Alice in Stretch & SqueezeLand: 15 Knife Map

August 16, 2012

Chapter Abstract

What is the order of orbit creation in the Lorenz attractor? The attractor is created by a tearing and squeezing mechanism since $g > 1$.

How are these orbits organized?

For attractors created by a stretch and fold mechanism $(g = 1)$ the logistic map $x' = f(x; a) = a - x^2$ provides useful insight.

For attractors created by a tear and squeeze mechanism $(g > 1)$ the knife map $y' = g(y; b) = b - |y|^{1/2}$ provides useful insight.

The two maps share many similarities and exhibit important differences.

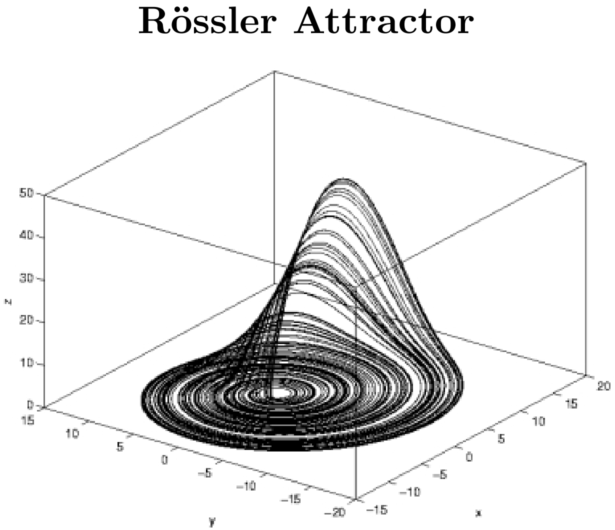
What We Did

1. Studied maps with 2 branches
2. L & R
3. Separated by a singularity
4. Models for Tearing Mechanism
5. Looked for "universality"
6. Searched for scaling

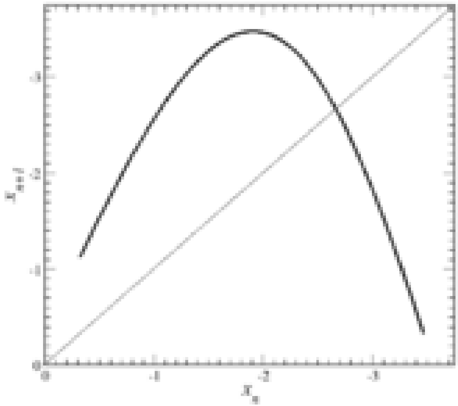
What We Found

1. Simple form: $x' = a - |x|^k$
2. $k = 2 \simeq$ folding; $k = \frac{1}{2} \simeq$ tearing
3. Localized global attractor
4. *Either* chaos or pd. 1 fixed point
5. Orbits of periods 1 and 2 organize systematics
6. Explosions
7. Prime and compound orbits
8. Local and Global focus points

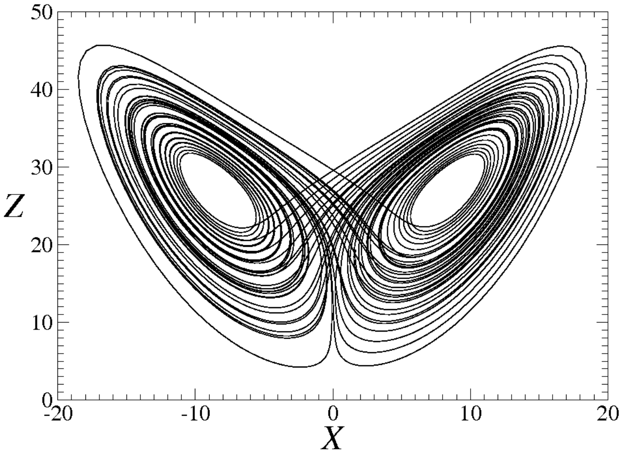
Rössler Attractor



Rössler Attractor - Return Map



Lorenz Attractor



Return Map for Lorenz Attractor

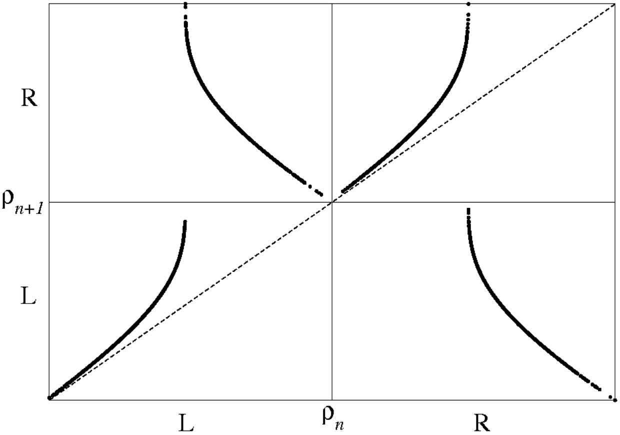
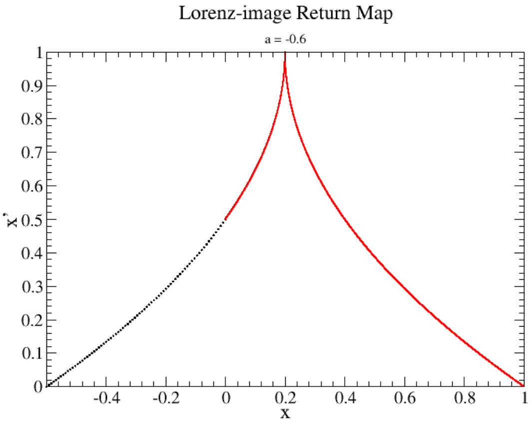
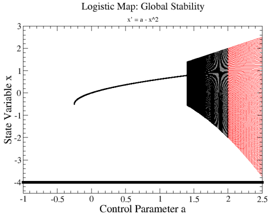
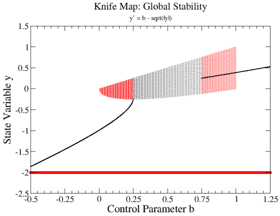


