Alice in Stretch & SqueezeLand
The Marvels of Topology and Chaos

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Abstract

Suppose you have data from a physical system that is behaving chaotically. What do you do? How do you analyze these data? What should you look for? What is the mechanism that generates chaos?

For a large class of systems an algorithm now exists for addressing each of these questions successively and successfully. We will go through the steps of this algorithm, showing how each works using experimental data and pointing out the connection with topology. In the process we will develop a classification scheme for strange attractors.

Outline

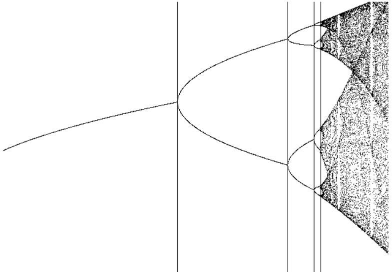
Background

J. R. Tredicce

Can you explain my data?

I dare you to explain my data!

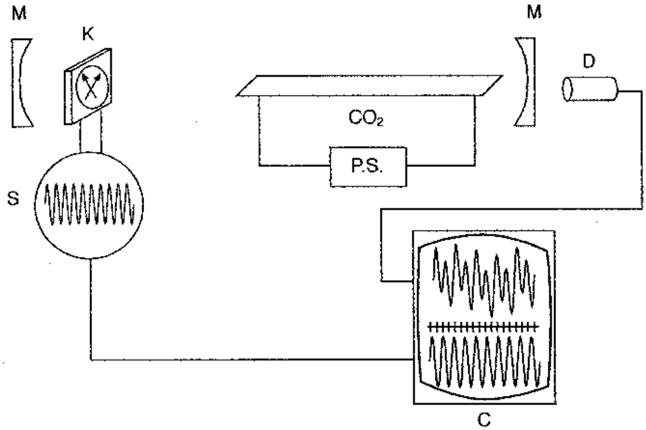
Where is Tredicce coming from?



Feigenbaum:

$$\alpha = 4.66920\ 16091\ .....$$
$$\delta = -2.50290\ 78750\ .....$$

Laser with Modulated Losses Experimental Arrangement



Original Objectives

Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The 4 Levels of Structure

- **Basis Sets of Orbits**
- **Branched Manifolds**
- **Bounding Tori**
- **Extrinsic Embeddings**

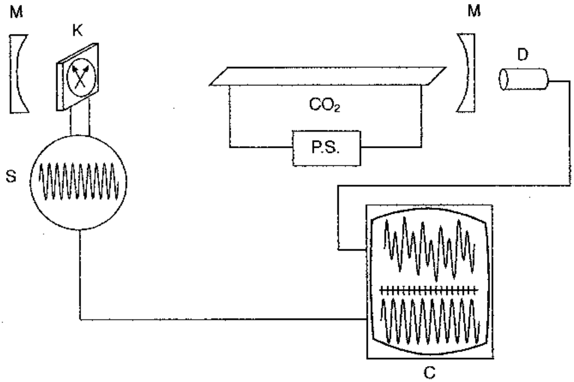
What Have We Learned?

1. Cover and Image Relations
2. Continuations: Analytical, Topological, Group
3. Cauchy Riemann & Clebsch-Gordonnery for Dynamical Systems
4. "Quantizing Chaos"
5. Representation Theory for Dynamical Systems

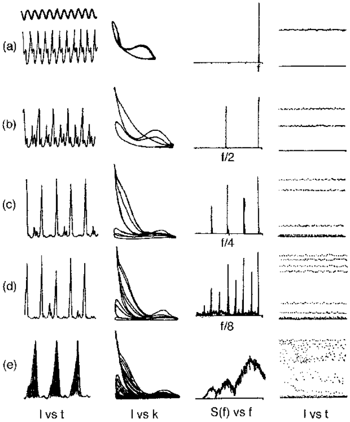
What Do We Need to Learn?

1. Higher Dimensions
2. Invariants
3. Mechanisms

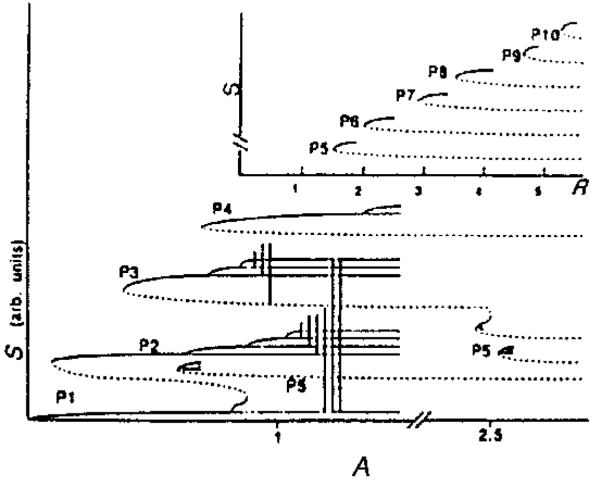
Laser Experimental Arrangement



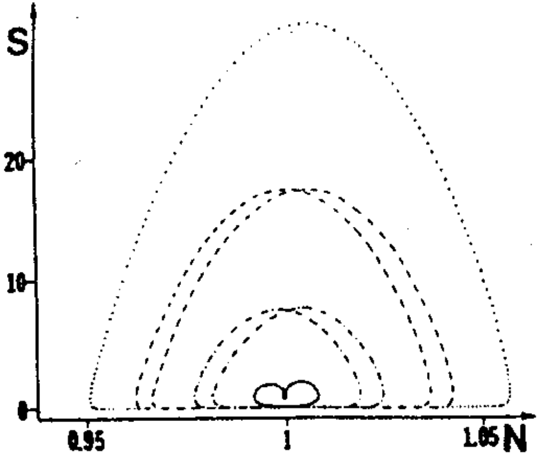
Oscilloscope Traces



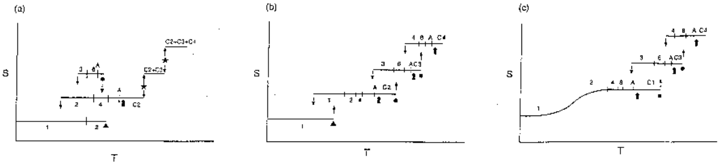
Bifurcation Schematics



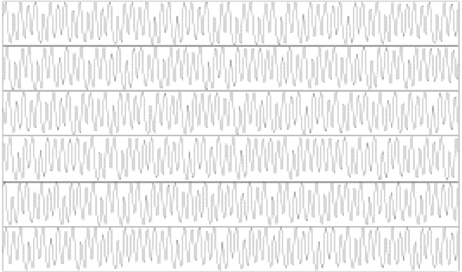
Coexisting Basins of Attraction

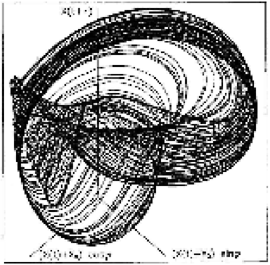


Bifurcation Perestroikas



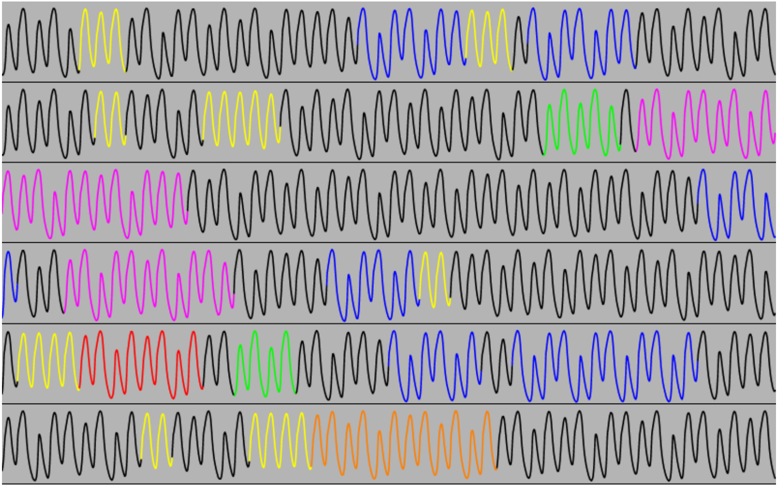
Experimental Data: LSA



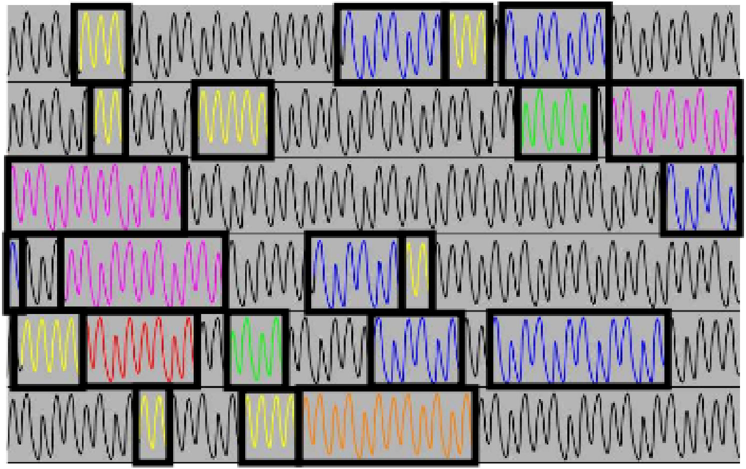


Lefranc - Cargese

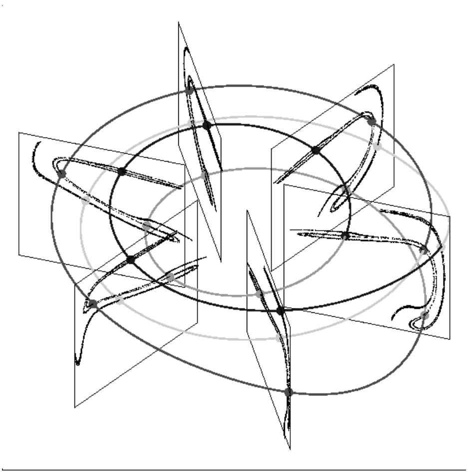
Experimental Data: LSA



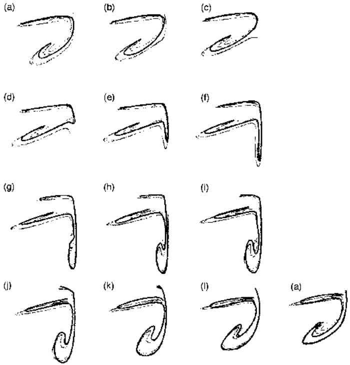
Experimental Data: LSA



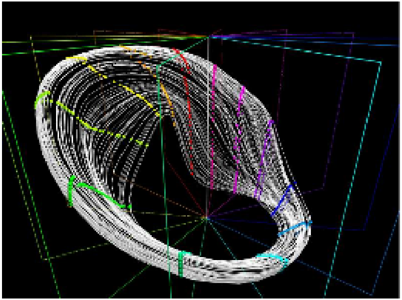
Stretching & Squeezing in a Torus



Rotating the Poincaré Section around the axis of the torus



Rotating the Poincaré Section around the axis of the torus



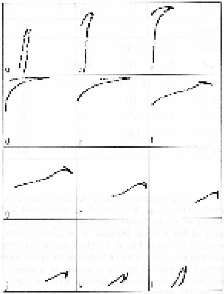
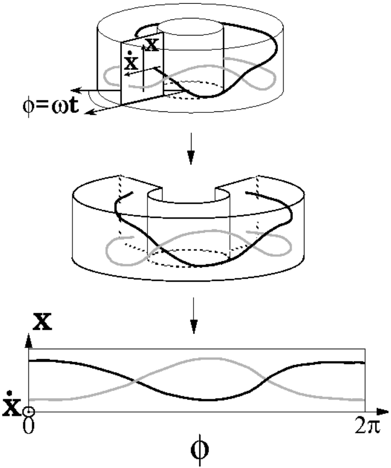


Figure 2. Left: Intersections of a chaotic attractor with a series of section planes are computed. Right: Their evolution from plane to plane shows the interplay of the stretching and squeezing mechanisms.

Lefranc - Cargese

Cutting Open a Torus

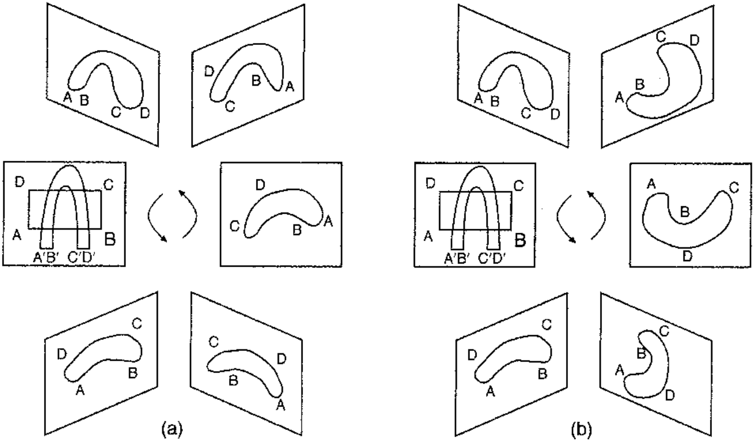


Alice in Stretch & SqueezeLand The Marvels of Topology and Chaos

Robert Gilmore

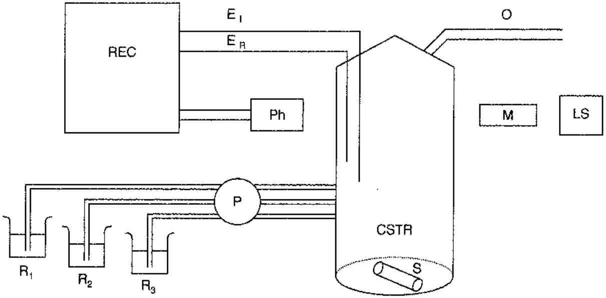
Introduction-01
Introduction-02
Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01

Global Torsion



A Chemical Experiment

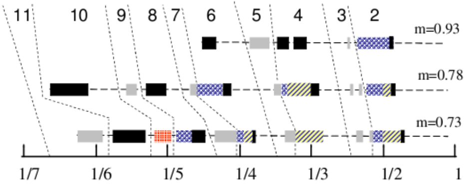
The Belousov-Zhabotinskii Reaction



The Lasers in Zaragosa

TABLE 1 – Folding processes characteristic of the different species of templates treated in this work

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						

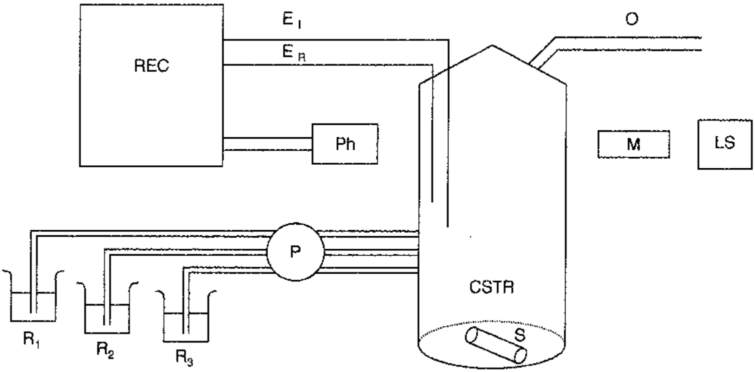


Used and Martin, Phys. Rev. E **82**, 016218 (2010)

TABLE 2 – Linking numbers between the UPOs extracted from the time series corresponding to pump modulation frequency f=4.25 KHz and modulation index m=0.73

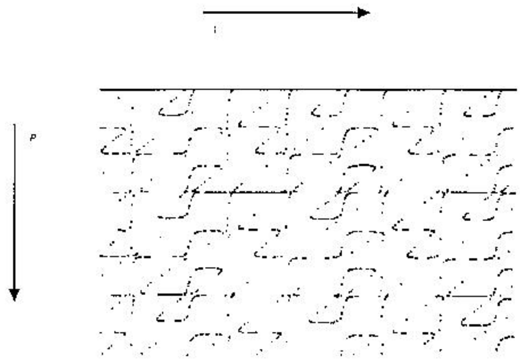
	0	10	**3a**	100	1000	10010	**6a**	1001010
0	0							
10	9	9						
3a	14	28	28					
100	14	28	42	28				
1000	18	37	56	56	55			
10010	23	*	70	*	92	92		
6a	28	56	*	*	112	*	139	
1001010	32	*	98	*	119	*	*	194

Belousov-Zhabotinskii Experimental Configuration

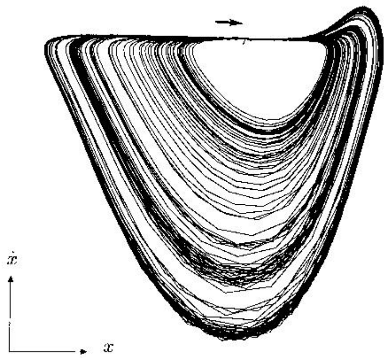


Close Returns Plot

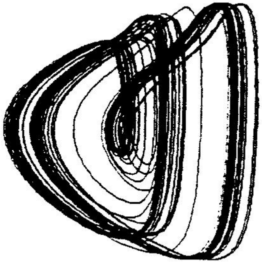
$$|x_i - x_{i+p}| < \epsilon \qquad \text{pixel} \rightarrow \text{black}$$



First Embedding Attempt: $x, \dot{x}, \ddot{x}$

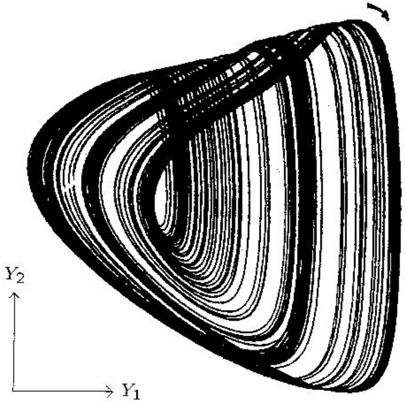


Second Embedding Attempt: $\int x, x, \dot{x}$



Nonstationary!

