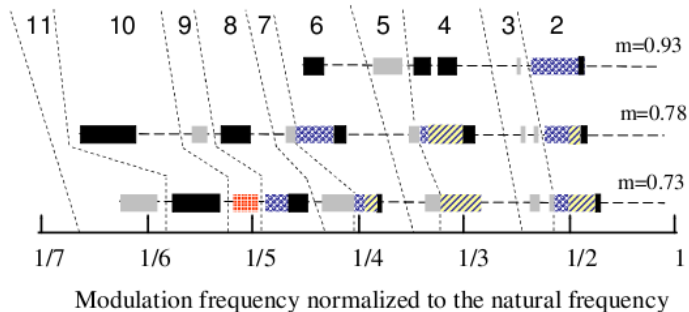
Used and Martin — Zaragosa

Perestroikas of Strange Attractors

The Topology
of Chaos

Robert
Gilmore

Evolution Under Parameter Change



Used and Martin — Zaragosa

Analysis of Nonstationary Data

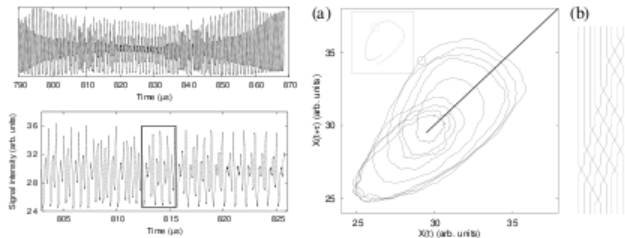


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

Lefranc - Cargese

Compare with Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.

Inequivalent Irreducible Representations

Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Global Torsion & Parity



(a)



$n=2$

(b)

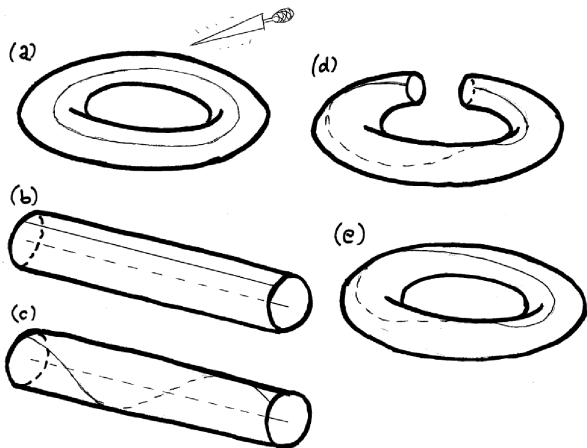


Parity=-1

(c)

Inequivalence in R^3

Inequivalence in R^3



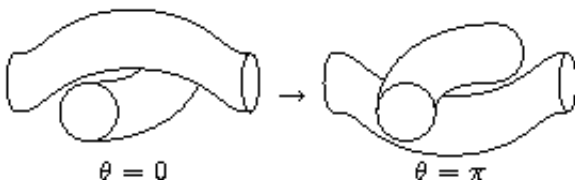
Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

Crossing Exchange in R^4



Parity reversal is also possible in R^4 by isotopy.

2 Twists = 1 Writhe = Identity



$$\mathbb{Z} \longrightarrow \mathbb{Z}_2$$

Global Torsion \longrightarrow Binary Op

Equivalences by Injection Obstructions to Isotopy

$$\begin{array}{ccccc} R^3 & \rightarrow & R^4 & \rightarrow & R^5 \\ \text{Global Torsion} & & \text{Global Torsion} & & \\ \text{Parity} & & & & \\ \text{Knot Type} & & & & \end{array}$$

There is one *Universal* reducible representation in R^N , $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

Can We Classify Strange Attractors?

Chemists have their classification.

Nuclear Physicists have their classification.

Particle Physicists have their classification.

Astronomers have their classification.