Image of Lorenz Return Map

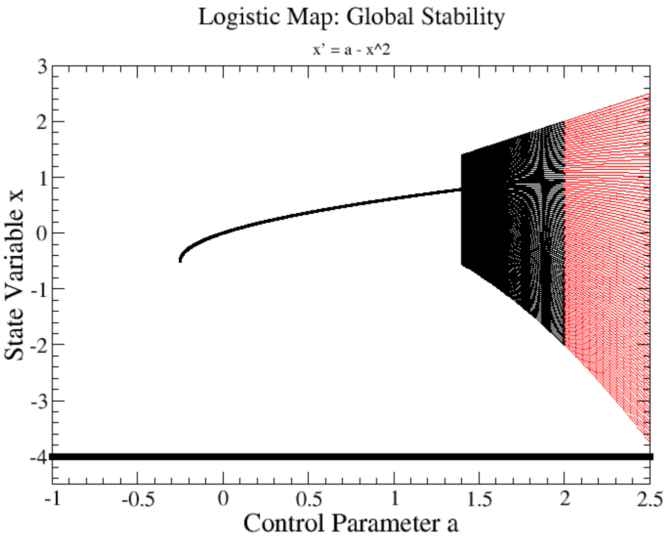


Stability Regions

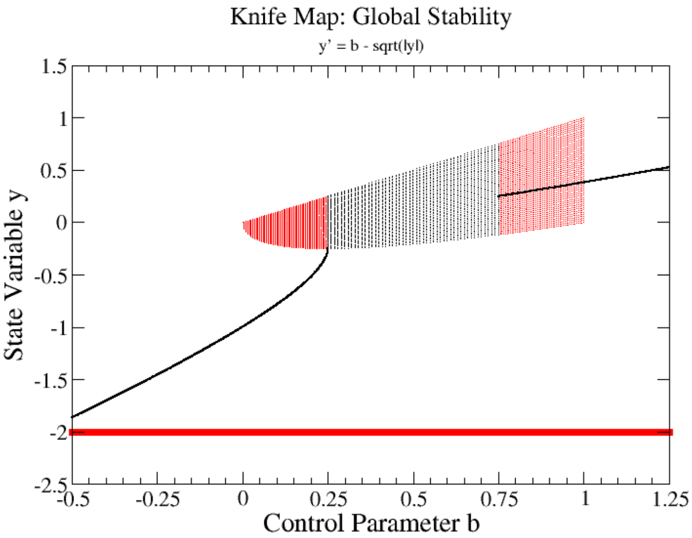




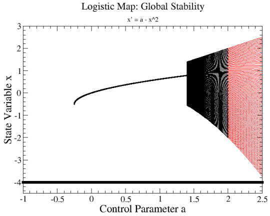
BigView: Logistic Map

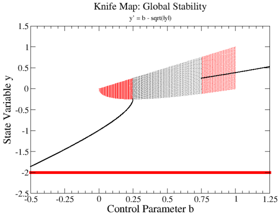


BigView: Knife Map

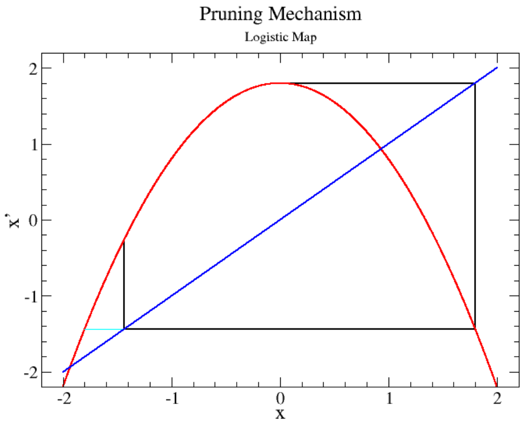


Stability Regions

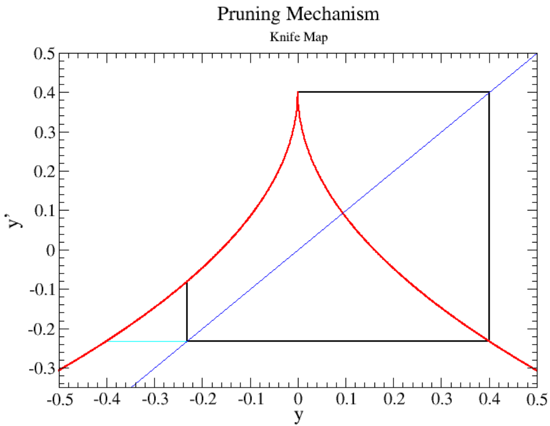




Return Map - Rössler Attractor



Return Map - Lorenz Image



Return Map Approximations

The Rossler return map is well approximated by the following maps:

$$
\begin{aligned}
x' &= \lambda x(1-x) \\
x' &= a - x^2 \\
x' &= 1 - \mu x^2 \\
x' &= 1 - \left| \frac{x-m}{w} \right|^2
\end{aligned}
$$

Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

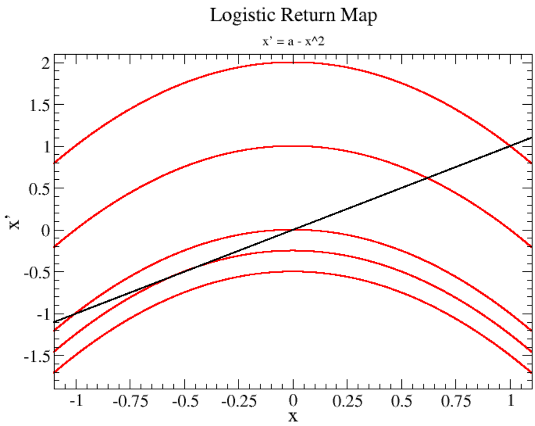
$$
\begin{aligned}
y' &= b - |y|^{1/2} \\
y' &= 1 - \mu |y|^{1/2} \\
y' &= 1 - \left| \frac{y - m}{w} \right|^{1/2}
\end{aligned}
$$

Comparison:

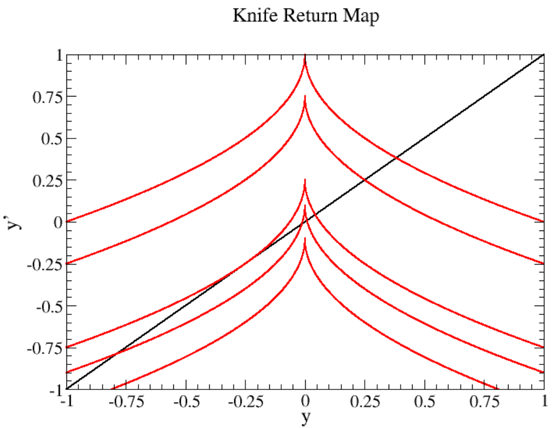
Logistic & Knife Maps

Logistic Map	Knife Map
$x' = f(x; a) = a - (\|x\|)^2$	$y' = f(y; b) = b - (\|y\|)^{1/2}$