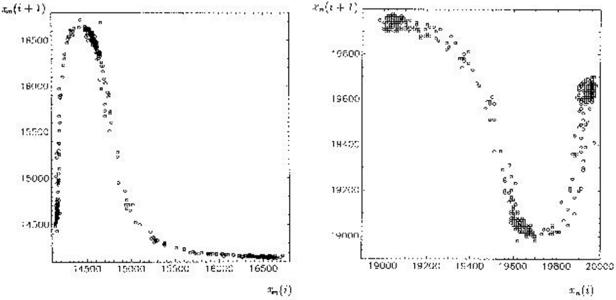
Third embedding attempt: $\int xe^{-t'/\tau}, x, \dot{x}$



Once you have an embedding:

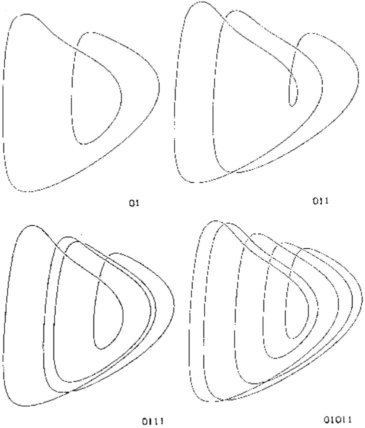
- Find a Poincaré Section
- Construct a First Return Map on the Section
- Introduce a Symbolic Encoding
- Encode all Unstable Periodic Orbits
- Find their Linking Numbers

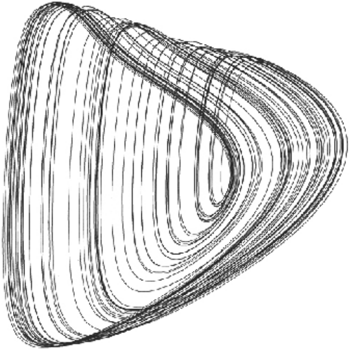
Two Symbols Suffice! 0 and 1



Embedded Periodic Orbits

Some Named Low-Period Orbits





Some Extracted and Reconstructed Periodic Orbits

Orbit	Name	Symbolics	Local Torsion	Self-Linking
1	1_1	1	1	0
2	2_1	01	1	1
3	3_1	011	2	2
4	4_1	0111	3	5
5	5_1	$01\ 011$	3	8
6	6_2	$011\ 0M1$	3	9
7	7_2	$(01)^2011$	4	16
8a	8_1	$(01)^20111$	5	23
8b	8_3	$01(011)^2$	5	21
9	9_3	$(01)^3011$	5	28
10a	10_6	$(011)^20101$	6	33
10b	10_6	$(011)^20111$	7	33
11	11_9	$01(011)^3$	7	40
13a		$(01)^2011\ 01\ 0111$	8	62
13b		$(01)^3011\ 0111$	8	60

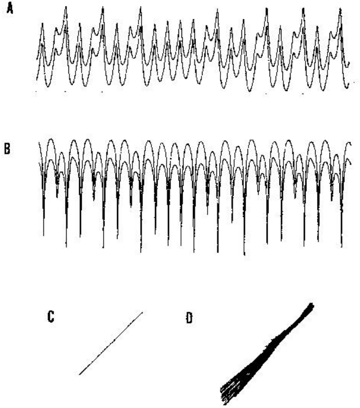
Table of Experimental Linking Numbers

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Testing the Result

(a), (c) y_1^m compared with y_1^d. (b), (d) y_3^m compared with y_3^d.



Chaos

Motion that is

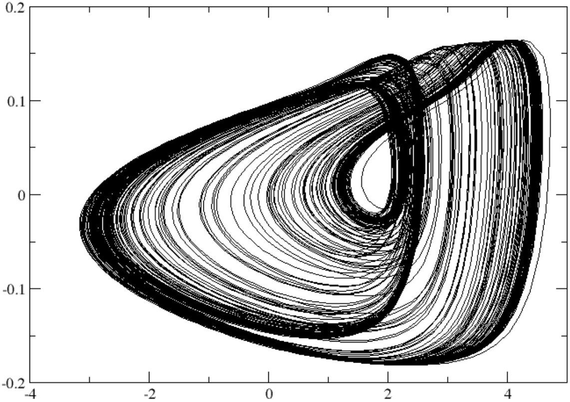
- **Deterministic:** $\frac{dx}{dt} = f(x)$
- **Recurrent**
- **Non Periodic**
- **Sensitive to Initial Conditions**

Strange Attractor

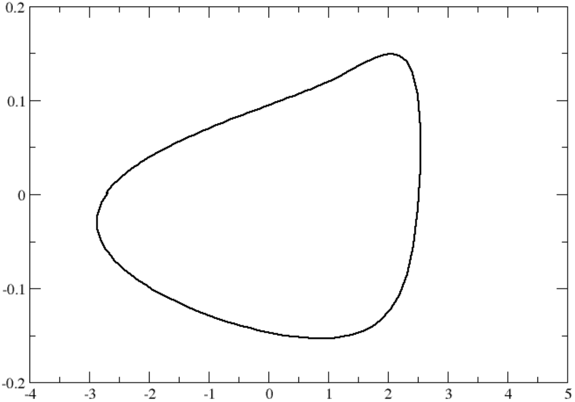
The Ω limit set of the flow. There are unstable periodic orbits "in" the strange attractor. They are

- "Abundant"
- Outline the Strange Attractor
- Are the Skeleton of the Strange Attractor

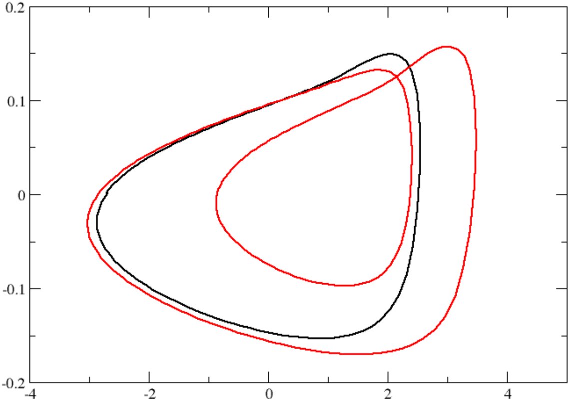
UPOs: Skeletons of Strange Attractors



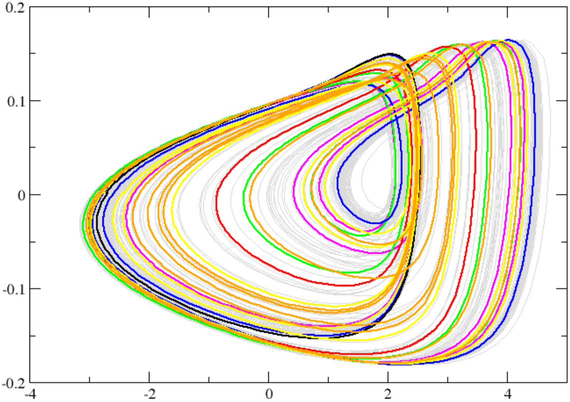
UPOs: Skeletons of Strange Attractors



UPOs: Skeletons of Strange Attractors



UPOs: Skeletons of Strange Attractors



UPOs Outline Strange Attractors

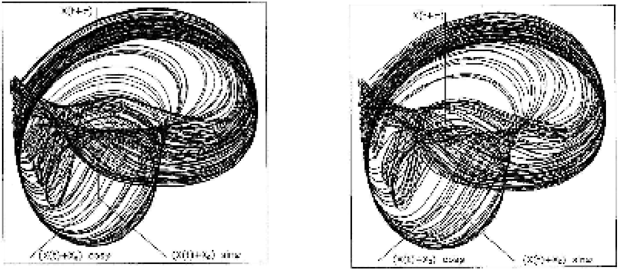


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

Organization of UPOs in R^3:

Gauss Linking Number

$$LN(A,B) = \frac{1}{4\pi} \oint \oint \frac{(\mathbf{r}_A - \mathbf{r}_B) \cdot d\mathbf{r}_A \times d\mathbf{r}_B}{|\mathbf{r}_A - \mathbf{r}_B|^3}$$

Interpretations of LN $\simeq$ # Mathematicians in World

Linking Number of Two UPOs

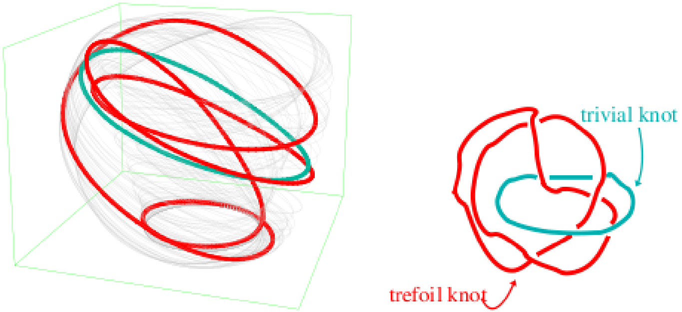
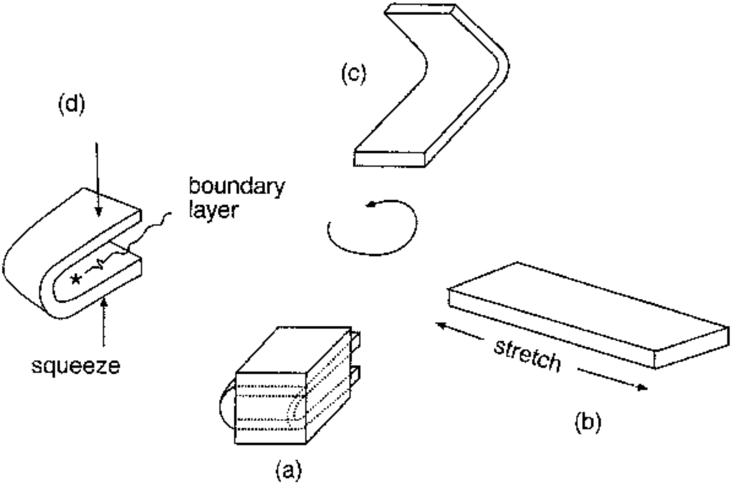


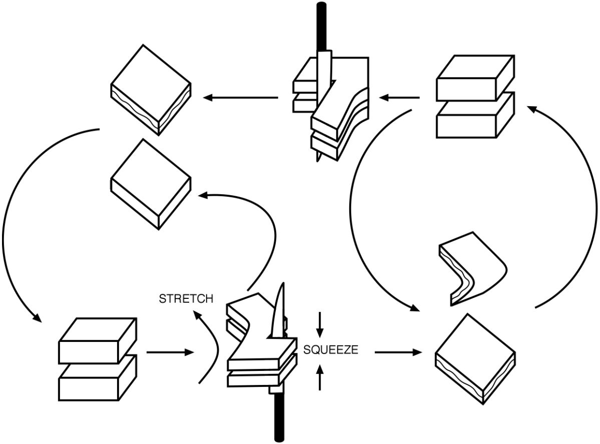
Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

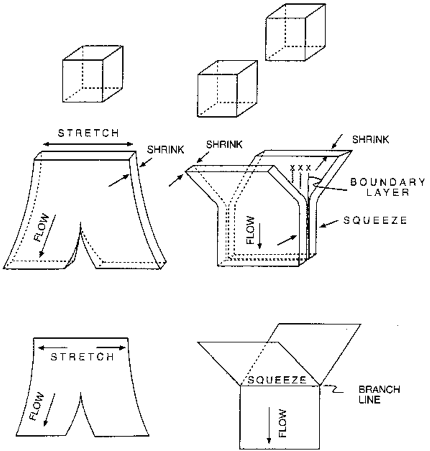
Stretching and Folding



Tearing and Squeezing



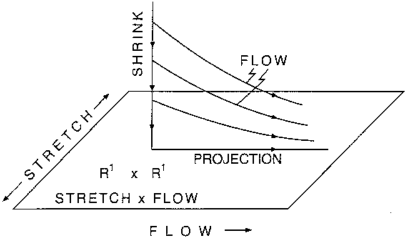
Stretching — Squeezing



Birman - Williams Projection

Identify x and y if

$$\lim_{t\to\infty} |x(t) - y(t)| \to 0$$



Birman - Williams Theorem

If:

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then:

Birman - Williams Theorem

If: **Certain Assumptions**

Then: **Specific Conclusions**

Assumptions, B-W Theorem

A flow $\Phi_t(x)$

- **on** R^n **is dissipative,** $\underline{n=3}$**, so that** $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 < 0.$
- **Generates a** **hyperbolic** **strange attractor** $\mathcal{SA}$

IMPORTANT: The underlined assumptions can be relaxed.

Conclusions, B-W Theorem

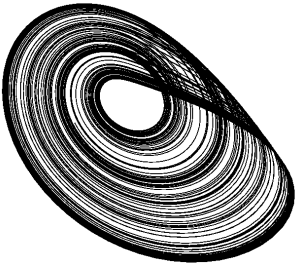
• The projection maps the strange attractor $\mathcal{SA}$ onto a 2-dimensional branched manifold $\mathcal{BM}$ and the flow $\Phi_t(x)$ on $\mathcal{SA}$ to a semiflow $\overline{\Phi}(x)_t$ on $\mathcal{BM}$.

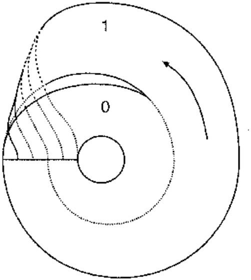
• UPOs of $\Phi_t(x)$ on $\mathcal{SA}$ are in 1-1 correspondence with UPOs of $\overline{\Phi}(x)_t$ on $\mathcal{BM}$. Moreover, every link of UPOs of $(\Phi_t(x), \mathcal{SA})$ is isotopic to the correspond link of UPOs of $(\overline{\Phi}(x)_t, \mathcal{BM})$.

Remark: "One of the few theorems useful to experimentalists."