Mendeleev's Table of the Chemical Elements

The Topology
of Chaos

Robert
Gilmore

Intro-01

Intro-02

Intro-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

PERIODIC TABLE OF THE ELEMENTS

<http://www.kf-split.hr/periodic/en/>

PERIOD	GROUP		RELATIVE ATOMIC MASS(1)																GROUP																	
	1	IA	GROUP IUPAC										13	IIIA	GROUP CAS						18	VIIIA														
			ATOMIC NUMBER												SYMBOL																					
			ELEMENT NAME																																	
1	1	1.0079											2	4.0026							2	4.0026														
1	1	H											2	He							2	He														
2	3	6.941	4	9.0122											5	10.811							10	20.180												
2	3	Li	4	Be											5	B	6	C	7	N	8	O	9	F	10	Ne										
2	3	LITHIUM	4	BERYLLIUM											5	BORON	6	CARBON	7	NITROGEN	8	OXYGEN	9	FLUORINE	10	NEON										
3	11	22.990	12	24.305											13	26.982	14	28.086	15	30.974	16	32.065	17	35.453	18	39.948										
3	11	Na	12	Mg											13	Al	14	Si	15	P	16	S	17	Cl	18	Ar										
3	11	SODIUM	12	MAGNESIUM											13	ALUMINUM	14	SILICON	15	PHOSPHORUS	16	SULFUR	17	CHLORINE	18	ARGON										
4	19	39.098	20	40.078	21	44.956	22	47.867	23	50.942	24	51.996	25	54.938	26	55.845	27	58.933	28	58.933	29	63.546	30	65.38	31	69.723	32	72.04	33	74.922	34	78.96	35	79.904	36	83.80
4	19	K	20	Ca	21	Sc	22	Ti	23	V	24	Cr	25	Mn	26	Fe	27	Co	28	Ni	29	Cu	30	Zn	31	Ga	32	Ge	33	As	34	Se	35	Br	36	Kr
4	19	POTASSIUM	20	CALCIUM	21	SCANDIUM	22	TITANIUM	23	VANADIUM	24	CHROMIUM	25	MANGANESE	26	IRON	27	COBALT	28	NICKEL	29	COPPER	30	ZINC	31	GALLIUM	32	GERMANIUM	33	ARSENIC	34	SELENIUM	35	BROMINE	36	KRYPTON
5	37	85.468	38	87.62	39	88.906	40	91.224	41	92.906	42	95.94	43	(98)	44	101.07	45	102.91	46	106.42	47	107.87	48	112.41	49	114.82	50	118.71	51	121.76	52	127.60	53	126.90	54	131.29
5	37	Rb	38	Sr	39	Y	40	Zr	41	Nb	42	Mo	43	Tc	44	Ru	45	Rh	46	Pd	47	Ag	48	Cd	49	In	50	Sn	51	Sb	52	Te	53	I	54	Xe
5	37	RUBIDIUM	38	STRONTIUM	39	YTRBIUM	40	ZIRCONIUM	41	NIOBIUM	42	MOLYBDENUM	43	TECHNETIUM	44	RUTHENIUM	45	RHODIUM	46	PALLADIUM	47	SILVER	48	CADMIUM	49	INDIUM	50	TIN	51	ANTIMONY	52	TELLEURIUM	53	IODINE	54	XENON
6	55	132.91	56	137.33	57-71	72	178.49	73	180.96	74	183.84	75	186.21	76	190.23	77	192.22	78	195.08	79	196.97	80	200.59	81	204.38	82	207.2	83	208.98	84	(209)	85	(210)	86	(222)	
6	55	Cs	56	Ba	57-71	72	Hf	73	Ta	74	W	75	Re	76	Os	77	Ir	78	Pt	79	Au	80	Hg	81	Tl	82	Pb	83	Bi	84	Po	85	At	86	Rn	
6	55	CAESIUM	56	BARIUM	57-71	72	HAFNIUM	73	TANTALUM	74	TUNGSTEN	75	RHENIUM	76	OSMIUM	77	IRIDIUM	78	PLATINUM	79	GOLD	80	MERCURY	81	THALLIUM	82	LEAD	83	BISMUTH	84	POLONIUM	85	ASTATINE	86	RADON	
7	87	(223)	88	(226)	89-103	104	(261)	105	(262)	106	(265)	107	(264)	108	(277)	109	(268)	110	(281)	111	(272)	112	(285)	113	(284)	114	(289)	115	(288)	116	(289)	117	(285)	118	(289)	
7	87	Fr	88	Ra	89-103	104	Rf	105	Db	106	Sg	107	Bh	108	Hs	109	Mt	110	Uu	111	Uu	112	Uu	113	Uu	114	Uu	115	Uu	116	Uu	117	Uu	118	Uu	
7	87	FRANCIUM	88	RADIUM	89-103	104	RIFRIDIUM	105	DOBRODIUM	106	SEABORGIUM	107	BOHRNIUM	108	HASSIUM	109	METIWIUM	110	UNUNNIUM	111	UNUNNIUM	112	UNUNNIUM	113	UNUNNIUM	114	UNUNNIUM	115	UNUNNIUM	116	UNUNNIUM	117	UNUNNIUM	118	UNUNNIUM	

(1) Pure Appl. Chem., 73, No. 4, 987-993 (2001)

Relative atomic mass is shown with five significant figures. For elements with no stable nuclides, the value enclosed in brackets indicates the mass number of the longest-lived isotope of the element.