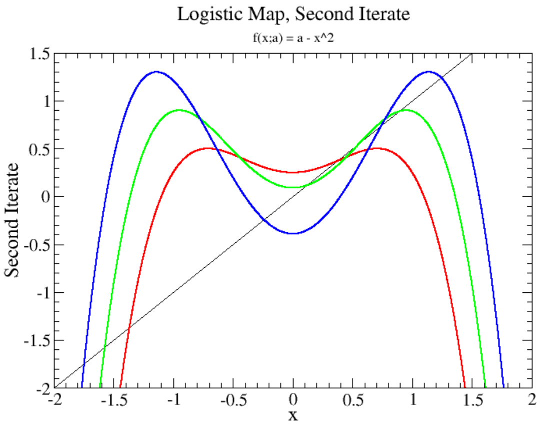
... for several values of a



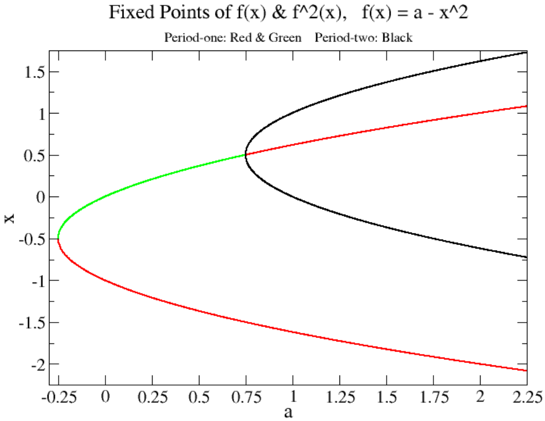
Knife Return Maps



Second Return Map

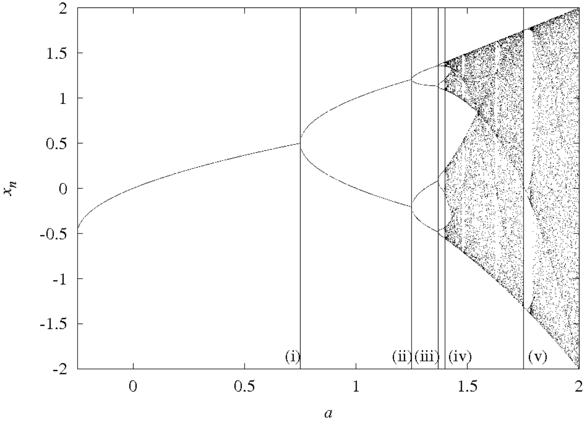


Period 1 & 2 Orbits - Logistic



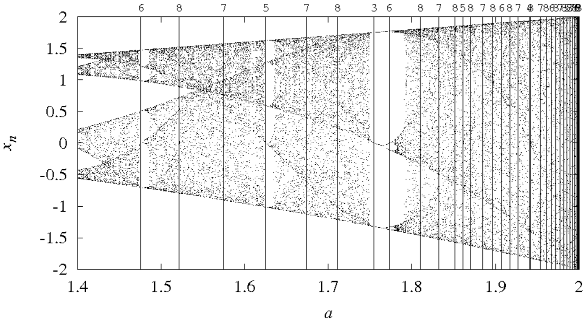
Bifurcation-01

Bifurcation Diagram

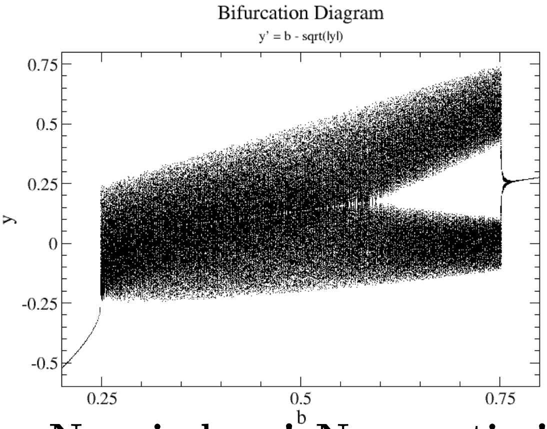


Bifurcation-02

.. Blow Up .... with Caustics

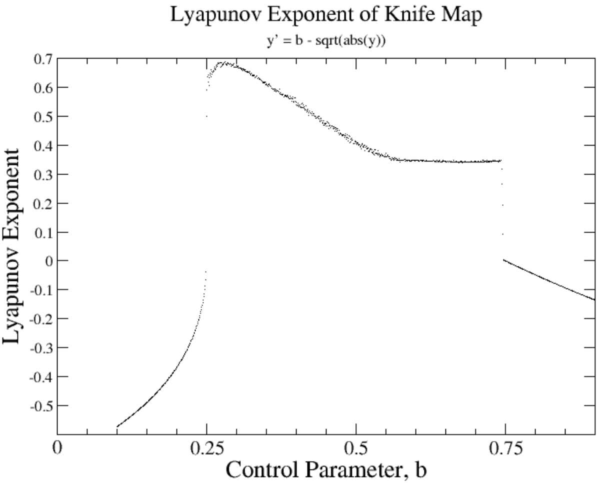


Knife Map - Bifurcation Diagram

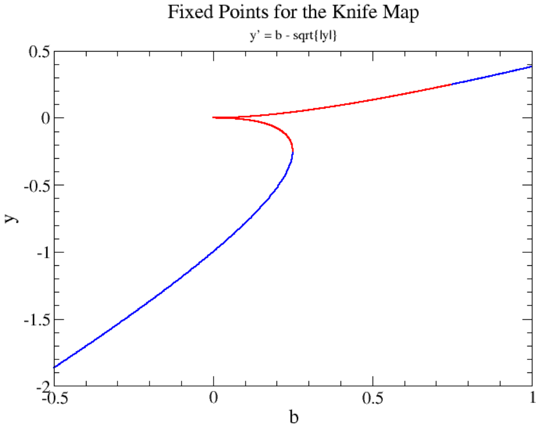


No windows! No caustics!

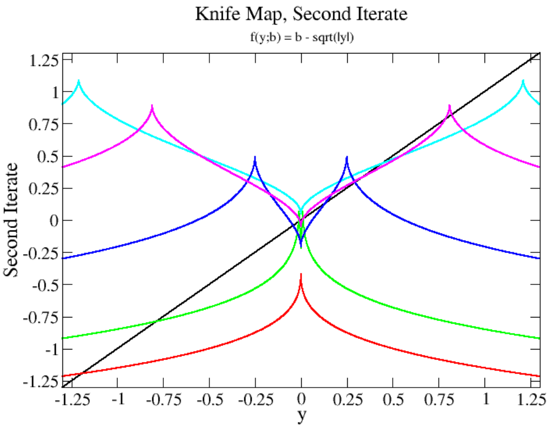
Knife Map - Lyapunov Exponent



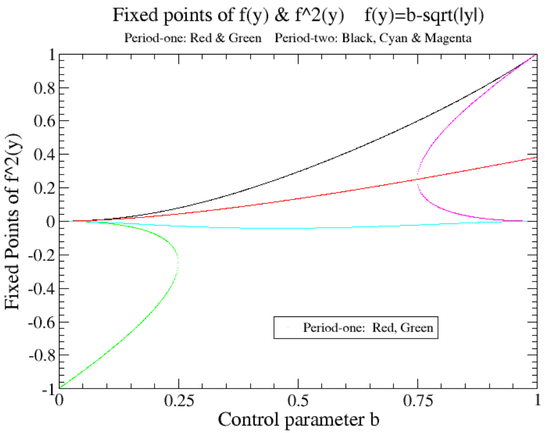
Fixed Points (Knife)



