The Topology of Chaos
Chapter 6: Bounding Tori

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September 2, 2008

Constraints on Branched Manifolds

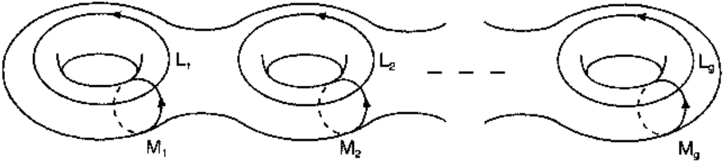
"Inflate" a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Surface Singularities

Flow field: three eigenvalues: +, 0, −

Vector field "perpendicular" to surface

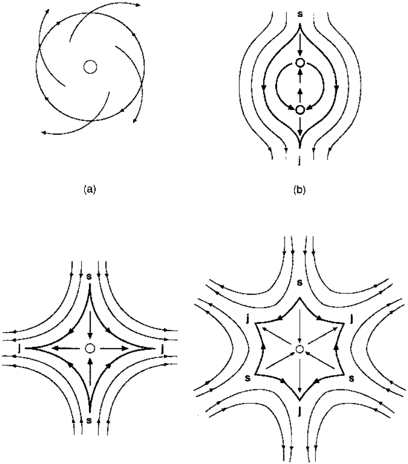
Einvalues on surface at fixed point: +, −

All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

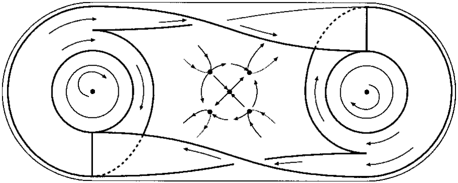
 # fixed points on surface = index = 2g - 2

Flow Near a Singularity

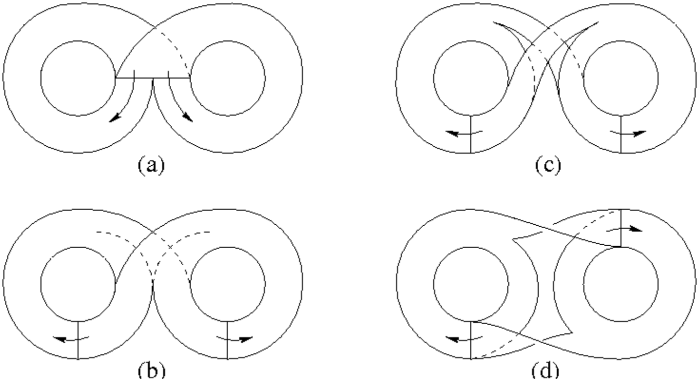


(a) (b) (c) (d)

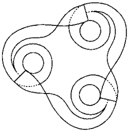
Torus Bounding Lorenz-like Flows

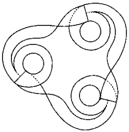


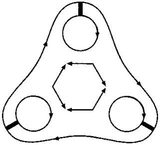
Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.


(a)


(b)


(c)

Bounding Tori contain all known Strange Attractors

Tab.1. All known strange attractors of dimension $d_L < 3$ are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g-1$ Orbit
Rossler, Duffing, Burke and Shaw	A_1	1
Various Lasers, Gateau Roule	A_1	1
Neuron with Subthreshold Oscillations	A_1	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	A_2	$(12)^2$
Multispiral attractors	A_n	$(12^{n-1})^2$
C_n Covers of Rossler	C_n	1^n
C_2 Cover of Lorenz$^{(a)}$	C_4	1^4
C_2 Cover of Lorenz$^{(b)}$	A_3	$(122)^2$
C_n Cover of Lorenz$^{(a)}$	C_{2n}	1^{2n}
C_n Cover of Lorenz$^{(b)}$	P_{n+1}	$(1n)^n$
$2 \rightarrow 1$ Image of Fig. 8 Branched Manifold	A_3	$(122)^2$
Fig. 8 Branched Manifold	P_5	$(14)^4$

$^{(a)}$ Rotation axis through origin.
$^{(b)}$ Rotation axis through one focus.

Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks

Transition matrix sum of two g-1 $\times$ g-1 matrices

One is cyclic g-1 $\times$ g-1 matrix

Other represents union of cycles

Labeling via (permutation) group theory

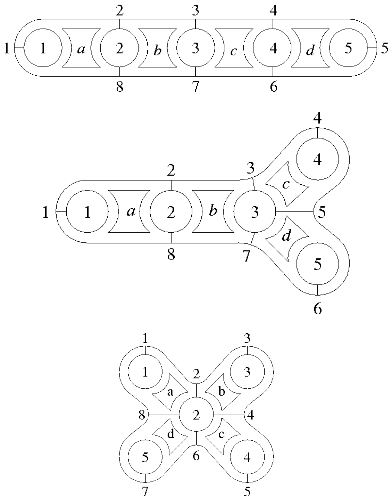
Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

g	m	$(p_1, p_2, \dots p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

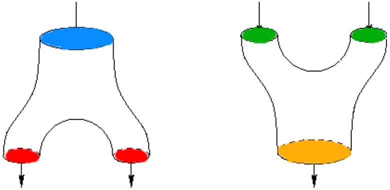
Motivation

Some Genus-9 Bounding Tori



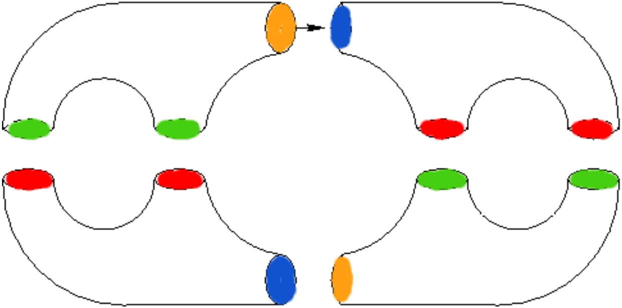
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Application: Lorenz Dynamics, g=3



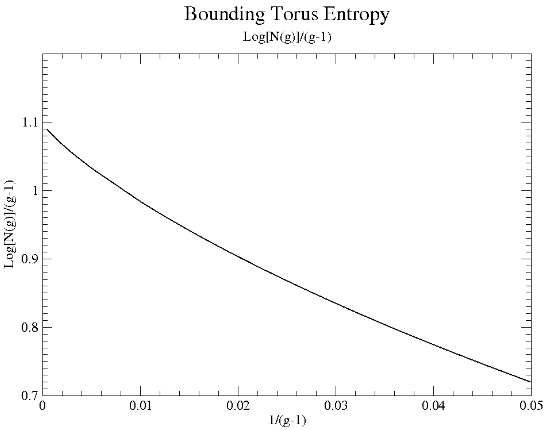
Construction of Poincaré Section

The Growth is Exponential

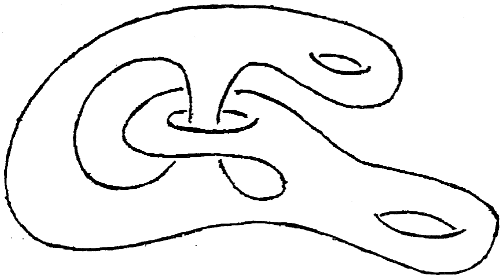
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

The Growth is Exponential
The Entropy is log 3



Extrinsic Embedding of Intrinsic Tori



Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.