However three such elements (Tl, Po, and U) do have a characteristic terrestrial isotopic composition, and for these an atomic weight is tabulated.

Editor: Aditya Varshni (adiva@rediffmail.com)

LANTHANIDE

57	138.91	58	140.12	59	140.91	60	144.24	61	(145)	62	150.36	63	151.96	64	157.25	65	158.93	66	162.50	67	164.93	68	167.26	69	168.93	70	173.04	71	174.97
La		Ce		Pr		Nd		Pm		Sm		Eu		Gd		Tb		Dy		Ho		Er		Tm		Yb		Lu	
LANTHANUM		CERIUM		PRASEODYMIUM		NEODYMIUM		PROMETHIUM		SAMARIUM		EUROPIUM		GADOLINIUM		TERBIUM		DYSPROSIUM		HOLMIUM		ERBIUM		THULIUM		YTTERIUM		LUTETIUM	

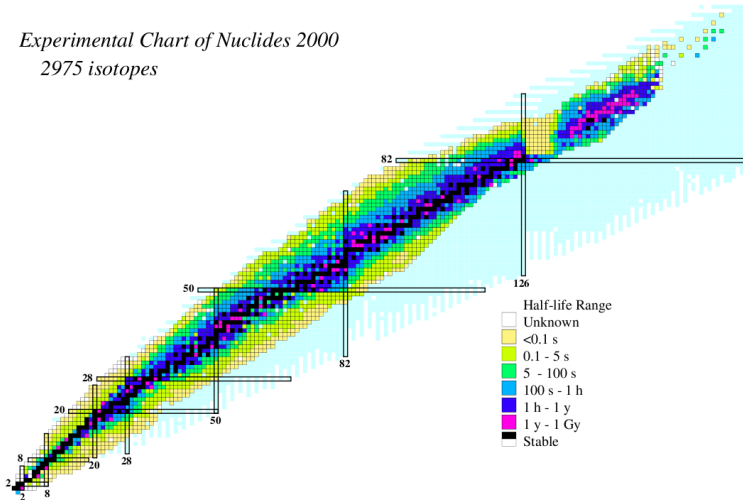
ACTINIDE

89	(227)	90	232.04	91	231.04	92	238.03	93	(237)	94	(244)	95	(243)	96	(247)	97	(247)	98	(251)	99	(252)	100	(257)	101	(258)	102	(259)	103	(262)
Ac		Th		Pa		U		Np		Pu		Am		Cm		Bk		Cf		Es		Fm		Md		No		Lr	
ACTINIUM		THORIUM		PROTACTINIUM		URANIUM		NEPTUNIUM		PLUTONIUM		AMERICIUM		CURSIUM		BERKELIUM		CALIFORNIUM		ENSGTENIUM		FERMIUM		MENDELEVIUM		NOBELIUM		LAWRENCIUM	

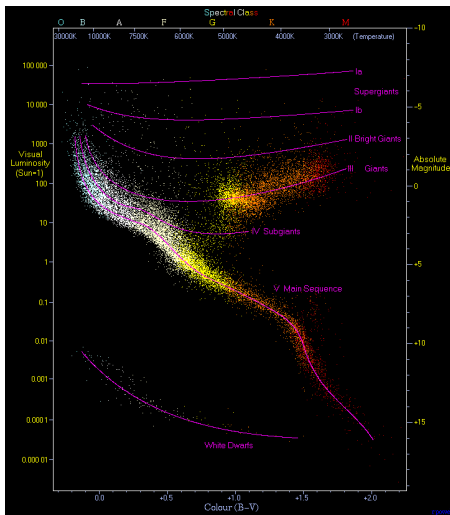
Copyright © 1999-2003 ENIG (enig@kf-split.hr)

Table of Atomic Nuclei

Experimental Chart of Nuclides 2000
2975 isotopes



Hertzsprung-Russell Diagram



The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

How Does it Work Out?

Chemical Elements	1 Integer: N_P
Atomic Nuclei	2 Integers: N_P, N_N
Stars	1 Continuous variable M + exceptions
Strange Attractors	????

An Experimental Study

What We Did

We analyzed data from a Laser with Saturable Absorber (LSA).

3 Different absorbers were used.

For each absorber data were taken under 6 - 10 operating conditions.