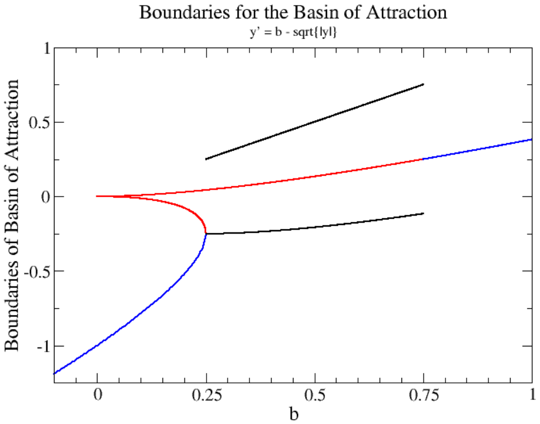
Second Iterates - Knife Map



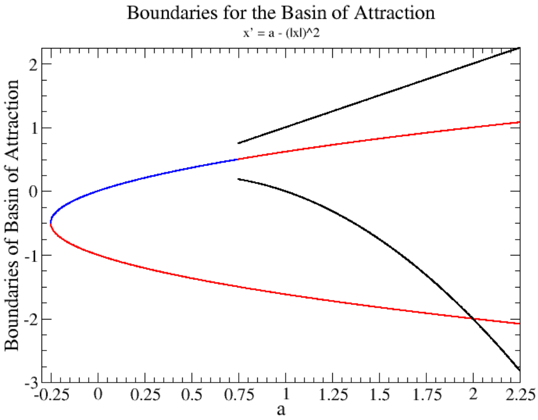
Period-One & Period-Two Orbits



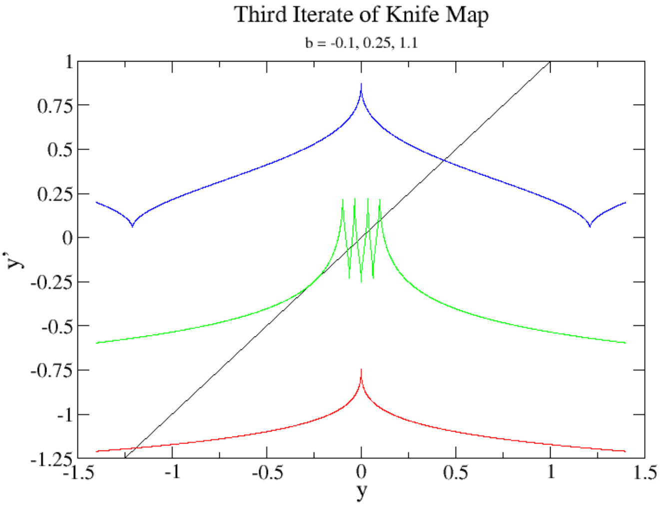
Attractor boundary (Knife)



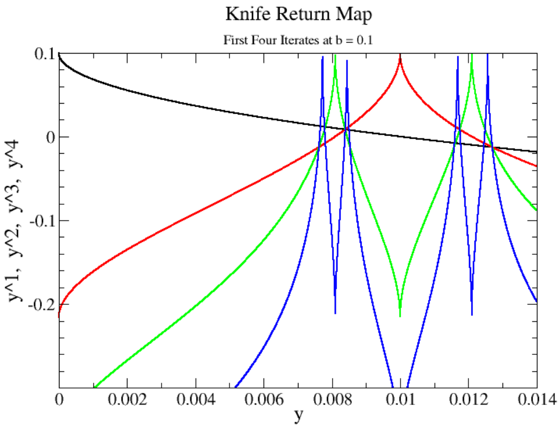
Attractor Boundaries - Logistic



Rite of Passage-01



Knife Map Iterates



Explosions-02

Table: Values $M^{(p)}$ of y where the pth iterate $f^{(p)}(y;b)$ has maxima. These locations are determined by a simple recursion relation (last line) where the indices $s_p = \pm 1$ are incoherent.

p	Number Max.	Coordinate Values
1	1	0
2	2	$\pm b^2$
3	4	$\pm(b \pm b^2)^2$
$\dots$	$\dots$	$\dots$
$p+1$	2^p	$M^{(p+1)} = s_p(b + M^{(p)})^2$

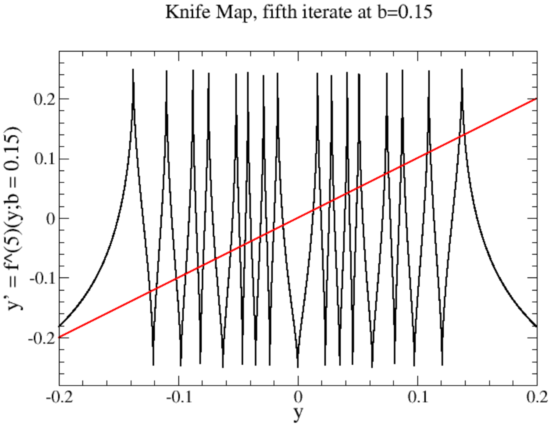
Explosions-03

As $p \to \infty$, with all $s_j = +1$, the abscissa of the rightmost point goes to a limit. The quadratic equation for this limit gives:

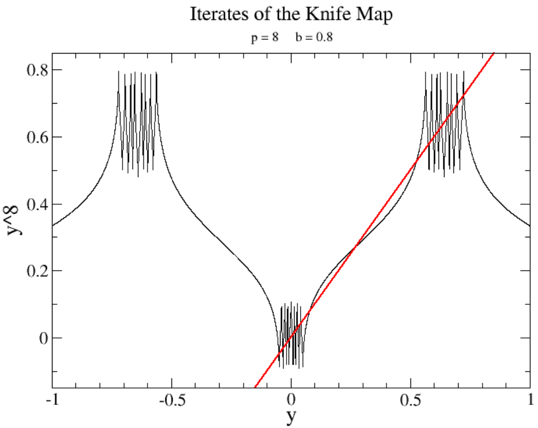
$$y(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b} \right)$$

At $b = \frac{1}{4}$ the bounding box is a square — beyond that the diagonal fails to intersect all the zig - zags. Orbits begin to get pruned away in singular saddle node bifurcations.