Rössler:

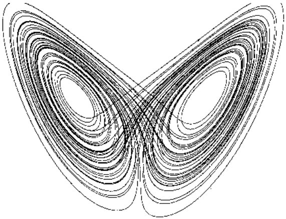
Attractor **Branched Manifold**

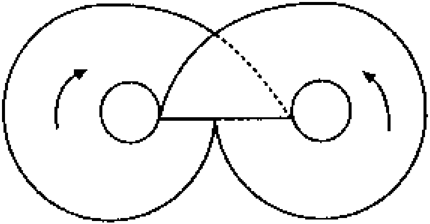




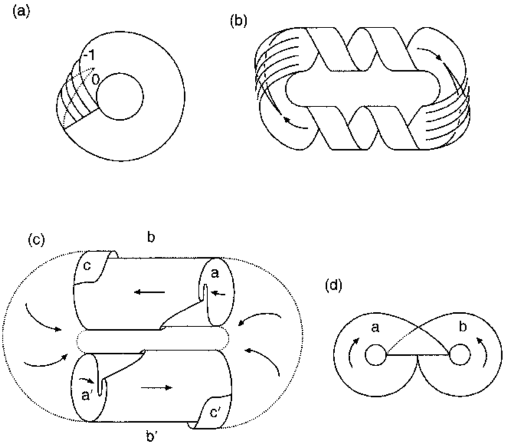
Lorenz:

Attractor	Branched Manifold



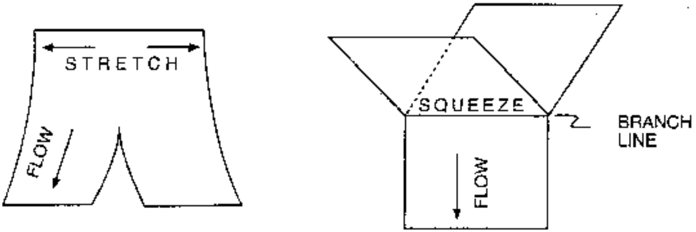


Inequivalent Branched Manifolds



Aufbau Princip for Branched Manifolds

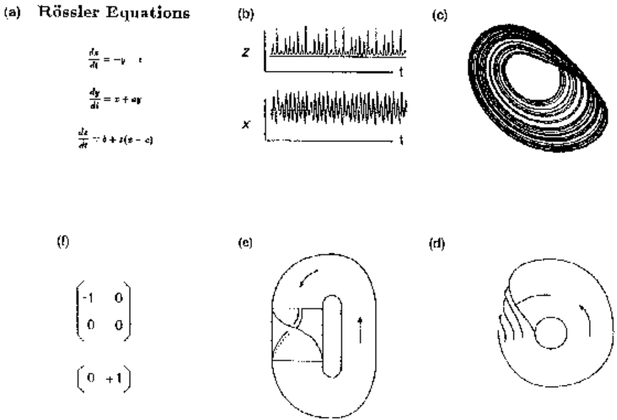
Any branched manifold can be built up from stretching and squeezing units



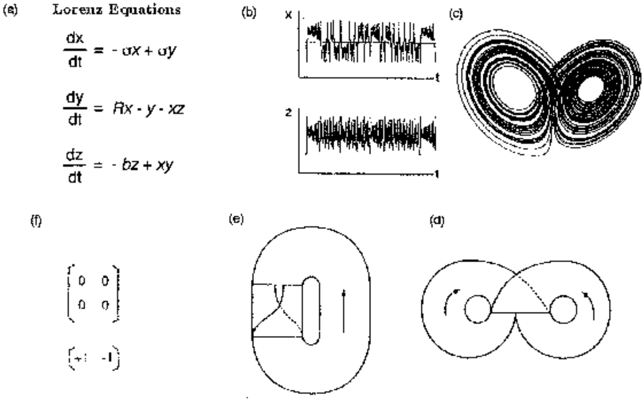
subject to the conditions:

- Outputs to Inputs
- No Free Ends

Rossler System



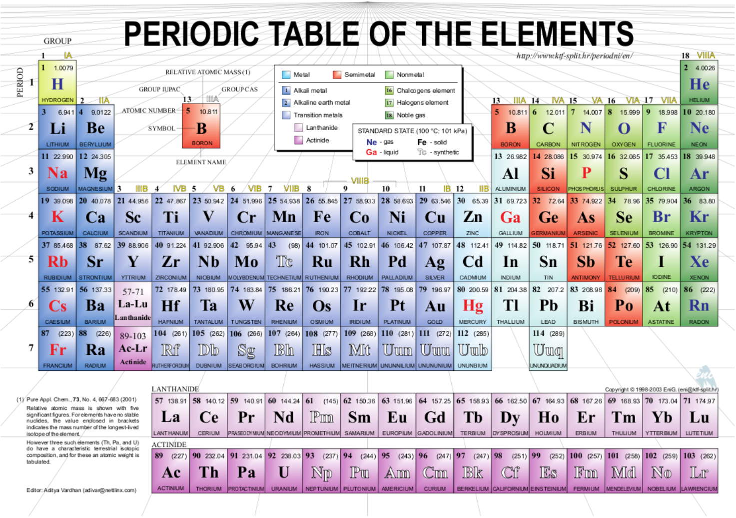
Lorenz System



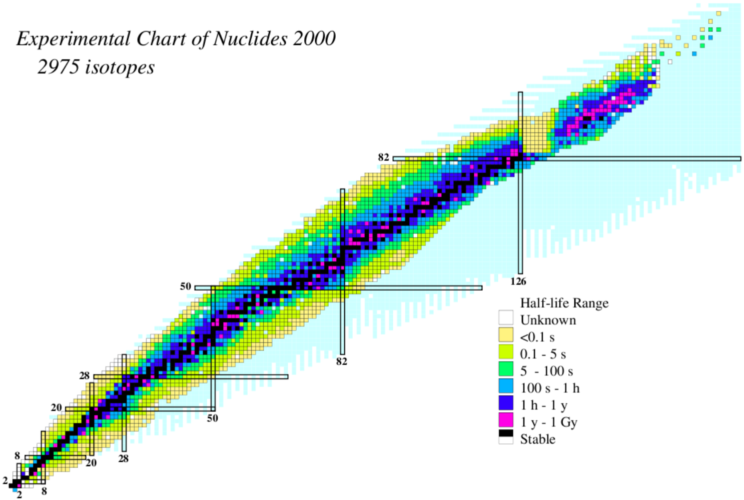
Poincaré Smiles at Us in R^3

- **Determine organization of UPOs** $\Rightarrow$
- **Determine branched manifold** $\Rightarrow$
- **Determine equivalence class of** $\mathcal{SA}$

We Like to be Organized



We Like to be Organized



Topological Analysis Program

Locate Periodic Orbits

Create an Embedding

Determine Topological Invariants (LN)

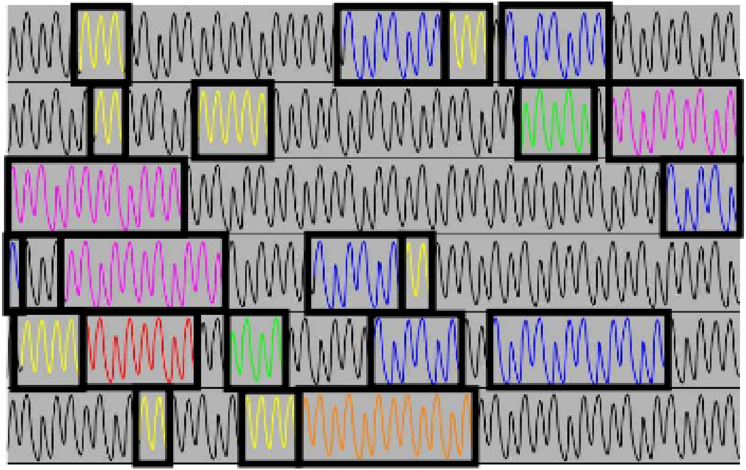
Identify a Branched Manifold

Verify the Branched Manifold

Model the Dynamics

Validate the Model

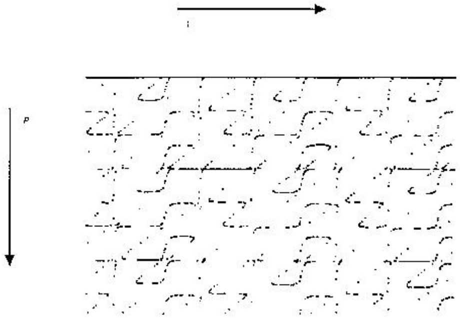
Method of Close Returns



Locate UPOs

Method of Close Returns

$$|x_i - x_{i+p}| < \epsilon, \quad \text{pixel} \rightarrow \text{black}$$



Embeddings

This is a tricky business. There are many problems ...

Many Methods: Time Delay, Differential, Hilbert Transforms, SVD, Mixtures, ...

Tests for Embeddings: Geometric, Dynamic, Topological†

None Good

We Demand a 3 Dimensional Embedding

Locate UPOs

Periodic Orbits Outline the Attractor

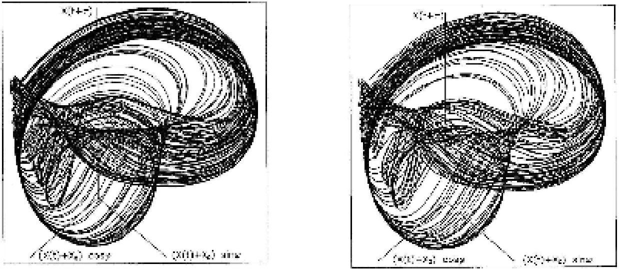


Figure 5. Left: a chaotic attractor reconstructed from a time series from a chaotic laser ; Right : Superposition of 12 periodic orbits of periods from 1 to 10.

Lefranc - Cargese

Linking Number of Orbit Pairs

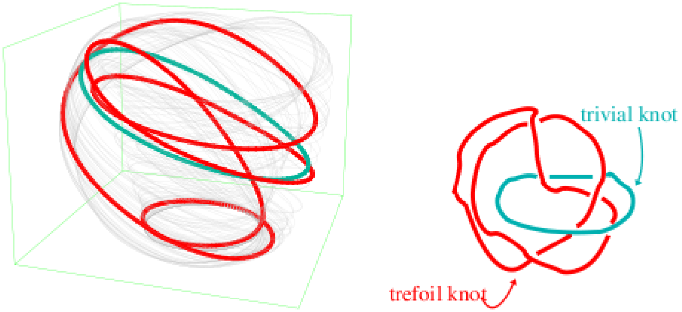


Figure 6. Left: two periodic orbits of periods 1 and 4 embedded in a strange attractor; Right: a link of two knots that is equivalent to the pair of periodic orbits up to continuous deformations without crossings.

Lefranc - Cargese

Compute Table of Expt'l LN

Table 7.2 Linking numbers for all the surrogate periodic orbits, to period 8, extracted from Belousov–Zhabotinskii data[a]

Orbit	Symbolics	1	2	3	4	5	6	7	8a	8b
1	1	0	1	1	2	2	2	3	4	3
2	01	1	1	2	3	4	4	5	6	6
3	011	1	2	2	4	5	6	7	8	8
4	0111	2	3	4	5	8	8	11	13	12
5	01 011	2	4	5	8	8	10	13	16	15
6	011 0M1	2	4	6	8	10	9	14	16	16
7	01 01 011	3	5	7	11	13	14	16	21	21
8a	01 01 0111	4	6	8	13	16	16	21	23	24
8b	01 011 011	3	6	8	12	15	16	21	24	21

[a] All indices are negative.

Compare w. LN From Various $\mathcal{BM}$

Table 2.1 Linking numbers for orbits to period five in Smale horseshoe dynamics.

	1s	1f	2_1	3f	3s	4_1	4_2f	4_2s	5_3f	5_3s	5_2f	5_2s	5_1f	5_1s
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	2	1	1	1	1	2	2	2	2
01	0	1	1	2	2	3	2	2	2	2	3	3	4	4
001	0	1	2	2	3	4	3	3	3	3	4	4	5	5
011	0	1	2	3	2	4	3	3	3	3	5	5	5	5
0111	0	2	3	4	4	5	4	4	4	4	7	7	8	8
0001	0	1	2	3	3	4	3	4	4	4	5	5	5	5
0011	0	1	2	3	3	4	4	3	4	4	5	5	5	5
00001	0	1	2	3	3	4	4	4	4	5	5	5	5	5
00011	0	1	2	3	3	4	4	4	5	4	5	5	5	5
00111	0	2	3	4	5	7	5	5	5	5	6	7	8	9
00101	0	2	3	4	5	7	5	5	5	5	7	6	8	9
01101	0	2	4	5	5	8	5	5	5	5	8	8	8	10
01111	0	2	4	5	5	8	5	5	5	5	9	9	10	8

Guess Branched Manifold

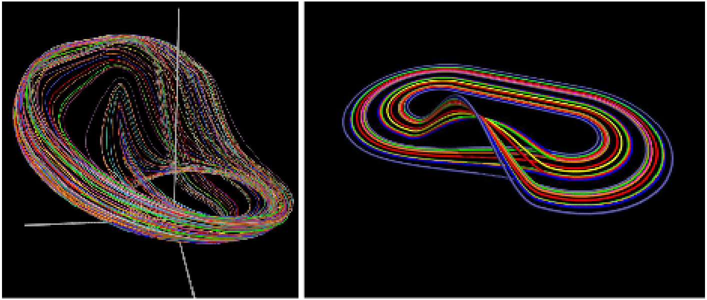


Figure 7. "Combing" the intertwined periodic orbits (left) reveals their systematic organization (right) created by the stretching and squeezing mechanisms.

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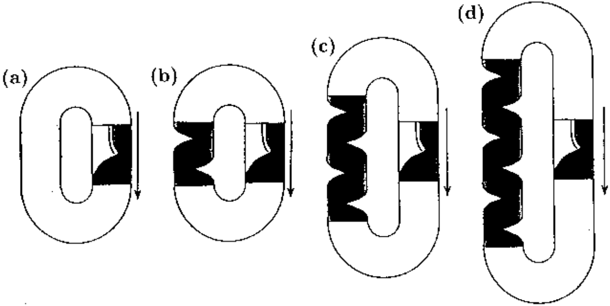
Identification & ‘Confirmation’

- $\mathcal{BM}$ Identified by LN of small number of orbits
- Table of LN GROSSLY overdetermined
- Predict LN of additional orbits
- Rejection criterion

What Do We Learn?

- $\mathcal{BM}$ Depends on Embedding
- Some things depend on embedding, some don't
- Depends on Embedding: Global Torsion, Parity, ..
- Independent of Embedding: Mechanism



Evolution Under Parameter Change

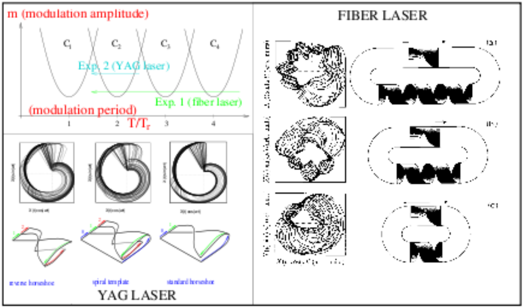
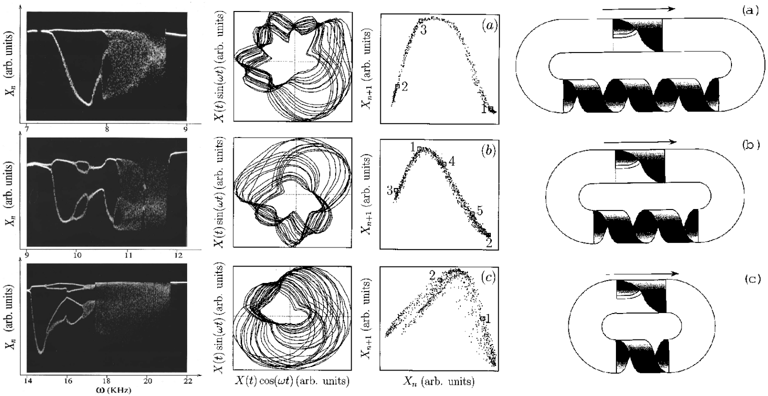


Figure 11. Various templates observed in two laser experiments. Top left: schematic representation of the parameter space of forced nonlinear oscillators showing resonance tongues. Right: templates observed in the fiber laser experiment: global torsion increases systematically from one tongue to the next [40]. Bottom left: templates observed in the YAG laser experiment (only the branches are shown): there is a variation in the topological organization across one chaotic tongue [39, 41].

Evolution Under Parameter Change



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An Unexpected Benefit

Analysis of Nonstationary Data

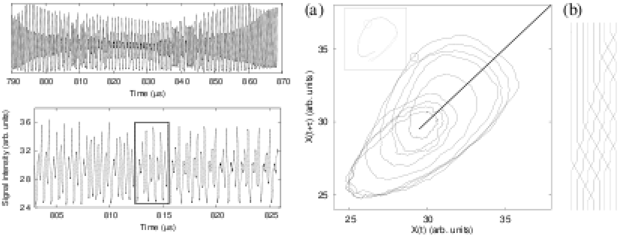


Figure 16. Top left: time series from an optical parametric oscillator showing a burst of irregular behavior. Bottom left: segment of the time series containing a periodic orbit of period 9. Right: embedding of the periodic orbit in a reconstructed phase space and representation of the braid realized by the orbit. The braid entropy is $h_T = 0.377$, showing that the underlying dynamics is chaotic. Reprinted from [61].

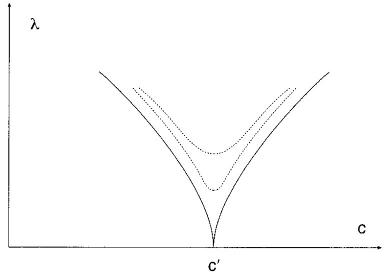
Lefranc - Cargese

Model the Dynamics

A hodgepodge of methods exist: # Methods $\simeq$ # Physicists

Validate the Model

Needed: Nonlinear analog of χ^2 test. OPPORTUNITY: Tests that depend on entrainment/synchronization.



