The Topology of Chaos
Chapter 3: Topology of Orbits

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Chaos

Motion that is

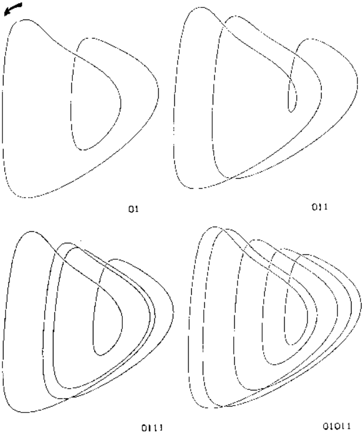
- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

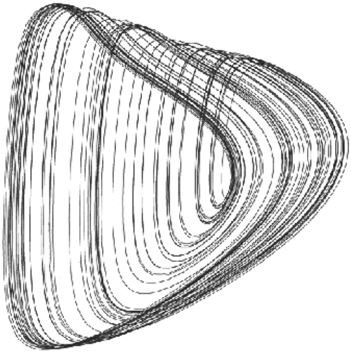
Strange Attractor

The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- **"Abundant"**
- **Outline the Strange Attractor**
- **Are the Skeleton of the Strange Attractor**

UPOs Outline Strange attractors





BZ reaction

UPOs Outline Strange attractors

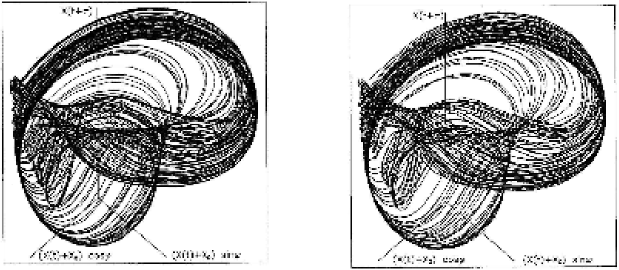


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

Organization of UPOs in R^3: Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

Linking Number of Two UPOs

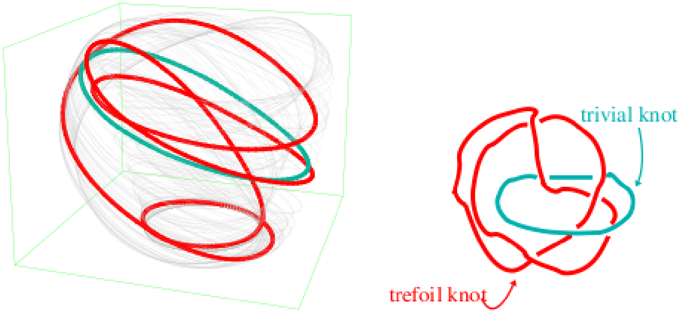
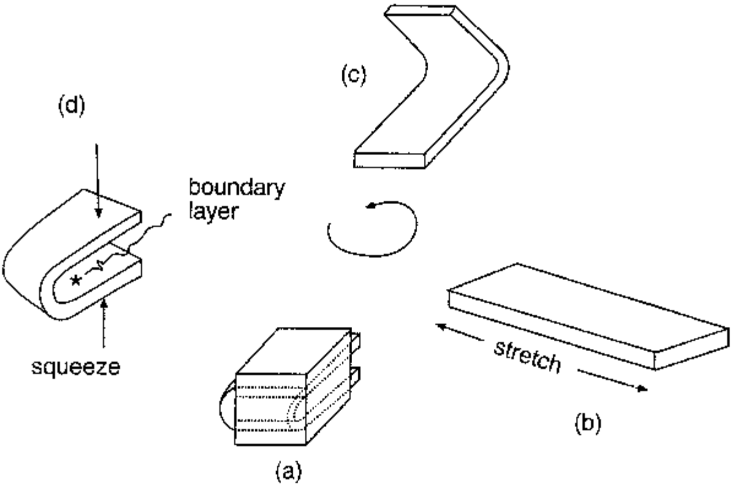


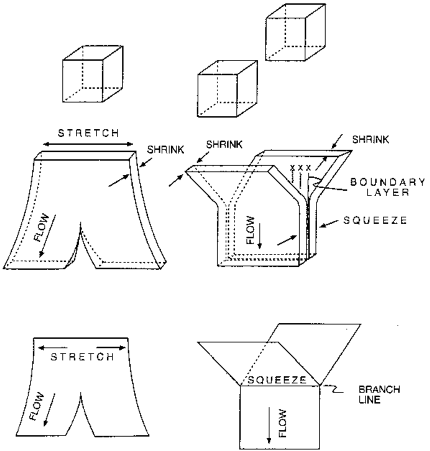
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

One Stretch-&-Squeeze Mechanism



Stretching — Squeezing

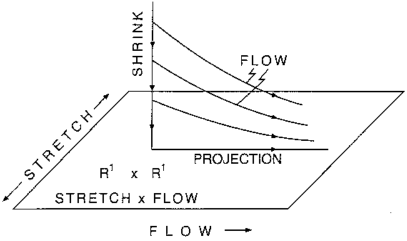


Collapse Along the Stable Manifold

Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n = 3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$
- **Generates a** **hyperbolic** **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