There was a total of 25 different data sets.

We wanted to “prove experimentally” that changing the absorber/operating condition served merely to push to flow around on the same branched manifold.

An Experimental Observation

What We Found

When certain orbits (UPOs) were present they were invariably accompanied by a specific set of other orbits.

This led us to propose that certain orbits ‘force’ other orbits.

Forcing is topological.

A discrete set of “basis orbits” serves to identify the complete collection of UPOs present in an attractor.

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

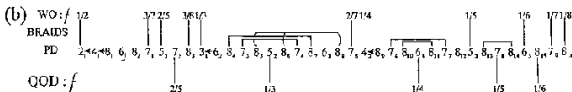
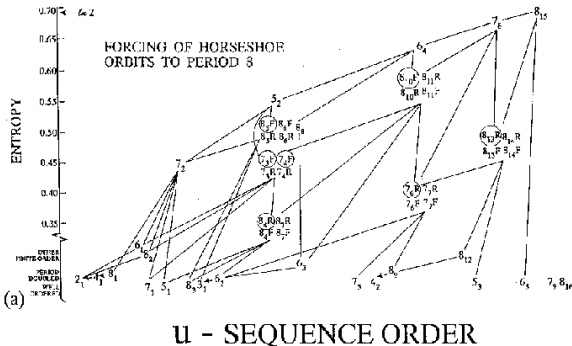
Exp'tal-08

Embed-01

Embed-02

Embed-03

Forcing Diagram - Horseshoe



Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required
- Higher dimension - ???

Constraints

Branched manifolds largely constrain the 'perestroikas' that forcing diagrams can undergo.

Is there some mechanism/structure that constrains the types of perestroikas that branched manifolds can undergo?

Constraints on Branched Manifolds

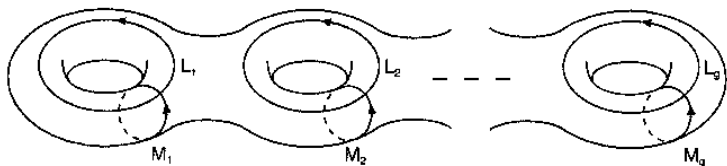
“Inflate” a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Tori are identified by genus g and dressed with a surface flow induced from that creating the strange attractor.

Surface Singularities

Flow field: three eigenvalues: +, 0, -

Vector field “perpendicular” to surface

Eigenvalues on surface at fixed point: +, -

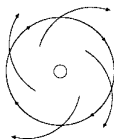
All singularities are regular saddles

$$\sum_{s.p.} (-1)^{\text{index}} = \chi(S) = 2 - 2g$$

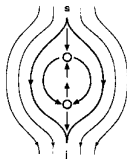
fixed points on surface = index = $2g - 2$

Singularities organize the surface flow dressing the torus

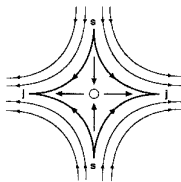
Flow Near a Singularity



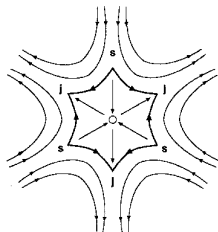
(a)



(b)



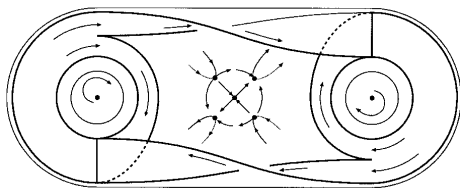
(c)



(d)

Some Bounding Tori

Torus Bounding Lorenz-like Flows



The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

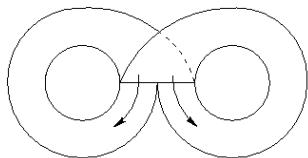
Exp'tal-08

Embed-01

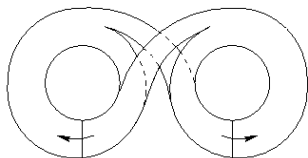
Embed-02

Embed-03

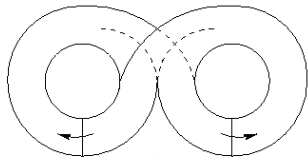
Twisting the Lorenz Attractor



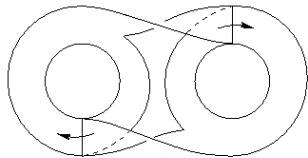
(a)



(c)



(b)



(d)

Constraints Provided by Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

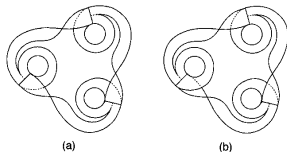
Exp'tal-08

Embed-01

Embed-02

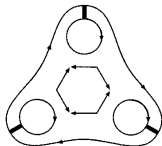
Embed-03

Two possible branched manifolds in the torus with $g=4$.