Compare with Original Objectives

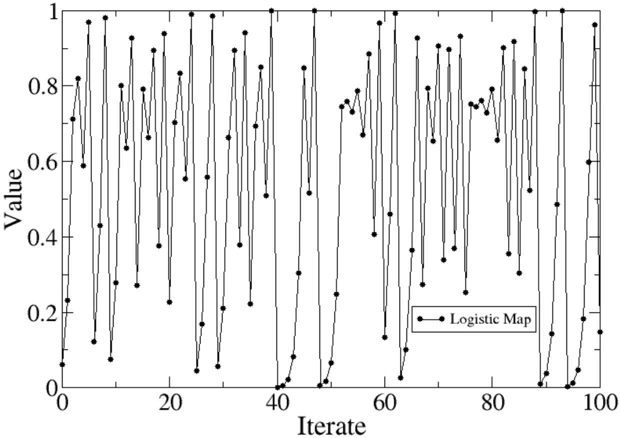
Construct a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

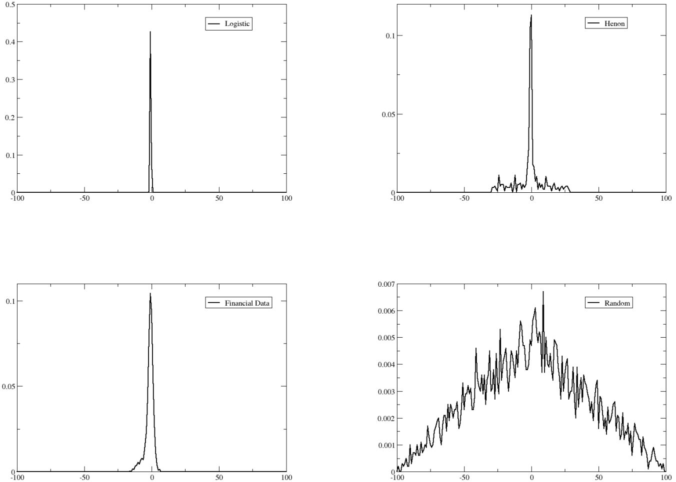
Determinism

How to predict the future from the past

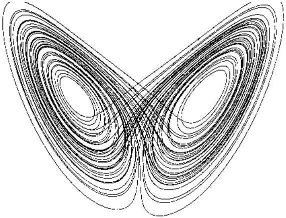


Some Prediction Results

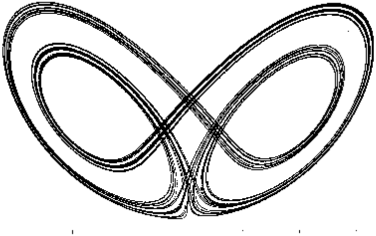
Tightly binned predictions suggest determinism



There Are Some Missing Orbits

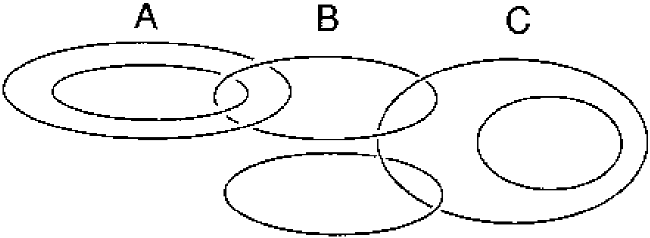


Lorenz



Shimizu-Morioka

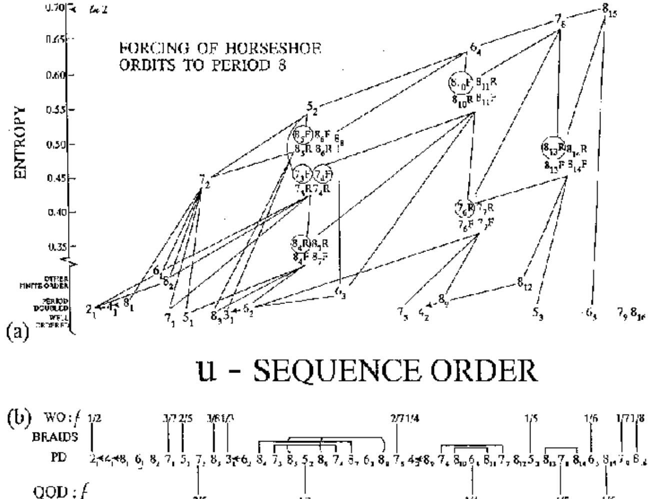
Orbit Forcing



A ⇒ B   B ⇒ C

A ⇒ C

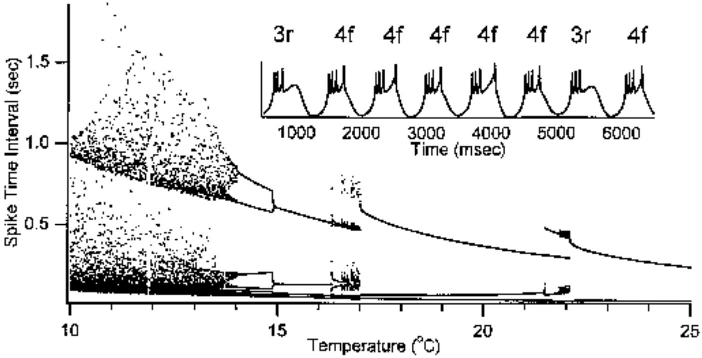
Forcing Diagram - Horseshoe



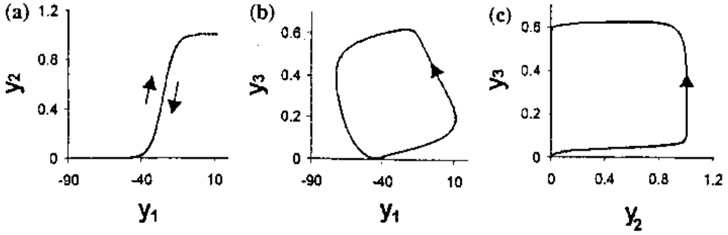
Status of Problem

- Horseshoe organization - active
- More folding - barely begun
- Circle forcing - even less known
- Higher genus - new ideas required

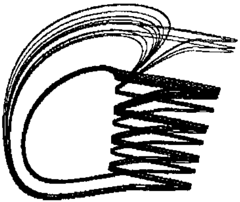
Is This Predictable or Not?



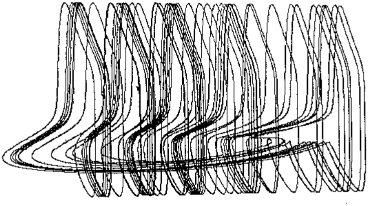
Some Variables are Strongly Correlated



The Attractor can be Projected in Many Ways

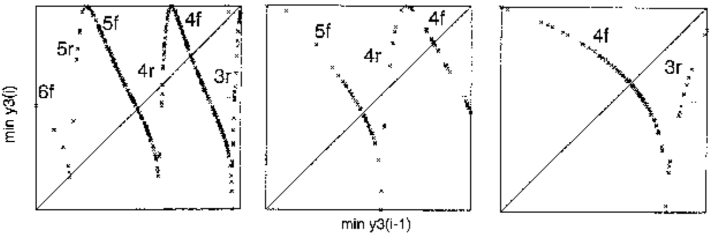


y_4-y_5 Plane



$\frac{y_4}{dt}$-y_5 Plane

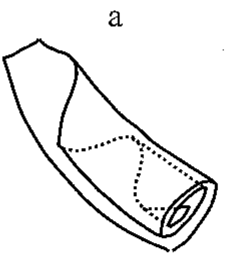
The Return Map "Drifts" with Temperature



$T = 12^o$ C $\quad T = 13.5^o$ C $\quad T = 16.5^o$ C

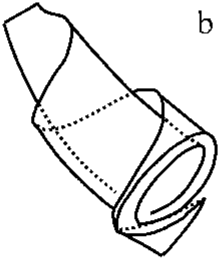
Scroll Templates

Outside to Inside



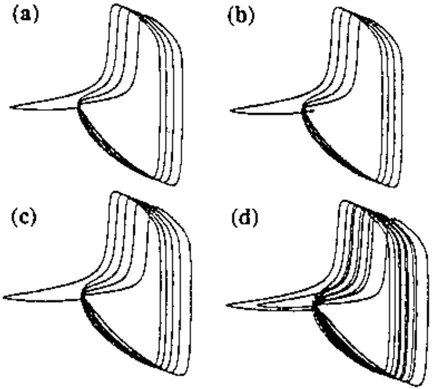
Branch	Array	0	1	2	3	4	5	6	7	8	9
0	+N-0	0	0	0	0	0	0	0	0	0	0
1	-N+0	0	1	2	2	2	2	2	2	2	2
2	+N-1	0	2	2	2	2	2	2	2	2	2
3	-N+1	0	2	2	3	4	4	4	4	4	4
4	+N-2	0	2	2	4	4	4	4	4	4	4
5	-N-2	0	2	2	4	4	5	6	6	6	6
6	-N-3	0	2	2	4	4	6	6	6	6	6
7	-N-3	0	2	2	4	4	6	6	7	8	8
8	-N-4	0	2	2	4	4	6	6	8	8	8
9	-N+4	0	2	2	4	4	6	6	8	8	9

Inside to Outside



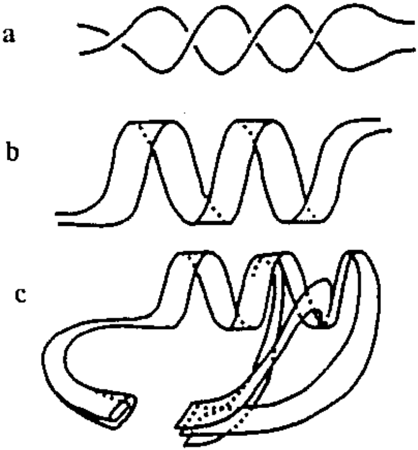
Branch	Array	0	1	2	3	4	5	6	7	8	9
0	0	0	0	2	2	4	4	6	6	8	8
1	-1	0	1	2	2	4	4	5	6	8	8
2	+1	2	2	2	2	4	4	6	6	8	8
3	-2	2	2	2	3	4	4	6	6	8	8
4	+2	4	4	4	4	4	4	6	6	8	8
5	-3	4	4	4	4	4	5	6	6	8	8
6	+3	6	6	6	6	6	6	6	6	8	8
7	-4	6	6	6	6	6	6	6	7	8	8
8	+4	8	8	8	8	8	8	8	8	8	8
9	-5	8	8	8	8	8	8	8	8	8	9

Some Periodic Orbits



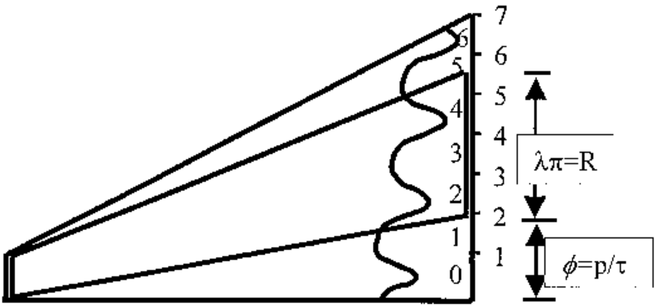
(a): $4f$ (b): $4r$ (c): $5f$ (d): $(4f, 5f)$

Steps in Constructing Scroll Template



Twist Relax Return

A Simple Two-Parameter Model of Chaotic Nerves



Φ = Drift λ = Stretch

Constraints

Branched manifolds largely constrain the 'perestroikas" that forcing diagrams can undergo.

Is there some mechanism/structure that constrains the types of perestroikas that branched manifolds can undergo?

Constraints on Branched Manifolds

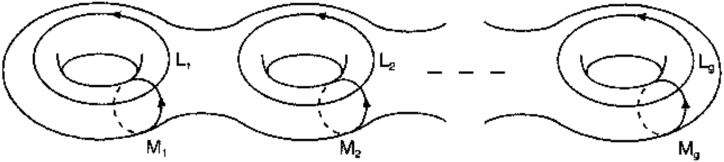
“Inflate” a strange attractor

Union of ϵ ball around each point

Boundary is surface of bounded 3D manifold

Torus that bounds strange attractor

Torus, Longitudes, Meridians



Tori are identified by genus g and dressed with a surface flow induced from that creating the strange attractor.

Surface Singularities

Flow field: three eigenvalues: +, 0, –

Vector field "perpendicular" to surface

Eigenvalues on surface at fixed point: +, –

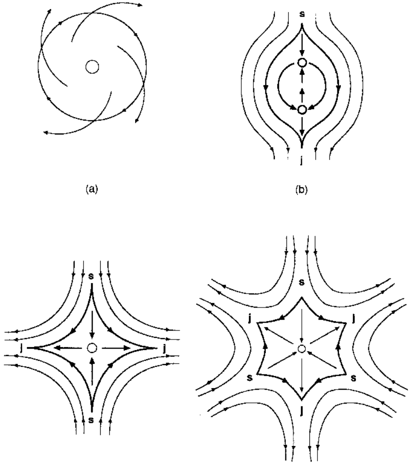
All singularities are regular saddles

$\sum_{s.p.}(-1)^{\text{index}} = \chi(S) = 2 - 2g$

 # fixed points on surface = index = 2g - 2

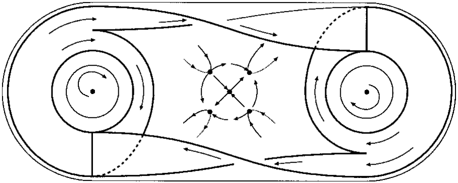
Singularities organize the surface flow dressing the torus

Flow Near a Singularity

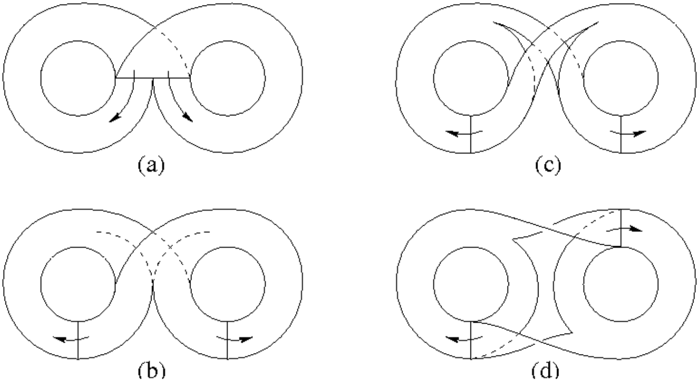


(a) (b) (c) (d)

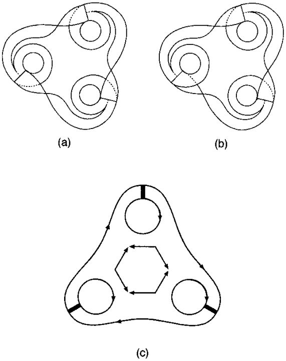
Torus Bounding Lorenz-like Flows



Twisting the Lorenz Attractor



Two possible branched manifolds in the torus with g=4.



Labeling Bounding Tori

Poincaré section is disjoint union of g-1 disks.

Transition matrix sum of two g-1 $\times$ g-1 matrices.

Both are g-1 $\times$ g-1 permutation matrices.

They identify mappings of Poincaré sections to P'sections.

Bounding tori labeled by (permutation) group theory.

Bounding Tori of Low Genus

TABLE I: Enumeration of canonical forms up to genus 9

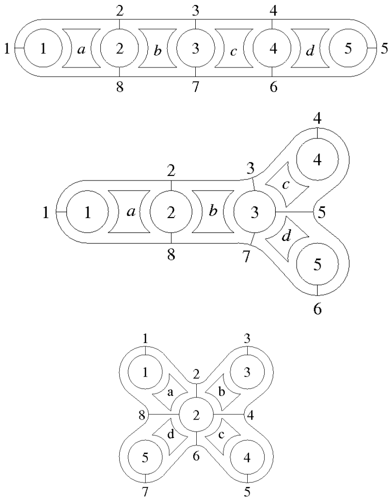
g	m	$(p_1, p_2, \dots p_m)$	$n_1 n_2 \dots n_{g-1}$
1	1	(0)	1
3	2	(2)	11
4	3	(3)	111
5	4	(4)	1111
5	3	(2,2)	1212
6	5	(5)	11111
6	4	(3,2)	12112
7	6	(6)	111111
7	5	(4,2)	112121
7	5	(3,3)	112112
7	4	(2,2,2)	122122
7	4	(2,2,2)	131313
8	7	(7)	1111111
8	6	(5,2)	1211112
8	6	(4,3)	1211121
8	5	(3,2,2)	1212212
8	5	(3,2,2)	1221221
8	5	(3,2,2)	1313131
9	8	(8)	11111111
9	7	(6,2)	11111212
9	7	(5,3)	11112112
9	7	(4,4)	11121112
9	6	(4,2,2)	11122122
9	6	(4,2,2)	11131313
9	6	(4,2,2)	11212212
9	6	(4,2,2)	12121212
9	6	(3,3,2)	11212122
9	6	(3,3,2)	11221122
9	6	(3,3,2)	11221212
9	6	(3,3,2)	11311313
9	5	(2,2,2,2)	12221222
9	5	(2,2,2,2)	12313132
9	5	(2,2,2,2)	14141414

The Growth is Exponential

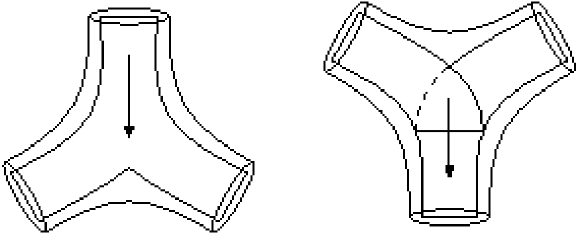
TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$
3	1	9	15	15	2211
4	1	10	28	16	5549
5	2	11	67	17	14290
6	2	12	145	18	36824
7	5	13	368	19	96347
8	6	14	870	20	252927

Some Genus-9 Bounding Tori



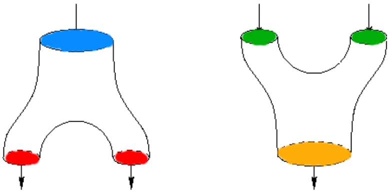
Aufbau Princip for Bounding Tori



These units ("pants, trinions") surround the stretching and squeezing units of branched manifolds.

Aufbau Princip for Bounding Tori

Any bounding torus can be built up from equal numbers of stretching and squeezing units



- **Outputs to Inputs**
- **No Free Ends**
- **Colorless**

Alice in Stretch & SqueezeLand The Marvels of Topology and Chaos

Robert Gilmore

Introduction-01

Introduction-02

Overview-01

Overview-02

Overview-03

Overview-04

Overview-05

Overview-06

Overview-07

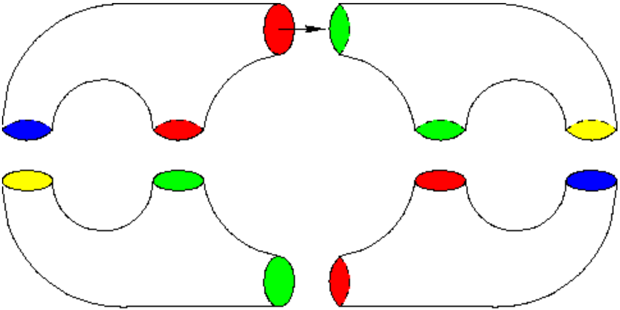
Experimental-01

Construction of Poincaré Section

P. S. = Union 

Components = g-1

Application: Lorenz Dynamics, g=3



Representation Theory for $g > 1$

Can we extend the representation theory for strange attractors "with a hole in the middle" (i.e., genus $= 1$) to higher-genus attractors?

Yes. The results are similar.

Begin as follows:

Alice in Stretch & SqueezeLand The Marvels of Topology and Chaos

Robert Gilmore

Introduction-01

Introduction-02

Overview-01

Overview-02

Overview-03

Overview-04

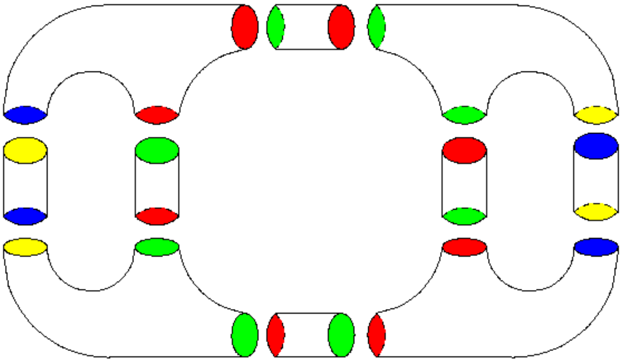
Overview-05

Overview-06

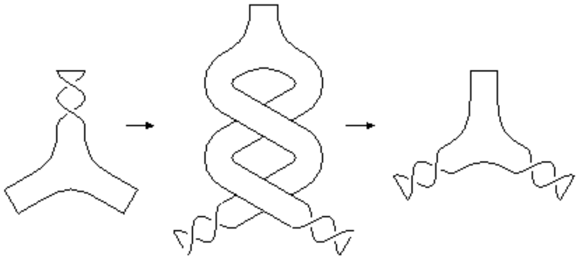
Overview-07

Experimental-01

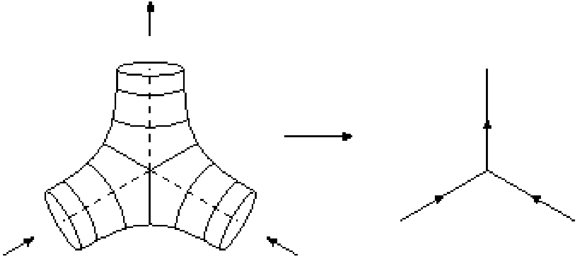
Application: Lorenz Dynamics, g=3



Embeddings

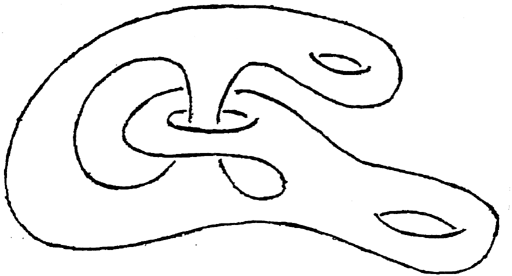


Preparations for Embedding tori into



Equivalent to embedding a specific class of directed networks into R^3

Extrinsic Embedding of Intrinsic Tori



A specific simple example.
Partial classification by links of homotopy group generators.
Nightmare Numbers are Expected.

Equivalences by Injection

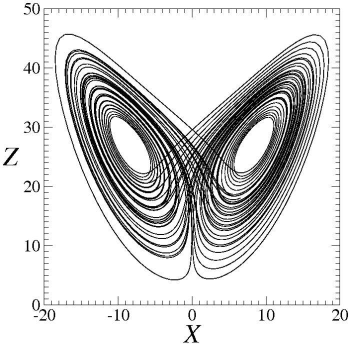
Obstructions to Isotopy

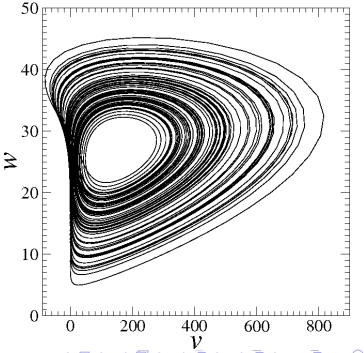
Index	R^3	R^4	R^5
Global Torsion	$Z^{\otimes 3(g-1)}$	$Z_2^{\otimes 2(g-1)}$	-
Parity	Z_2	-	-
Knot Type	Gen. KT.	-	-

In R^5 all representations (embeddings) of a genus-g strange attractor become equivalent under isotopy.

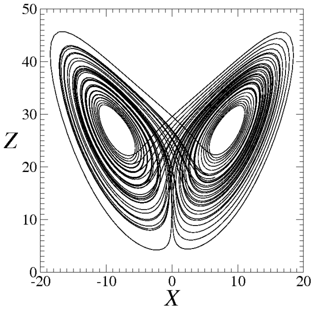
Modding Out a Rotation Symmetry

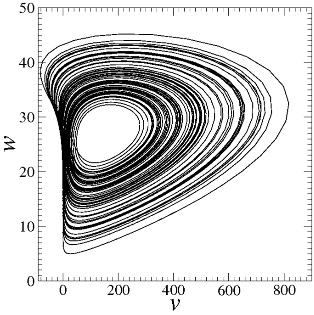
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$

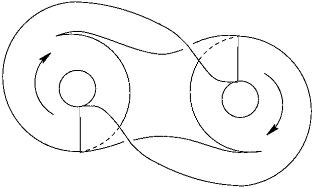


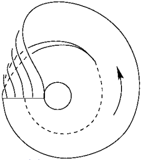


Lorenz Attractor and Its Image



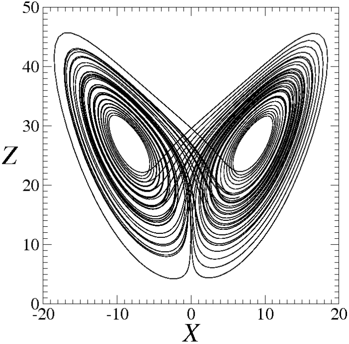


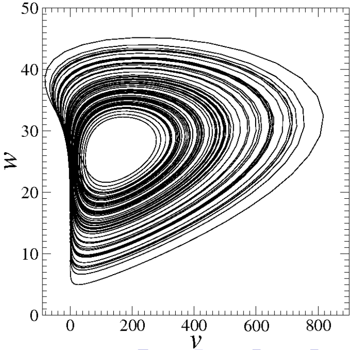




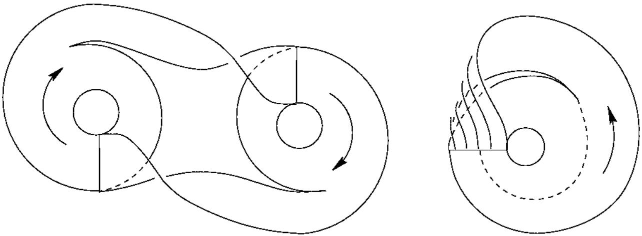
Creating a Cover with Symmetry

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \leftarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Re\ (X+iY)^2 \\ Im\ (X+iY)^2 \\ Z \end{pmatrix}$$

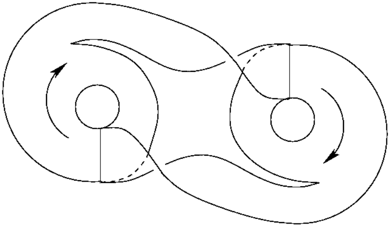


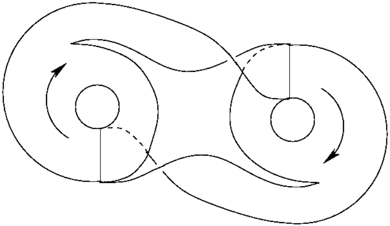


Cover-Image Branched Manifolds



Two Two-fold Lifts
Different Symmetry

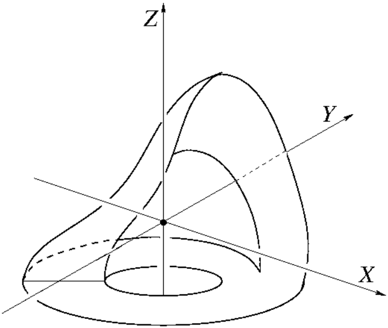




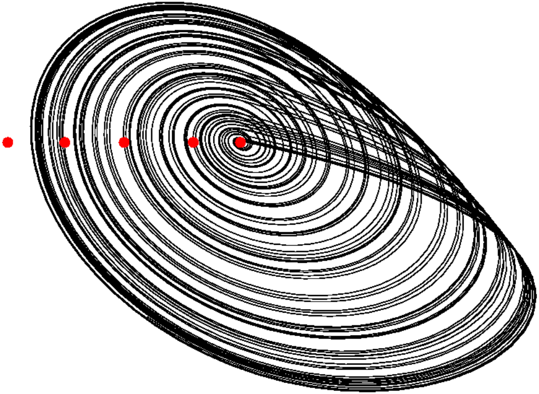
Rotation Symmetry

Inversion Symmetry

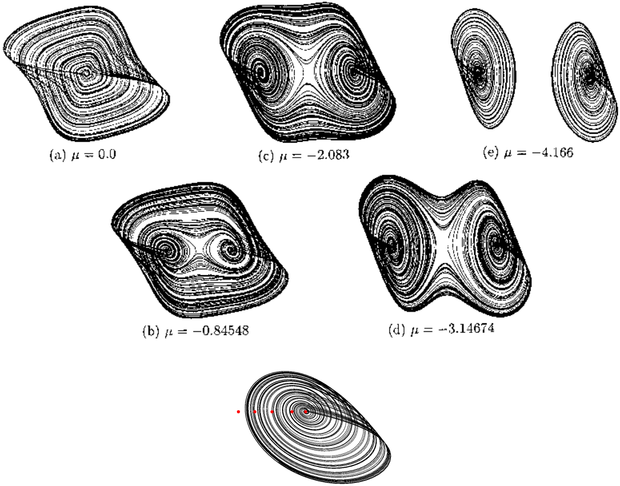
Topological Index: Choose Group
Choose Rotation Axis (Singular Set)



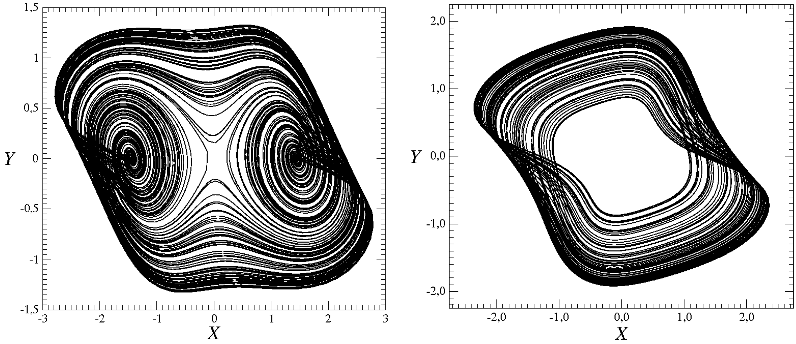
Different Rotation Axes Produce Different (Nonisotopic) Lifts



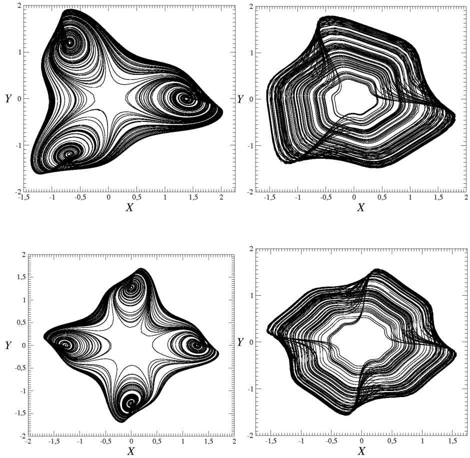
Nonisotopic Locally Diffeomorphic Lifts



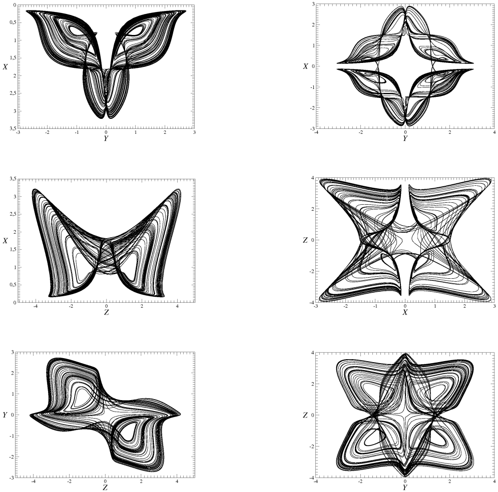
Two Two-fold Covers
Same Symmetry



Three-fold, Four-fold Covers



Two Inequivalent Lifts with V_4 Symmetry



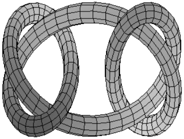
Algorithm

- Construct Invariant Polynomials, Syzygies, Radicals
- Construct Singular Sets
- Determine Topological Indices
- Construct Spectrum of Structurally Stable Covers
- Structurally Unstable Covers Interpolate

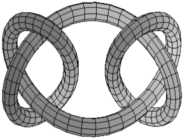
Symmetries Due to Symmetry

- Schur's Lemmas & Equivariant Dynamics
- Cauchy Riemann Symmetries
- Clebsch-Gordon Symmetries
- Continuations
  - Analytic Continuation
  - Topological Continuation
  - Group Continuation

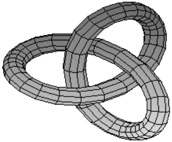
Covers of a Trefoil Torus



Granny Knot

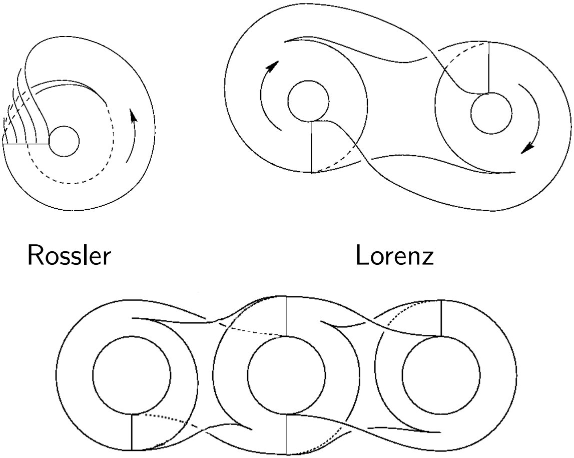


Square Knot



Trefoil Knot

Covers of Covers of Covers

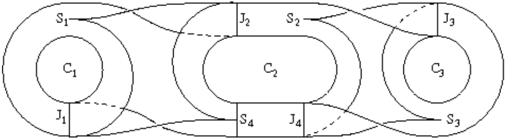


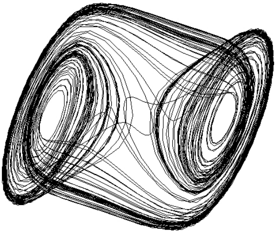
Rossler

Lorenz

Ghrist

EveryKnot Lives Here



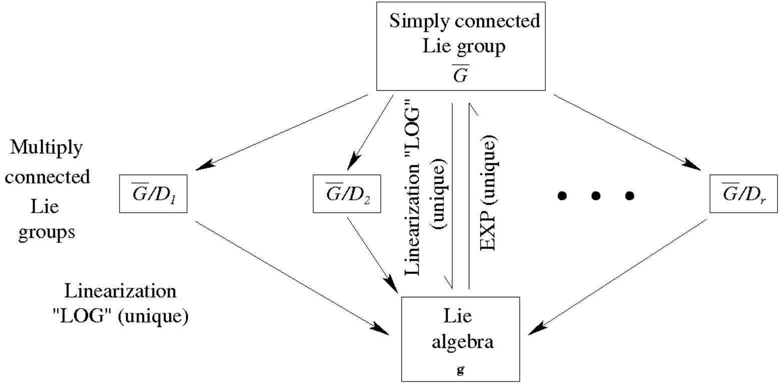


Local Stuff

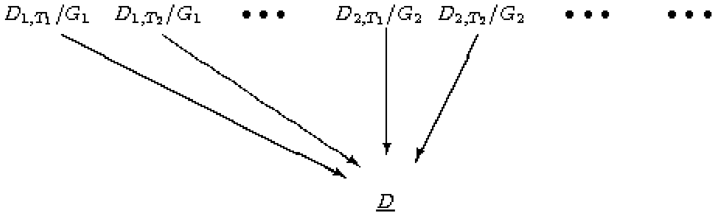
Groups:
Local Isomorphisms
Cartan's Theorem

Dynamical Systems:
Local Diffeomorphisms
??? Anything Useful ???