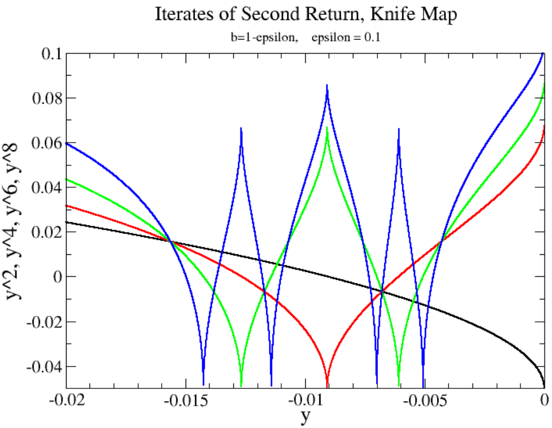
Structural Stability: $0 < b < \frac{1}{4}$



End Play - Near $b = 1$

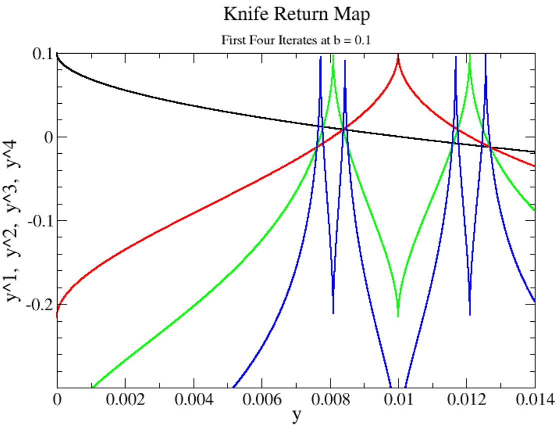


Iterates Near $b = 1$

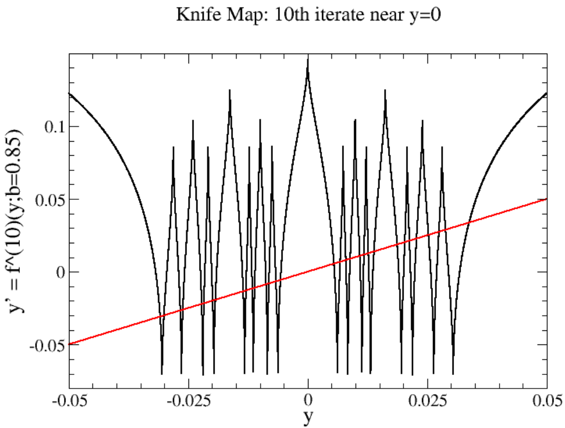


implosion1

Note Scaling Relations

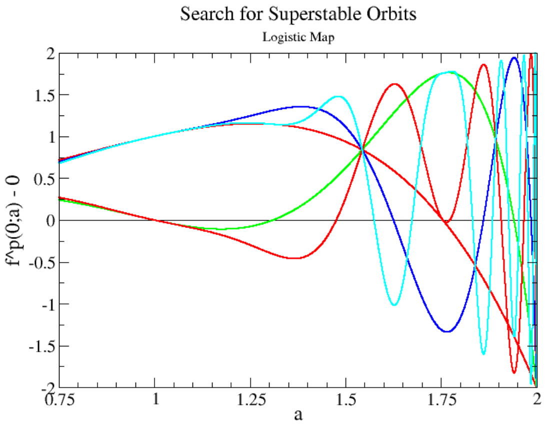


Structural Stability: $\frac{3}{4} < b < 1$

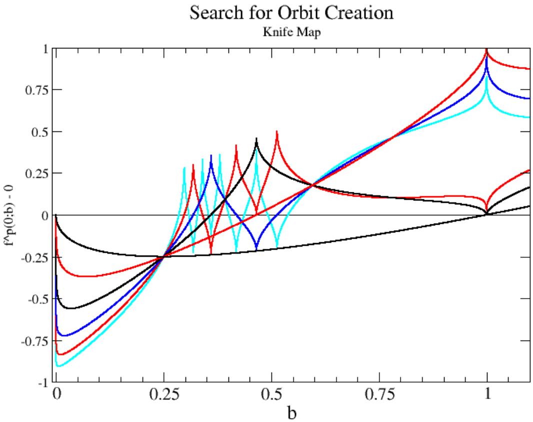


Hunt for Saddle-Node Bifurcations

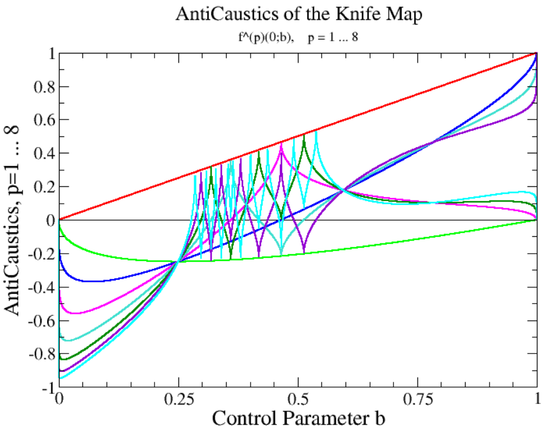
Caustic Crossings



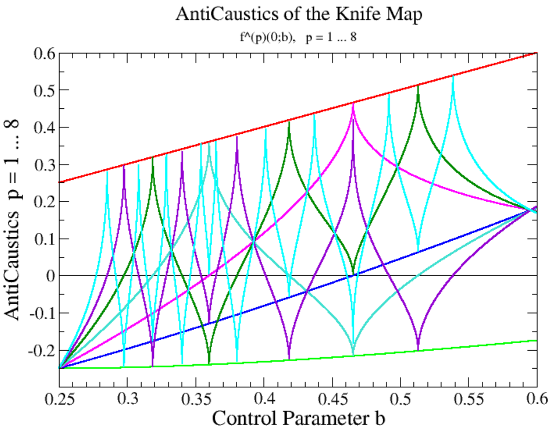
Hunt for Singular SNBs



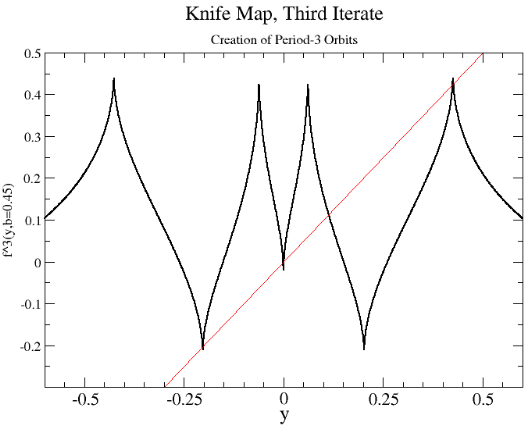
Anti Caustic Crossings



Anti Caustic Crossings: Expansion



Period Three Singular SNB



Renormalization-02

Local expression near $y = 0$ for the period-three explosion:

$$h(y;b) = f^{(3)}(y;b) = b - \sqrt{|b - \sqrt{|b - \sqrt{|y|}|}|}$$

$$h(b_3 + \epsilon; y) \to \left(b_3 - \sqrt{\sqrt{b_3} - b_3}\right) +$$

$$\left(1 + \frac{2\sqrt{b_3} - 1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\epsilon + \left(\frac{1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\sqrt{|y|}$$

Renormalization-03

Renormalization for the period-three explosion.

$$y' = h(y; b_3 + \epsilon) \to \Delta(b - b_3) + \alpha\sqrt{|y|} =$$

$$1.286974759(b - b_3) + 0.7869747590\sqrt{|y|}$$

$$z' = (\Delta/\alpha^2)(b_3 - b) - \sqrt{|z|}$$

