Alice in Stretch & SqueezeLand: 11 Bounding Tori

August 12, 2012

Chapter Abstract

Each strange attractor in R^3 is contained in a 3D submanifold in R^3.

The boundary of this 3D submanifold is a torus.

Tori are classified by *genus*, and bounding tori are also dressed by the flow induced on the surface.

Bounding tori are classified by permutation group elements.

They are also built up by an Aufbau Principle.

The global Poincaré surface of section of a genus-g bounding torus consists of $g-1$ disconnected disks.

Constraints

Branched manifolds largely constrain the 'perestroikas" that forcing diagrams can undergo.

Is there some mechanism/structure that constrains the types of perestroikas that branched manifolds can undergo?

Constraints on Branched Manifolds

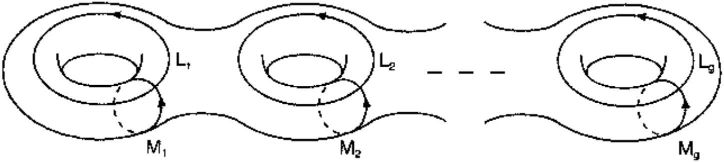
“Inflate” a strange attractor

Union of ϵ balls around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Tori are identified by genus g and dressed with a surface flow induced from that creating the strange attractor.

Surface Singularities

Flow field: three eigenvalues: +, 0, –

Vector field “perpendicular” to surface

Eigenvalues on surface at fixed point: +, –

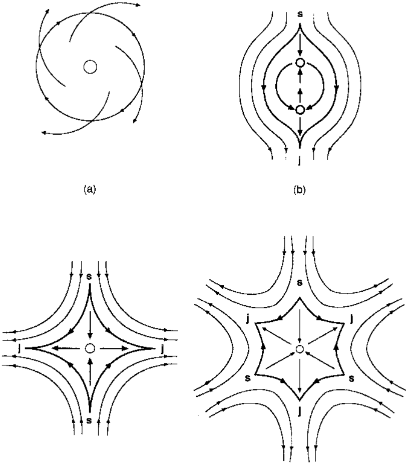
All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

 # fixed points on surface = index = 2g - 2

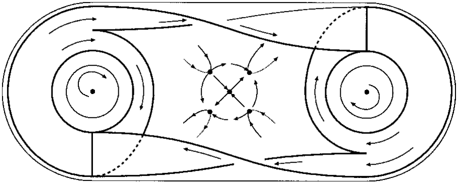
Singularities organize the surface flow dressing the torus

Flow Near a Singularity

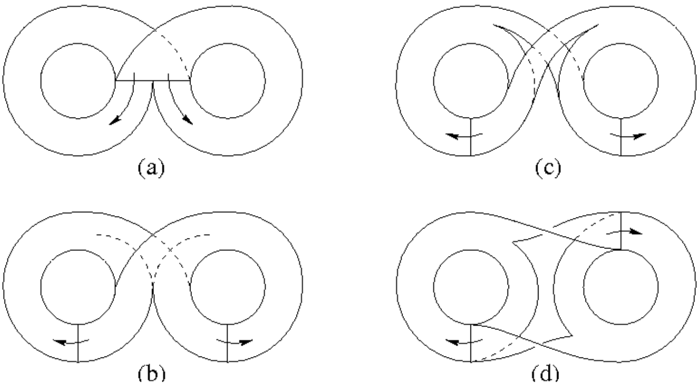


(a) (b) (c) (d)

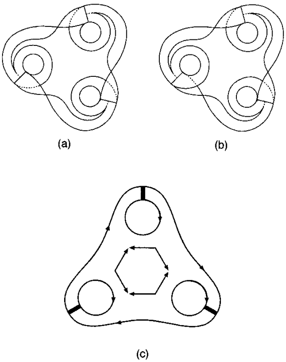
Torus Bounding Lorenz-like Flows



Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.



Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks.

Transition matrix sum of two g-1 $\times$ g-1 matrices.

Both are g-1 $\times$ g-1 permutation matrices.

They identify mappings of Poincaré sections to P'sections.

Bounding tori labeled by (permutation) group theory.