Cartan's Theorem for Lie Groups



Locally Diffeomorphic Covers of $\underline{D}$



$\underline{D}$: Universal Image Dynamical System

Local Isomorphisms & Diffeomorphisms

Local Isomorphisms & Diffeomorphisms

Lie Groups

Local Isomorphisms & Diffeomorphisms

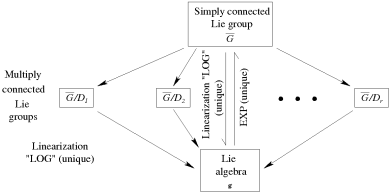
Lie Groups

Local Isomorphisms

Local Isomorphisms & Diffeomorphisms

Lie Groups

Local Isomorphisms

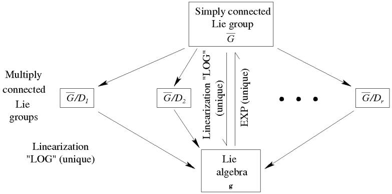


Local Isomorphisms & Diffeomorphisms

Lie Groups

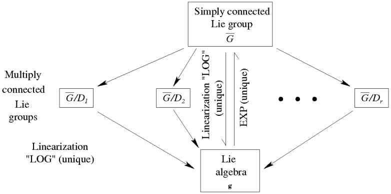
Dynamical Systems

Local Isomorphisms



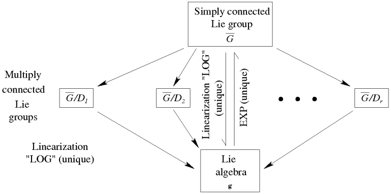
Local Isomorphisms & Diffeomorphisms

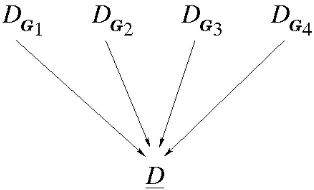
Lie Groups	Dynamical Systems
Local Isomorphisms	Local Diffeos



Local Isomorphisms & Diffeomorphisms

Lie Groups	Dynamical Systems
Local Isomorphisms	Local Diffeos





Rotating the Attractor

$$\frac{d}{dt}\left[\begin{array}{c} X \\ Y \end{array}\right] = \left[\begin{array}{c} F_1(X,Y) \\ F_2(X,Y) \end{array}\right] + \left[\begin{array}{c} a_1 \sin(\omega_d t + \phi_1) \\ a_2 \sin(\omega_d t + \phi_2) \end{array}\right]$$

$$\left[\begin{array}{c} u(t) \\ v(t) \end{array}\right] = \left[\begin{array}{cc} \cos \Omega t & -\sin \Omega t \\ \sin \Omega t & \cos \Omega t \end{array}\right] \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right]$$

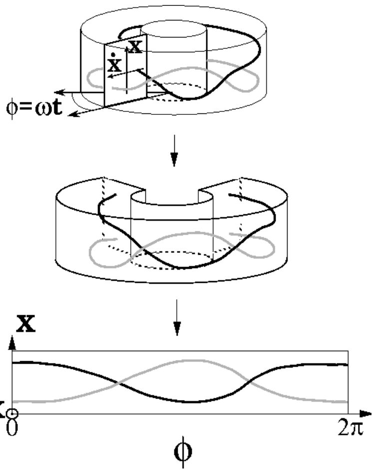
$$\frac{d}{dt}\left[\begin{array}{c} u \\ v \end{array}\right] = R\mathbf{F}(R^{-1}\mathbf{u}) + R\mathbf{t} + \Omega \left[\begin{array}{c} -v \\ +u \end{array}\right]$$

$\Omega = n\ \omega_d$ — Global Diffeomorphisms

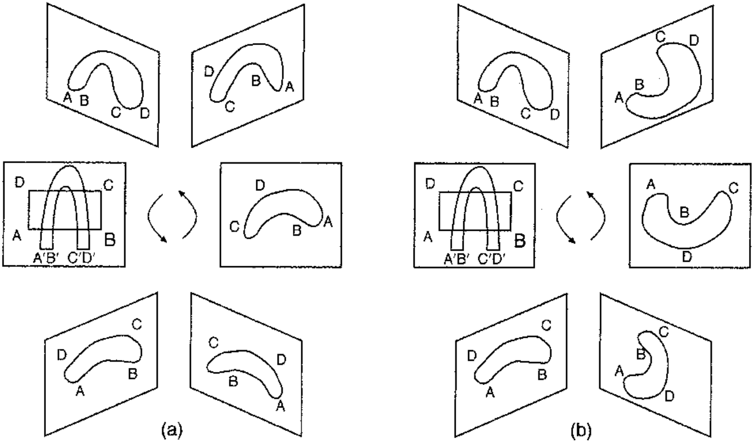
$q\ \Omega = p\ \omega_d$ — Local Diffeomorphisms (p-fold covers)

Another Visualization

Cutting Open a Torus



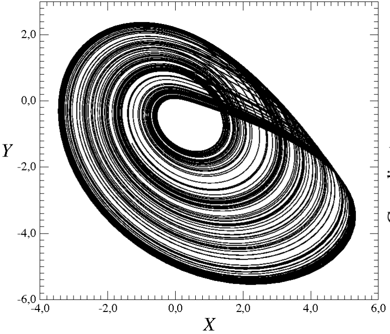
Global Torsion

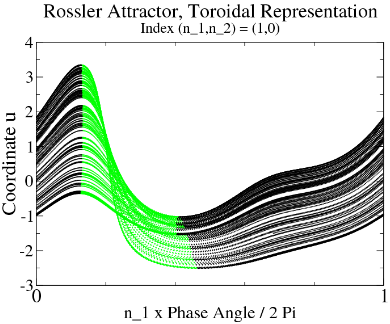


(a) (b)

Rossler Attractor: Two Representations

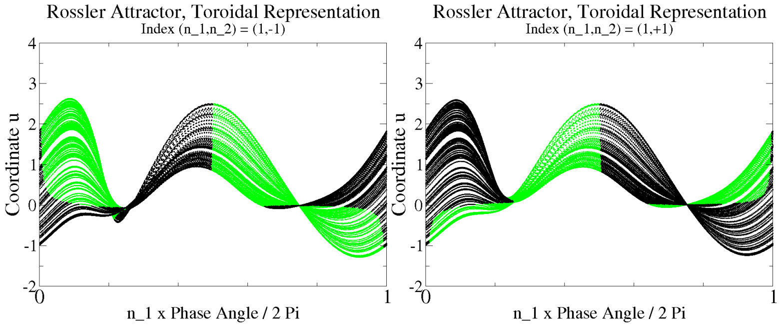
R^3 $\qquad$ $D^2 \times S^1$





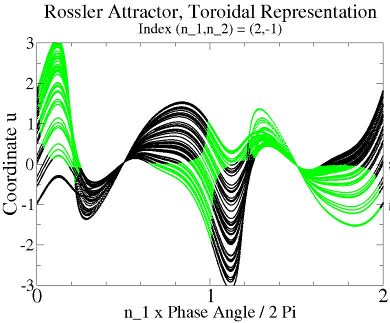
Rossler Attractor:

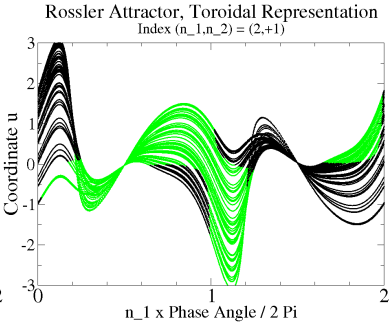
Two More Representations with $n = \pm 1$



Rossler Attractor:

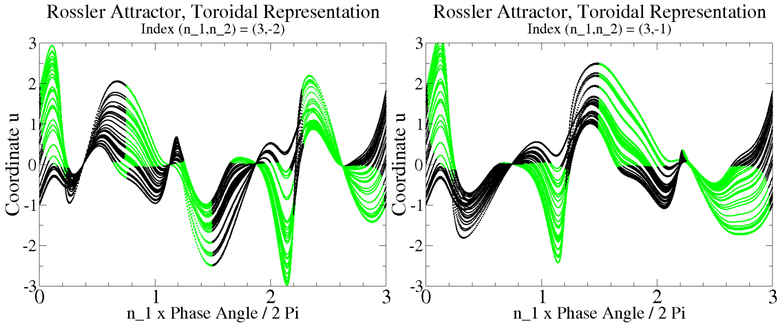
Two Two-Fold Covers with $p/q = \pm 1/2$





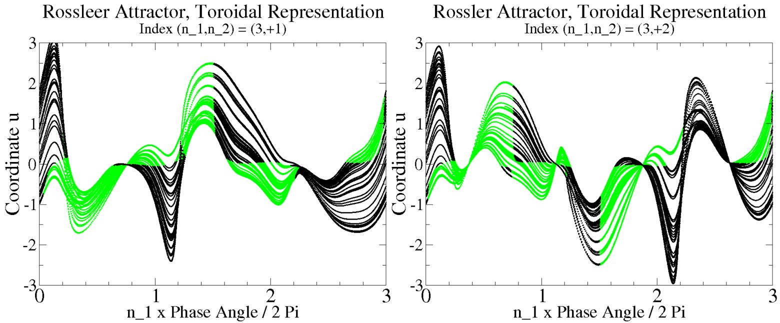
Rossler Attractor:

Two Three-Fold Covers with $p/q = -2/3, -1/3$



Rossler Attractor:

And Even More Covers (with $p/q = +1/3, +2/3$)



Angular Momentum and Energy

$$L(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} X dY - Y dX \qquad K(0) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) dt$$

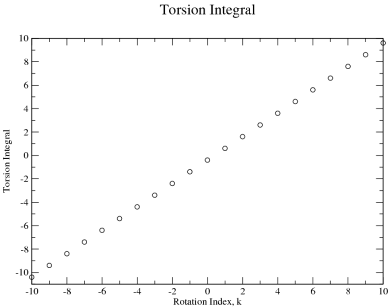
$$L(\Omega) = \langle u\dot{v} - v\dot{u} \rangle \qquad K(\Omega) = \langle \frac{1}{2}(\dot{u}^2 + \dot{v}^2) \rangle$$

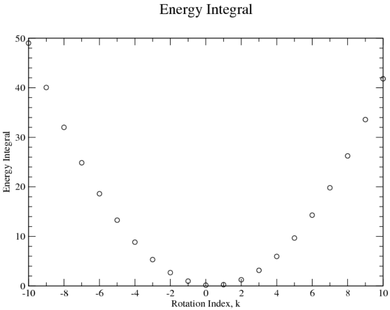
$$= L(0) + \Omega \langle R^2 \rangle \qquad = K(0) + \Omega L(0) + \frac{1}{2}\Omega^2 \langle R^2 \rangle$$

$$\langle R^2 \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (X^2 + Y^2) dt = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} (u^2 + v^2) dt$$

Energy and Angular Momentum

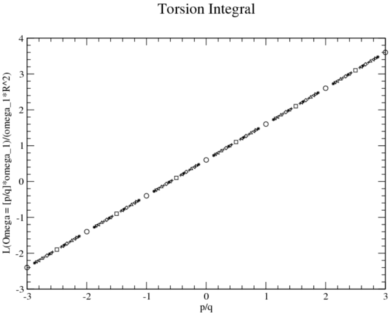
Diffeomorphic, Quantum Number n

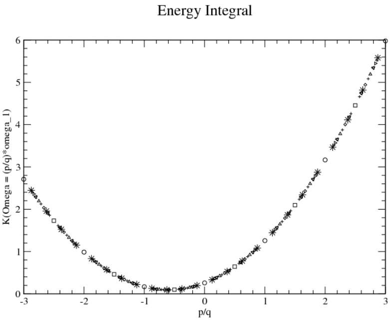




Energy and Angular Momentum

Subharmonics, Quantum Numbers p/q





Representations

An embedding creates a diffeomorphism between an ('invisible') dynamics in someone's laboratory and a ('visible') attractor in somebody's computer.

Embeddings provide a *representation* of an attractor.

Equivalence is by Isotopy.

Irreducible is by Dimension

Representations

We know about representations from studies of groups and algebras.

We use this knowledge as a guiding light.

Inequivalent Irreducible Representations

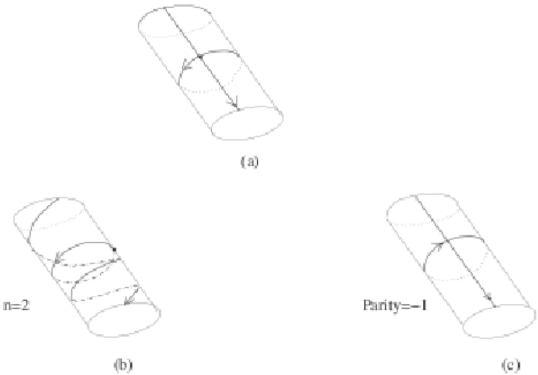
Irreducible Representations of 3-dimensional Genus-one attractors are distinguished by three topological labels:

Parity	P
Global Torsion	N
Knot Type	KT

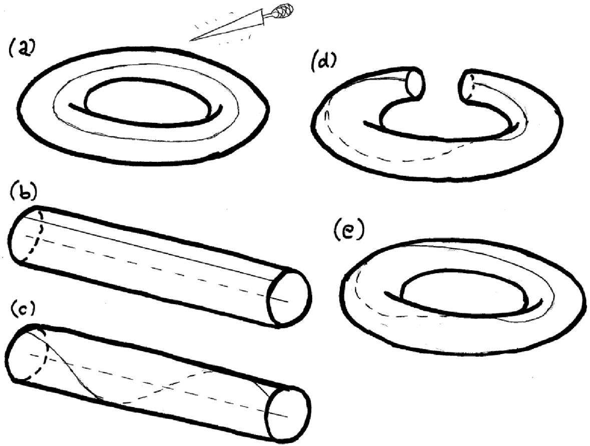
$$\Gamma^{P,N,KT}(\mathcal{SA})$$

Mechanism (stretch & fold, stretch & roll) is an invariant of embedding. It is independent of the representation labels.

Global Torsion & Parity



Inequivalence in R^3



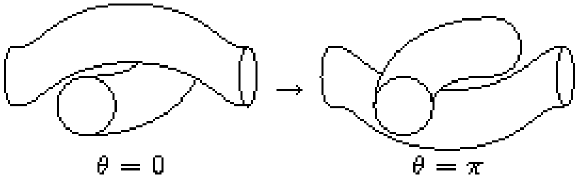
Equivalent Reducible Representations

Topological indices (P,N,KT) are obstructions to isotopy for embeddings of minimum dimension (irreducible representations).

Are these obstructions removed by injections into higher dimensions (reducible representations)?

Systematically?

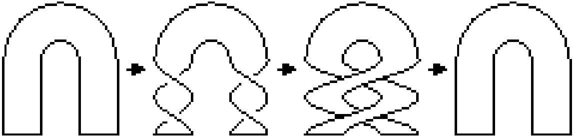
Crossing Exchange in R^4



Parity reversal is also possible in R^4 by isotopy.