Renormalization Algorithm: K10*

1. Write down the symbol sequence for the primary period-p orbit: $K10* = K\sigma_1\sigma_2\cdots\sigma_{p-1}$.
2. Make the identification $\sigma = +1 \rightarrow s = +1, \sigma = 0 \rightarrow s = -1$.
3. Construct $f^{(p)}(b;y) \rightarrow$

$$b - \sqrt{s_{p-1}(b - \cdots \sqrt{s_2(b - \sqrt{s_1(b - \sqrt{y})})} \cdots)}$$

4. Taylor expand this function to terms linear in b and $\sqrt{y}$ and determine the value of b for which the constant term vanishes.

Equations: K10*

For the saddle node pair $5_2 = K1001$ this algorithm gives

$$b - \sqrt{(+1)(b - \sqrt{(-1)(b - \sqrt{(-1)(b - \sqrt{(+1)(b - \sqrt{y}))))}}}}$$

The constant term vanishes for $b = 0.418656$, and for this value of b

$$y' = \Delta(b - b_{5_2}) + \alpha\sqrt{|y|} = -3.231180\Delta b - 1.983690\sqrt{|y|}$$

Results: K10* to Period 6

$$y' = \Delta(b - b_c) + \alpha\sqrt{|y|} \qquad y', y \simeq 0$$

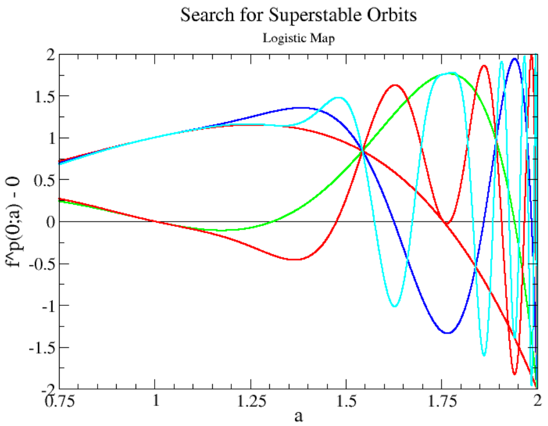
Orbit	Symbolics	b_c	Δ	α
3_1	$K10$	0.465571	1.286974	0.786974
4_2	$K100$	0.360157	2.624703	1.180563
5_3	$K1000$	0.318897	4.647225	1.664335
5_2	$K1001$	0.418656	−3.231180	−1.983690
5_1	$K1011$	0.513175	2.628970	1.509712
6_5	$K10000$	0.297846	7.481728	2.233184
6_4	$K10001$	0.340328	−8.535145	−3.639587
6_3	$K10011$	0.380540	7.596535	3.574548