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

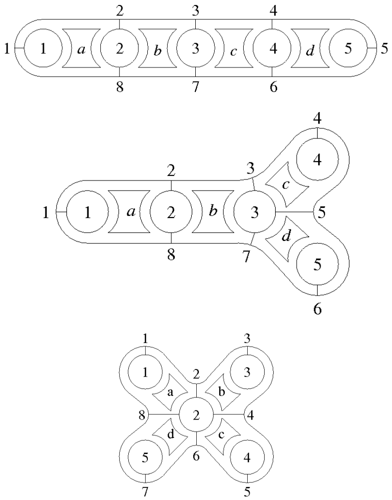
g	m	$(p_1, p_2, \dots p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

The Growth is Exponential

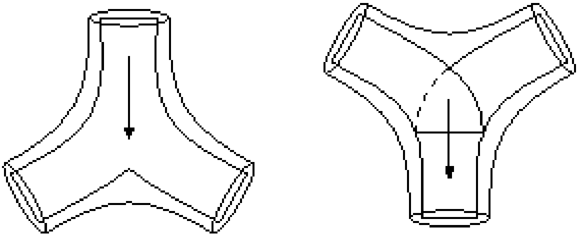
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

Some Genus-9 Bounding Tori



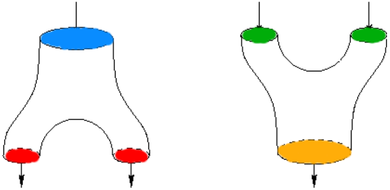
Aufbau Princip for Bounding Tori



These units ("pants, trinions") surround the stretching and squeezing units of branched manifolds.

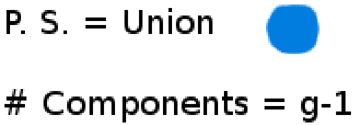
Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units

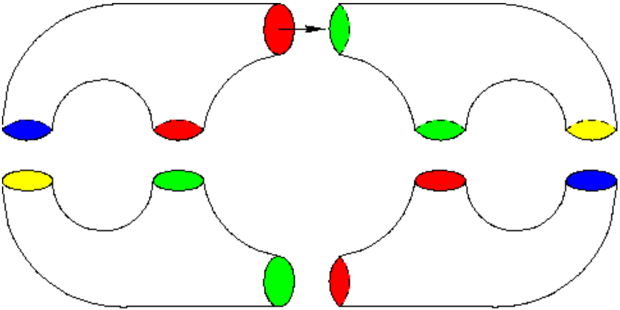


- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Construction of Poincaré Section



Application: Lorenz Dynamics, g=3



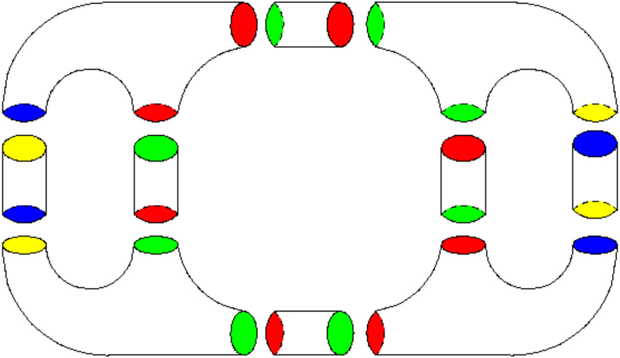
Representation Theory for $g > 1$

Can we extend the representation theory for strange attractors "with a hole in the middle" (i.e., genus $= 1$) to higher-genus attractors?

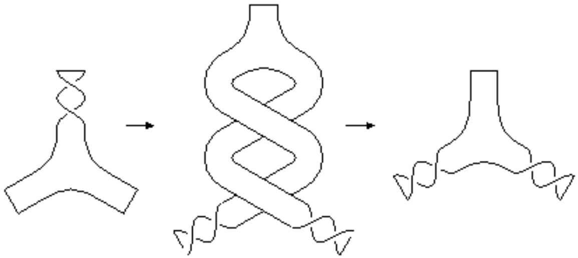
Yes. The results are similar.

Begin as follows:

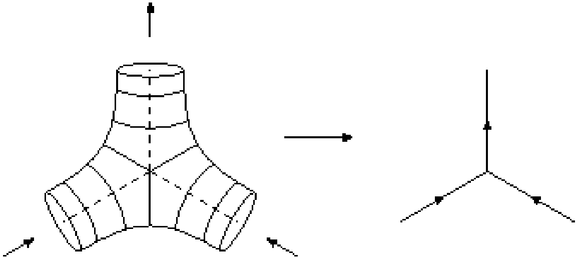
Application: Lorenz Dynamics, g=3



Embeddings

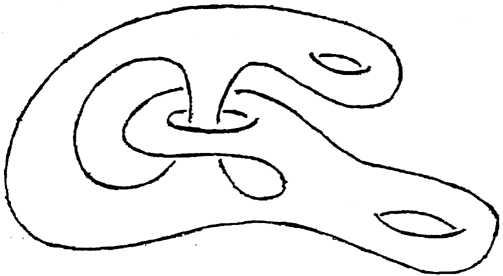


Redcution to Networks



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Intrinsic Tori



A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

Equivalences by Injection

Obstructions to Isotopy

Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus-g strange attractor become equivalent under isotopy.