Isotopies

2 Twists = 1 Writhe = Identity



$$Z \longrightarrow Z_2$$

Global Torsion $\longrightarrow$ Binary Op

Equivalences by Injection

Obstructions to Isotopy

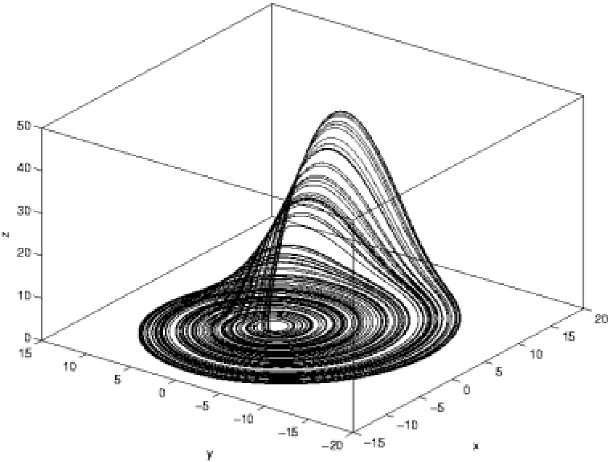
R^3	$\rightarrow$	R^4	$\rightarrow$	R^5
Global Torsion		Global Torsion		
Parity				
Knot Type				

There is one *Universal* reducible representation in R^N, $N \geq 5$.
In R^N the only topological invariant is *mechanism*.

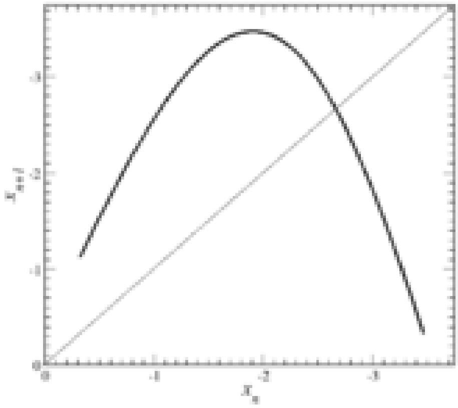
What We Did

What We Found

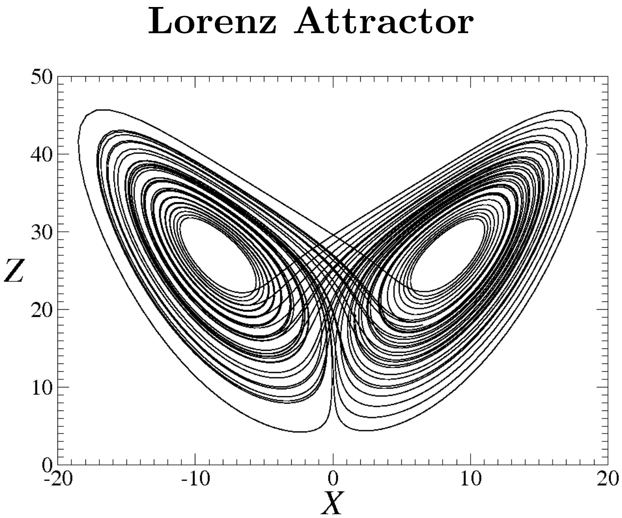
Rössler Attractor



Rössler Attractor - Return Map



Basis Set of Orbits



Return Map for Lorenz Attractor

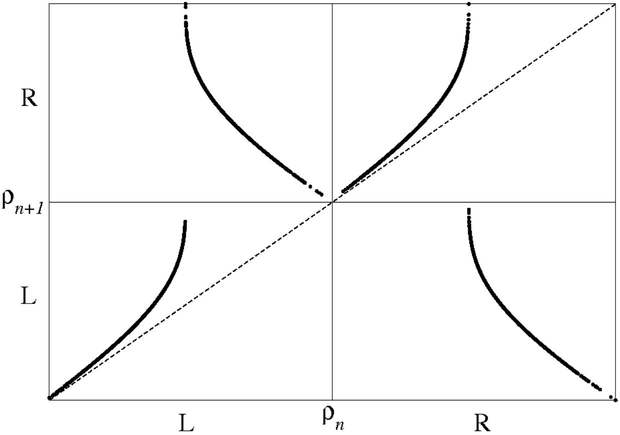
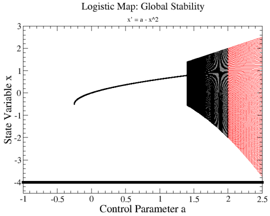


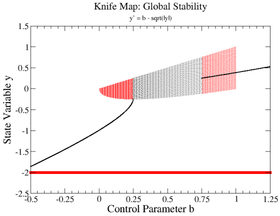
Image of Lorenz Return Map

To be supplied

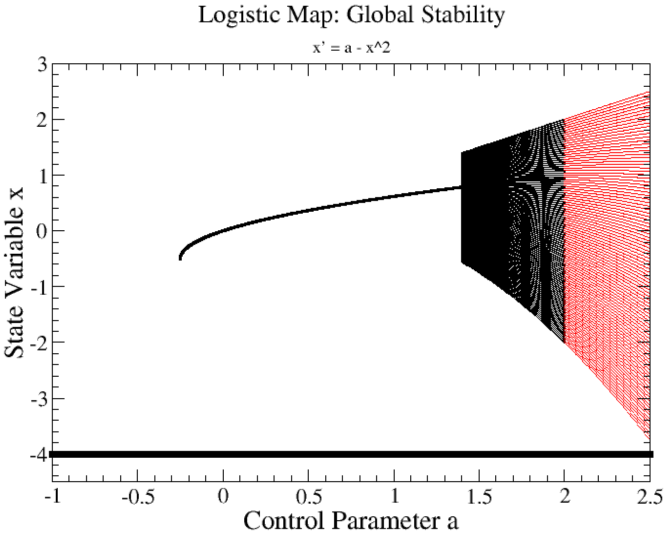
Comparison-01

Stability Regions

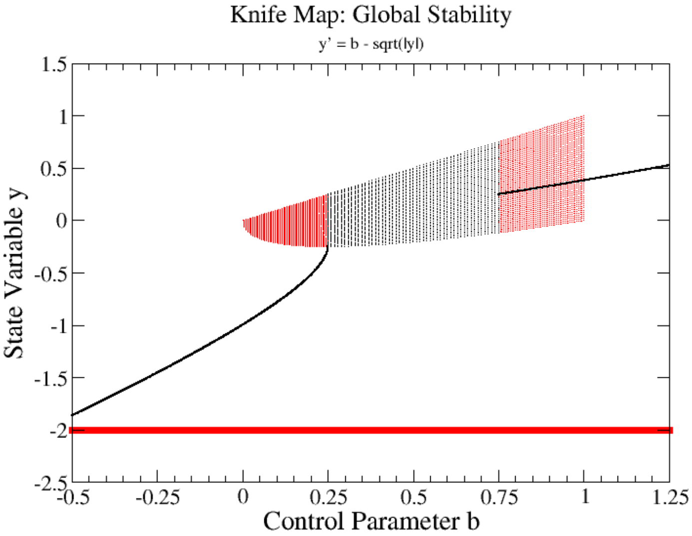




BigView: Logistic Map

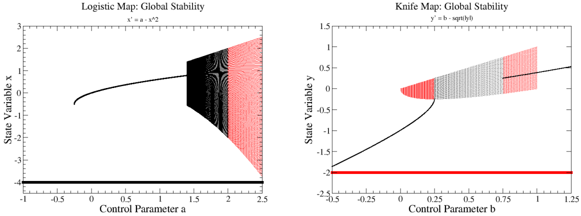


BigView: Knife Map

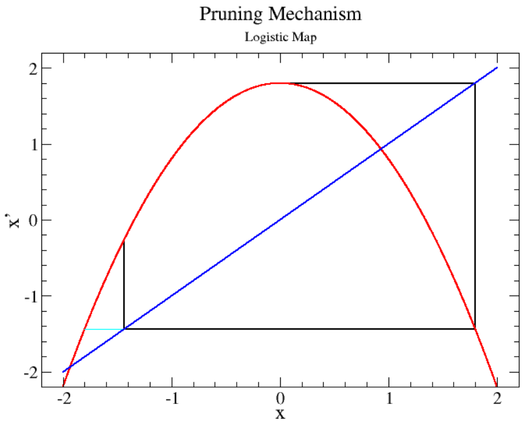


Comparison-02

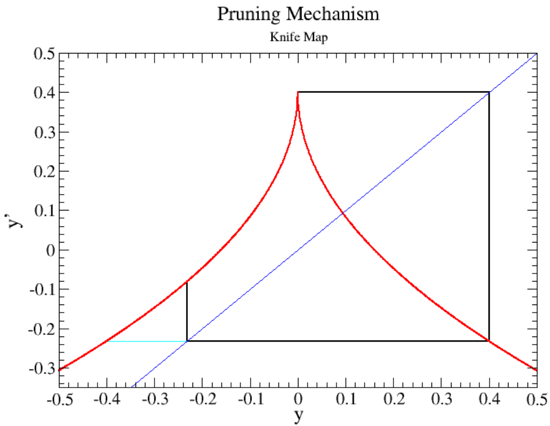
Stability Regions



Return Map - Rössler Attractor



Return Map - Lorenz Image



Return Map Approximations

The Rossler return map is well approximated by the following maps:

$$
\begin{aligned}
x' &= \lambda x(1-x) \\
x' &= a - x^2 \\
x' &= 1 - \mu x^2 \\
x' &= 1 - \left| \frac{x-m}{w} \right|^2
\end{aligned}
$$

Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

$$
\begin{aligned}
y' &= b - |y|^{1/2} \\
y' &= 1 - \mu |y|^{1/2} \\
y' &= 1 - \left| \frac{y - m}{w} \right|^{1/2}
\end{aligned}
$$

Comparison:

Logistic & Knife Maps

Logistic Map	*Knife Map*
$x' = f(x;a) = a - (\lvert x \rvert)^2$	$y' = f(y;b) = b - (\lvert y \rvert)^{1/2}$

Logistic-04

... for several values of a

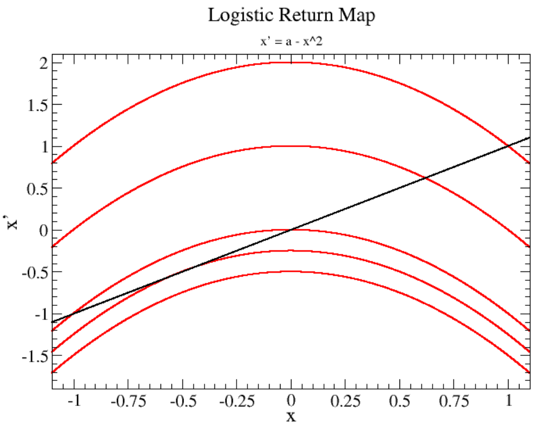
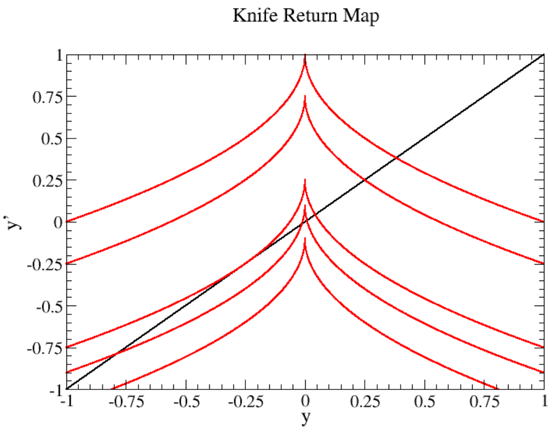
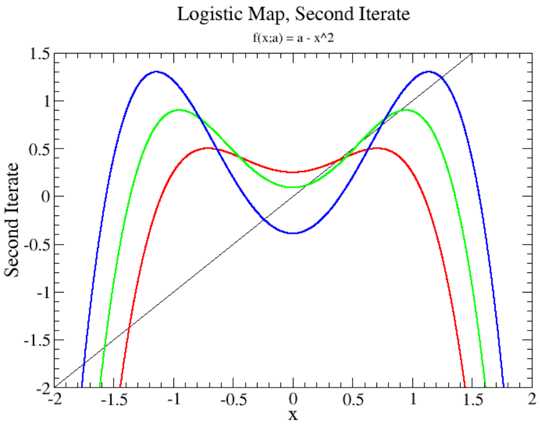


Image Lorenz-04

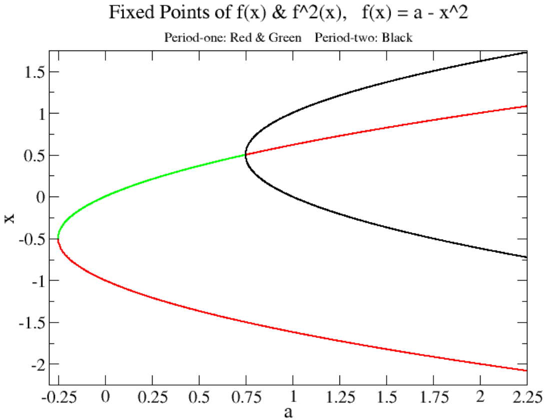
Knife Return Maps



Second Return Map

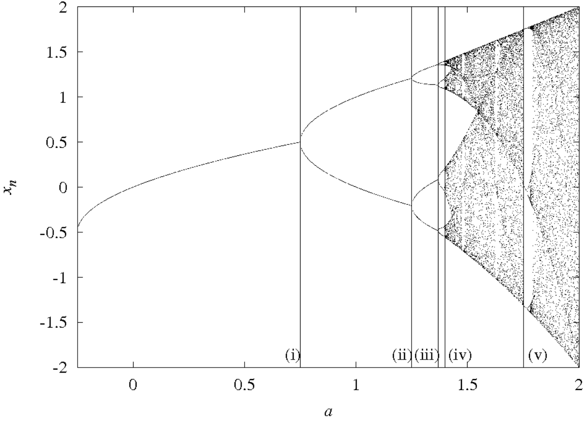


Period 1 & 2 Orbits - Logistic

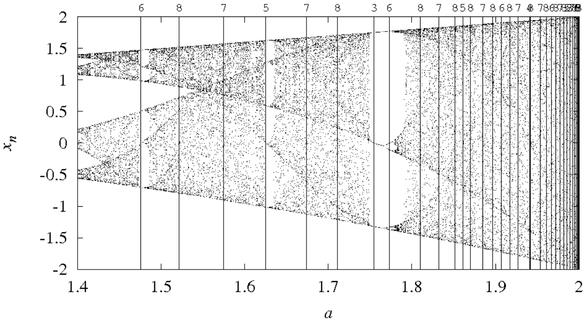


Bifurcation-01

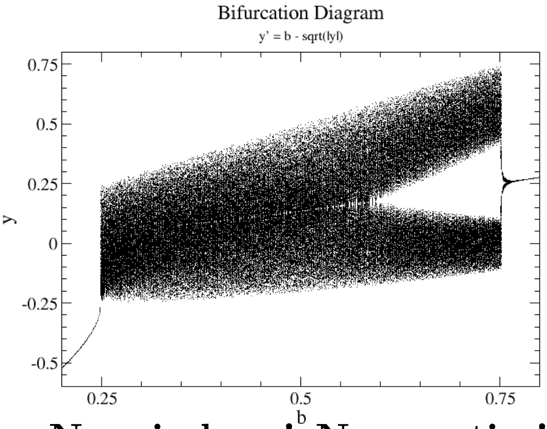
Bifurcation Diagram



.. Blow Up .... with Caustics

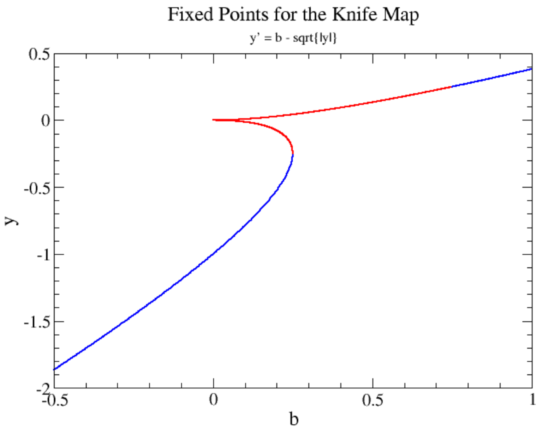


Knife Map - Bifurcation Diagram

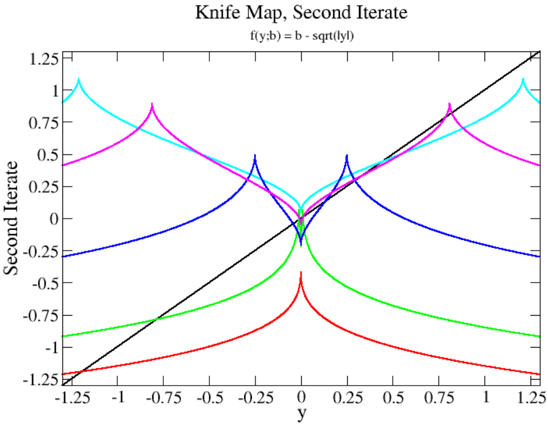


No windows! No caustics!