Renormalization-07

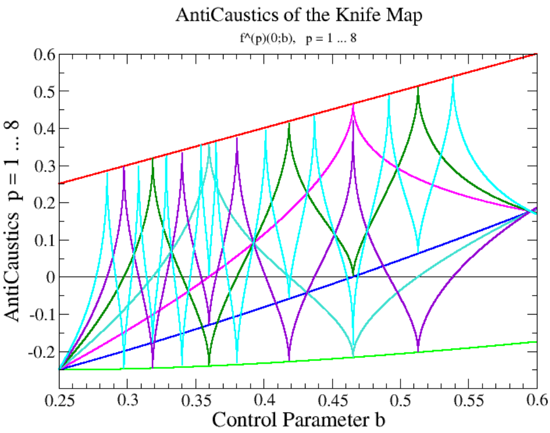
Renormalization for the final period-two explosion.

$$f^{(2)}(1-\epsilon, y) \simeq -\frac{\epsilon}{2} + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)\sqrt{|y|} \tag{1}$$

Hunt for Saddle-Node Bifurcations



Hunt for S. Saddle-Node Bifurcations



Breakpoints

Table: Important parameter values for global stability and unstable periodic orbit behavior.

Global Stability	Unstable Orbits
	0.0
1/4	1/4
	0.5957439420
3/4	
	0.7825988587
	1.0

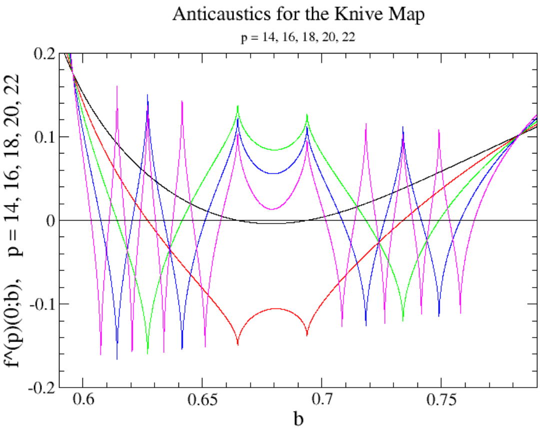
U Sequence

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)[a]

Name	Bifurcation	Name	Bifurcation	Name	Bifurcation
$^0_1$	$1_1[s_1]$	00101 $^0_1$1	$7_3[s_7^3]$	0001 $^0_1$1	$6_4[s_6^3]$
01	$2_1[s_1 \times 2^1]$	001010 $^0_1$1	$8_5[s_8^4]$	000111 $^0_1$1	$8_{11}[s_8^9]$
0111	$4_1[s_1 \times 2^2]$	001 $^0_1$1	$5_2[s_5^2]$	00011 $^0_1$1	$7_7[s_7^7]$
01010111	$8_1[s_1 \times 2^3]$	001110 $^0_1$1	$8_6[s_8^5]$	000110 $^0_1$1	$8_{12}[s_8^{10}]$
0111 $^0_1$1	$6_1[s_6^1]$	00111 $^0_1$1	$7_4[s_7^4]$	000 $^0_1$1	$5_3[s_5^3]$
011111 $^0_1$1	$8_2[s_8^1]$	001111 $^0_1$1	$8_7[s_8^6]$	000010 $^0_1$1	$8_{13}[s_8^{11}]$
01111 $^0_1$1	$7_1[s_7^1]$	0011 $^0_1$1	$6_3[s_6^2]$	00001 $^0_1$1	$7_8[s_7^8]$
011 $^0_1$1	$5_1[s_5^1]$	001101 $^0_1$1	$8_8[s_8^7]$	000011 $^0_1$1	$8_{14}[s_8^{12}]$
01101 $^0_1$1	$7_2[s_7^2]$	00110 $^0_1$1	$7_5[s_7^5]$	0000 $^0_1$1	$6_5[s_6^4]$
011011 $^0_1$1	$8_3[s_8^2]$	00 $^0_1$1	$4_2[s_4^1]$	000001 $^0_1$1	$8_{15}[s_8^{13}]$
0 $^0_1$1	$3_1[s_3]$	00010011	$8_9[s_4^1 \times 2^1]$	00000 $^0_1$1	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	00010 $^0_1$1	$7_6[s_7^6]$	000000 $^0_1$1	$8_{16}[s_8^{14}]$
001011 $^0_1$1	$8_4[s_8^3]$	000101 $^0_1$1	$8_{10}[s_8^8]$		

[a] The notation P_i refers to the ith bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the ith saddle-node bifurcation of period P is denoted s_P^i, and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i.

Symbol Exchange Near Endplay



Symbol Exchange Near Endplay

- Symbols 0, 1 created at $b = 0$
- New orbit, (11), created at $b = \frac{3}{4}$
- Symbol pair - 11 -, replaced by - (11) - as $b \to 1$
- Implosions begin at $b = 0.5957...$, end at midpoint
- Explosions begin at midpoint, end at $b = 0.7825..$
- Implosions and explosions symmetrically matched