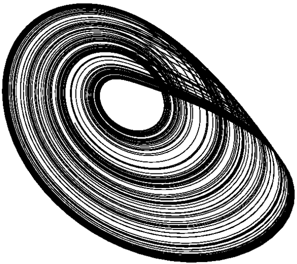
- **The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.**
- **UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.**

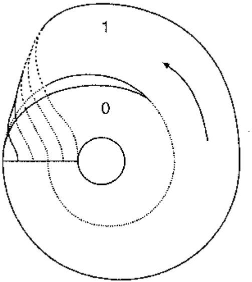
Remark: "One of the few theorems useful to experimentalists."

A Very Common Mechanism

Rössler:

Attractor **Branched Manifold**

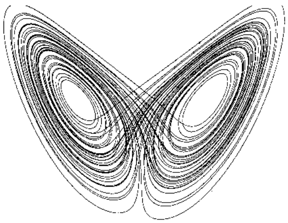


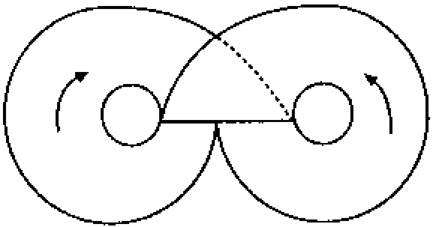


A Mechanism with Symmetry

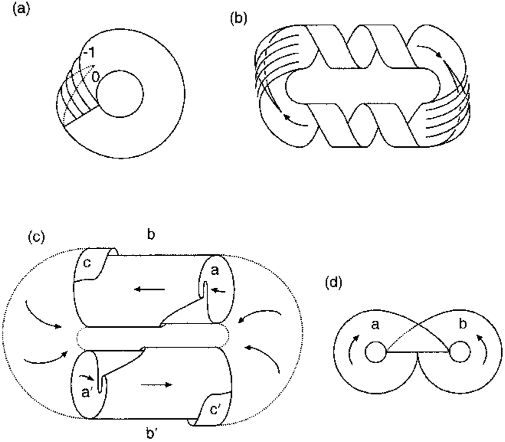
Lorenz:

Attractor **Branched Manifold**

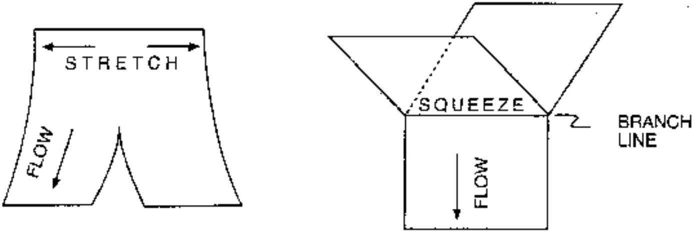




Inequivalent Branched Manifolds



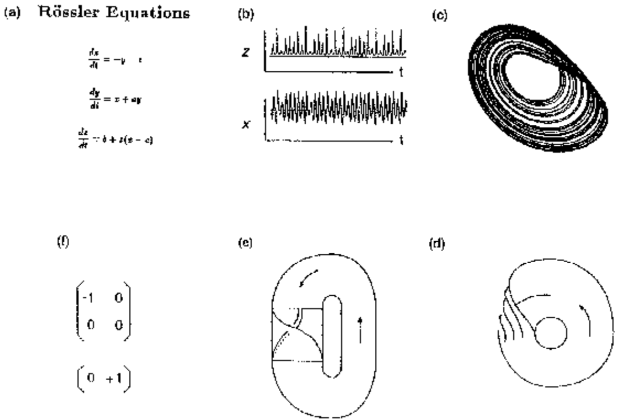
Any branched manifold can be built up from stretching and squeezing units



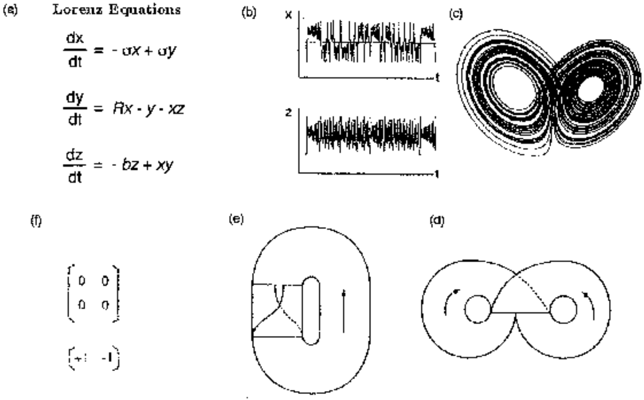
subject to the conditions:

- **Outputs to Inputs**
- **No Free Ends**

Rossler System



Lorenz System



Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$