(a)

(b)



(c)

Labeling Bounding Tori

Poincaré section is disjoint union of $g-1$ disks.

Transition matrix sum of two $g-1 \times g-1$ matrices.

Both are $g-1 \times g-1$ permutation matrices.

They identify mappings of Poincaré sections to P'sections.

Bounding tori labeled by (permutation) group theory.

Some Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

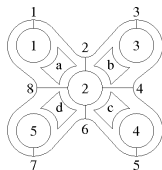
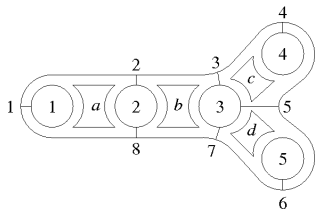
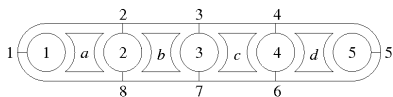
g	m	(p_1, p_2, \dots, p_m)	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11313133
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

The Growth is Exponential

TABLE I: Number of canonical bounding tori as a function of genus, g .

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

Some Genus-9 Bounding Tori



Aufbau Princip for Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

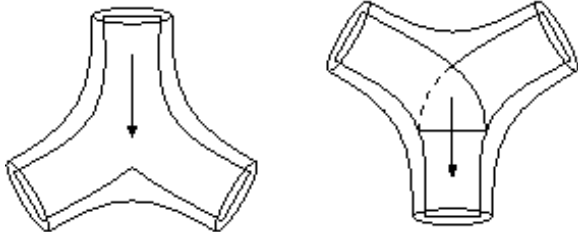
Exp'tal-08

Embed-01

Embed-02

Embed-03

Aufbau Princip for Bounding Tori



These units ("pants, trinions") surround the stretching and squeezing units of branched manifolds.

Aufbau Princip for Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

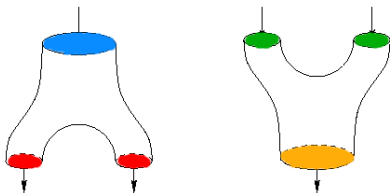
Exp'tal-08

Embed-01

Embed-02


Embed-03

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Construction of Poincaré Section

P. S. = Union 

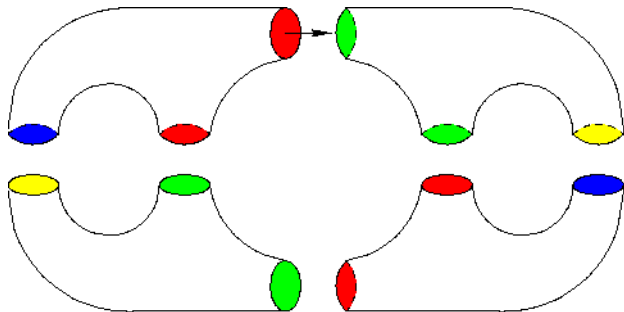
Components = $g-1$

Aufbau Princip for Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Application: Lorenz Dynamics, $g=3$



Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03

Representation Theory for $g > 1$

Can we extend the representation theory for strange attractors “with a hole in the middle” (i.e., genus = 1) to higher-genus attractors?

Yes. The results are similar.

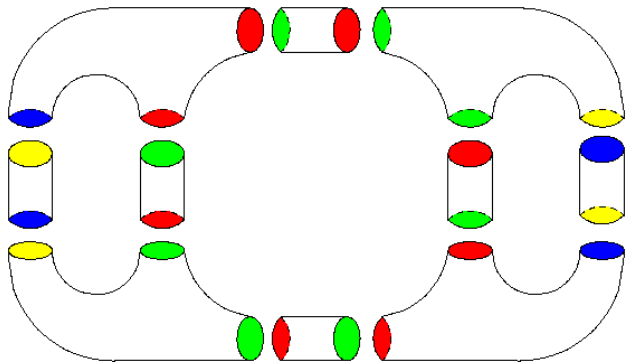
Begin as follows:

Aufbau Princip for Bounding Tori

The Topology
of Chaos

Robert
Gilmore

Application: Lorenz Dynamics, $g=3$



Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

Exp'tal-07

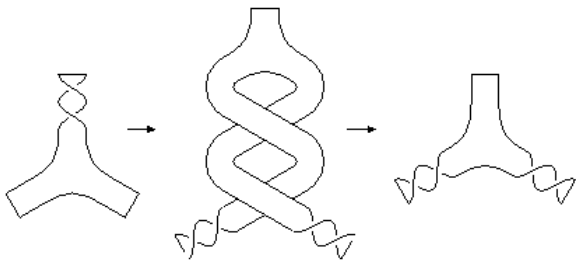
Exp'tal-08

Embed-01

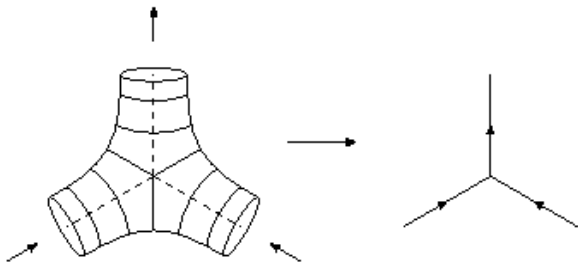
Embed-02

Embed-03

Embeddings

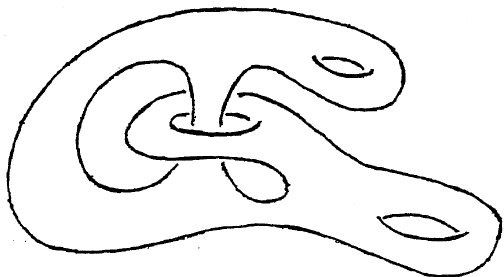


Preparations for Embedding tori into



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Intrinsic Tori



A specific simple example.

Partial classification by links of homotopy group generators.

Nightmare Numbers are Expected.

Equivalences by Injection

Obstructions to Isotopy

Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus- g strange attractor become equivalent under isotopy.

Summary

**1 Question Answered \Rightarrow
2 Questions Raised**

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

**There is now a classification theory
for low-dimensional strange attractors.**

- ① It is topological
- ② It has a hierarchy of 4 levels
- ③ Each is discrete
- ④ There is rigidity and degrees of freedom
- ⑤ It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

The Classification Theory has 4 Levels of Structure

① Basis Sets of Orbits

The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds

The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori

The Classification Theory has 4 Levels of Structure

- 1 Basis Sets of Orbits
- 2 Branched Manifolds
- 3 Bounding Tori
- 4 Extrinsic Embeddings

Four Levels of Structure

The Topology of Chaos

Robert Gilmore

Intro.-01

Intro.-02

Intro.-03

Exp'tal-01

Exp'tal-02

Exp'tal-03

Exp'tal-04

Exp'tal-05

Exp'tal-06

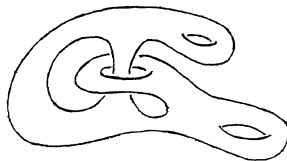
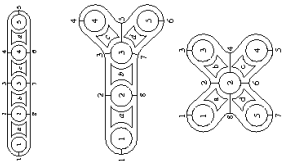
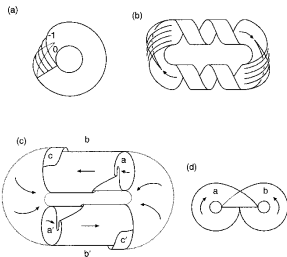
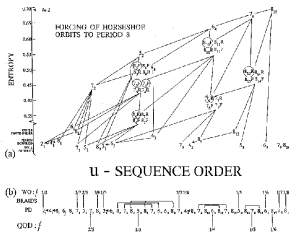
Exp'tal-07

Exp'tal-08

Embed-01

Embed-02

Embed-03



Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

Answered Questions

There is a Representation Theory for Strange Attractors

There is a complete set of representation labels for strange attractors of any genus g .

The labels are complete and discrete.

Representations can become equivalent when immersed in higher dimension.

All representations (embeddings) of a 3-dimensional strange attractor become isotopic (equivalent) in R^5 .

The *Universal Representation* of an attractor in R^5 identifies mechanism. No embedding artifacts are left.

The topological index in R^5 that identifies mechanism remains to be discovered.

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos = Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section = union of $g - 1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuation
- Cauchy-Riemann symmetries
- Quantizing Chaos

We hope to find:

- Robust topological invariants for R^N , $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \rightarrow D^2$,
 $n \times D^2 \rightarrow n \times D^2$ (e.g., Lorenz), $D^N \rightarrow D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points
(0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy