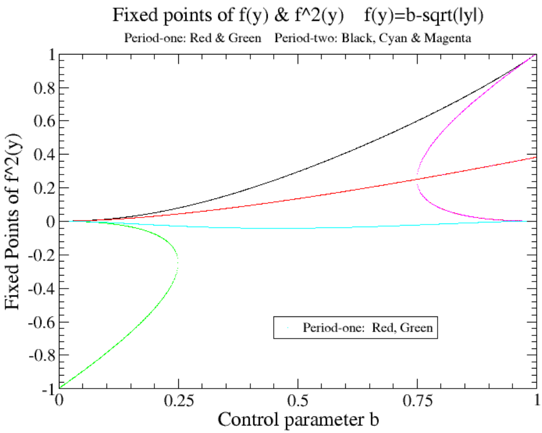
Fixed Points (Knife)



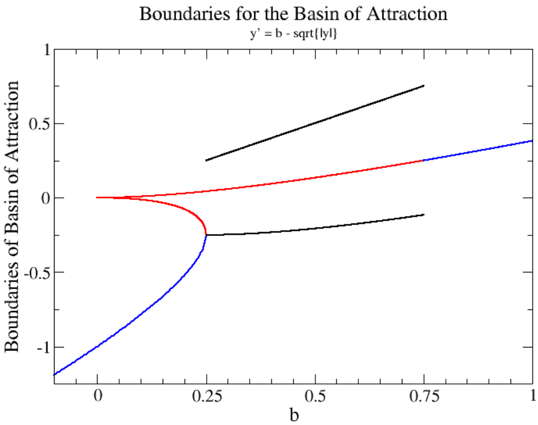
Second Iterates - Knife Map



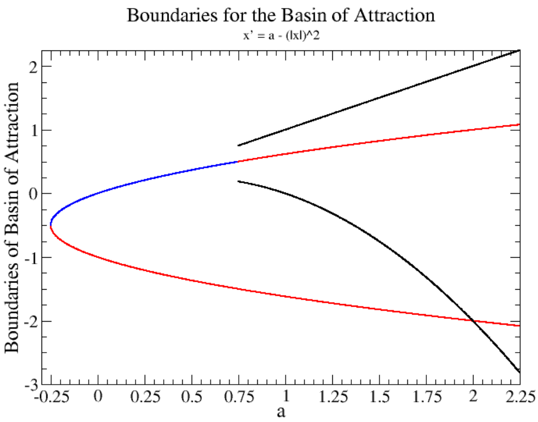
Period-One & Period-Two Orbits



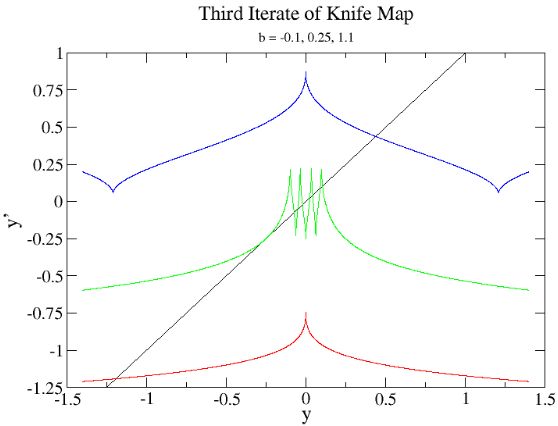
Attractor boundary (Knife)



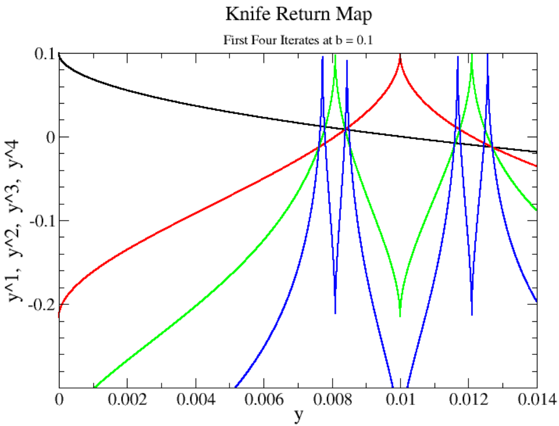
Attractor Boundaries - Logistic



Forcing Diagram - Horseshoe



Forcing Diagram - Horseshoe



Explosions-02

Table: Values $M^{(p)}$ of y where the pth iterate $f^{(p)}(y;b)$ has maxima. These locations are determined by a simple recursion relation (last line) where the indices $s_p = \pm 1$ are incoherent.

p	Number Max.	Coordinate Values
1	1	0
2	2	$\pm b^2$
3	4	$\pm(b \pm b^2)^2$
$\cdots$	$\cdots$	$\cdots$
$p+1$	2^p	$M^{(p+1)} = s_p(b + M^{(p)})^2$

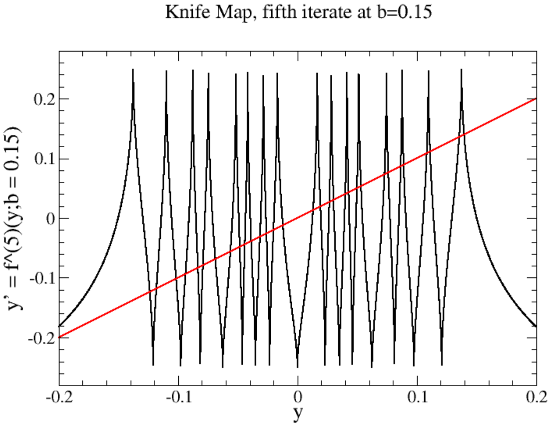
Explosions-03

As $p \to \infty$, with all $s_j = +1$, the abscissa of the rightmost point goes to a limit. The quadratic equation for this limit gives:

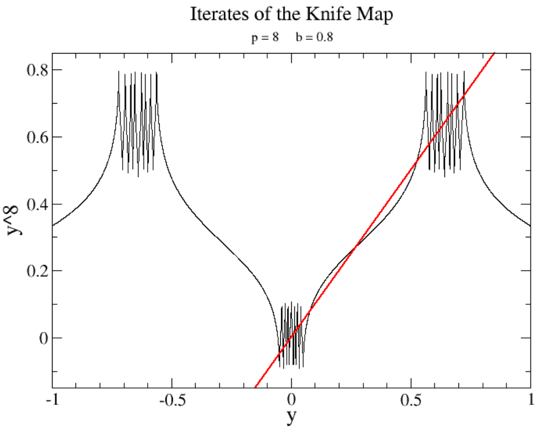
$$y(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b}\right)$$

At $b = \frac{1}{4}$ the bounding box is a square — beyond that the diagonal fails to intersect all the zig - zags. Orbits begin to gen pruned away in singulr saddle node bifurcations.

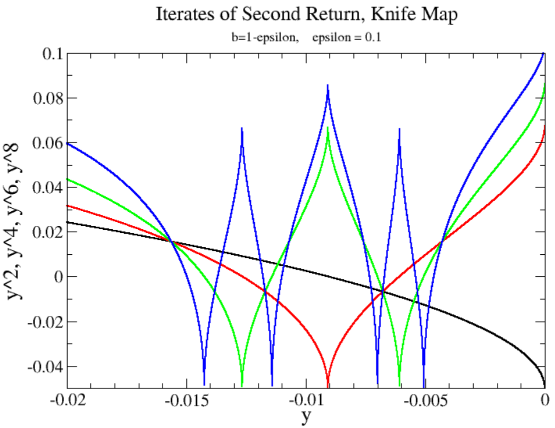
Structural Stability: $0 < b < \frac{1}{4}$



End Play - Near $b = 1$

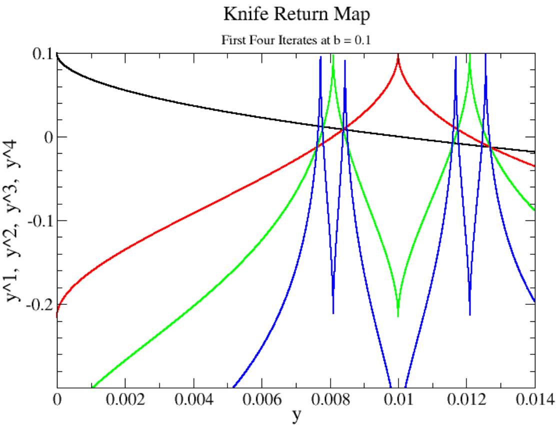


Iterates Near $b = 1$

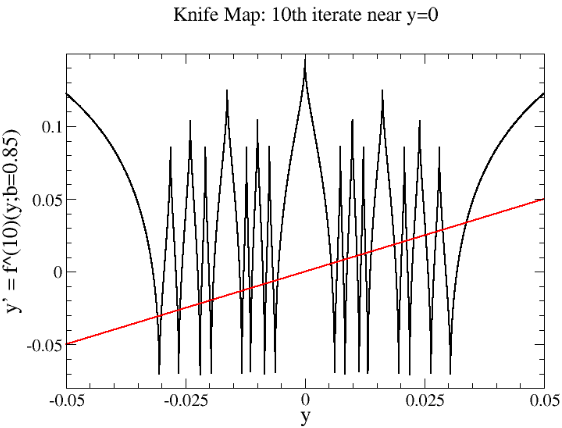


implosion1

Note Scaling Relations

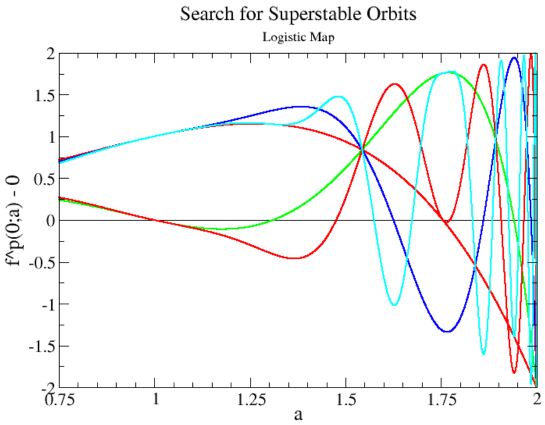


Structural Stability: $\frac{3}{4} < b < 1$

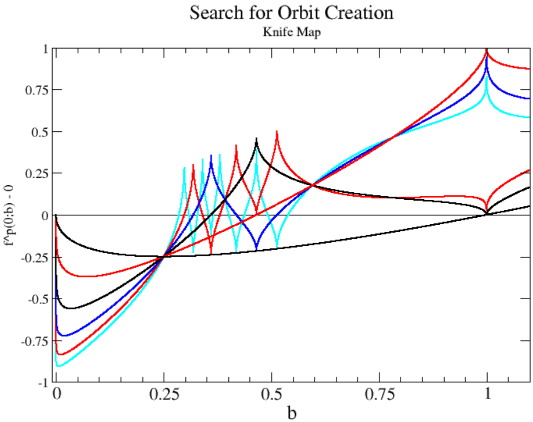


Hunt for Saddle-Node Bifurcations

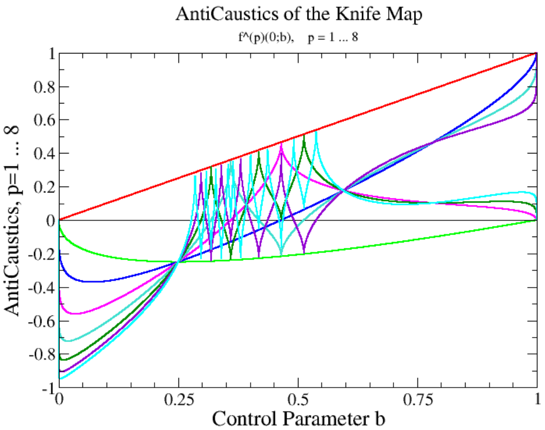
Caustic Crossings



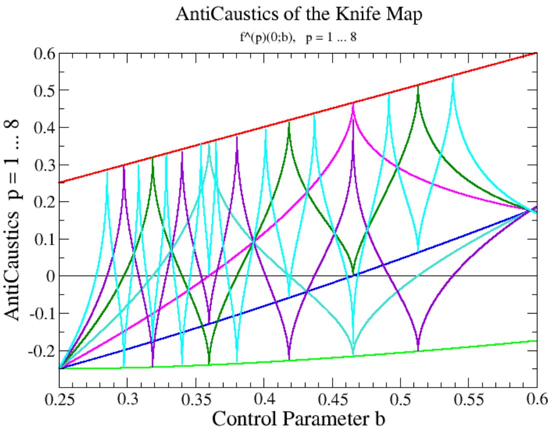
Hunt for Singular SNBs



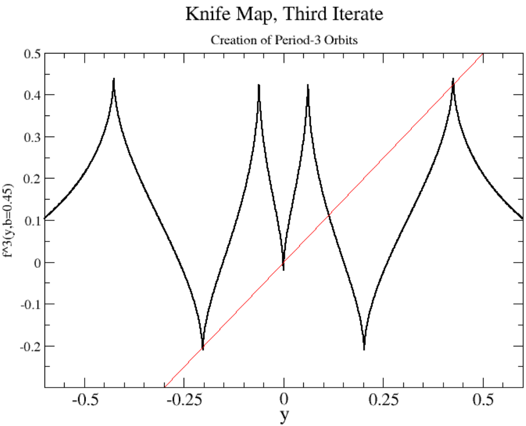
Anti Caustic Crossings



Anti Caustic Crossings: Expansion



Period Three Singular SNB



Renormalization-02

Local expression near $y = 0$ for the period-three explosion:

$$h(y;b) = f^{(3)}(y;b) = b - \sqrt{|b - \sqrt{|b - \sqrt{|y|}|}|}$$

$$h(b_3+\epsilon; y) \rightarrow \left(b_3 - \sqrt{\sqrt{b_3} - b_3}\right) +$$

$$\left(1 + \frac{2\sqrt{b_3} - 1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\epsilon + \left(\frac{1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\sqrt{|y|}$$

Renormalization-03

Renormalization for the period-three explosion.

$$y' = h(y; b_3 + \epsilon) \rightarrow \Delta(b - b_3) + \alpha\sqrt{|y|} =$$

$$1.286974759(b - b_3) + 0.7869747590\sqrt{|y|}$$

$$z' = (\Delta/\alpha^2)(b_3 - b) - \sqrt{|z|}$$

Renormalization Algorithm: K10*

1. Write down the symbol sequence for the primary period-p orbit: $K10* = K\sigma_1\sigma_2\cdots\sigma_{p-1}$.
2. Make the identification $\sigma = +1 \to s = +1, \sigma = 0 \to s = -1$.
3. Construct $f^{(p)}(b;y) \to$

$$b - \sqrt{s_{p-1}(b - \cdots \sqrt{s_2(b - \sqrt{s_1(b - \sqrt{y})})} \cdots)}$$

4. Taylor expand this function to terms linear in b and $\sqrt{y}$ and determine the value of b for which the constant term vanishes.

Equations: K10*

For the saddle node pair $5_2 = K1001$ this algorithm gives

$$b - \sqrt{(+1)(b - \sqrt{(-1)(b - \sqrt{(-1)(b - \sqrt{(+1)(b - \sqrt{y}))))}}}}$$

The constant term vanishes for $b = 0.418656$, and for this value of b

$$y' = \Delta(b - b_{5_2}) + \alpha\sqrt{|y|} = -3.231180\Delta b - 1.983690\sqrt{|y|}$$

Results: K10* to Period 6

$$y' = \Delta(b - b_c) + \alpha\sqrt{|y|} \qquad y', y \simeq 0$$

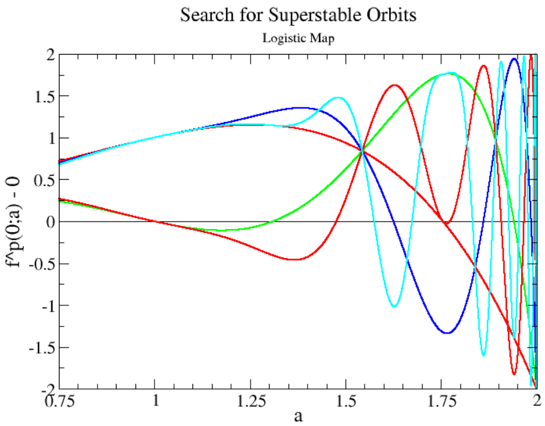
Orbit	Symbolics	b_c	Δ	α
3_1	$K10$	0.465571	1.286974	0.786974
4_2	$K100$	0.360157	2.624703	1.180563
5_3	$K1000$	0.318897	4.647225	1.664335
5_2	$K1001$	0.418656	−3.231180	−1.983690
5_1	$K1011$	0.513175	2.628970	1.509712
6_5	$K10000$	0.297846	7.481728	2.233184
6_4	$K10001$	0.340328	−8.535145	−3.639587
6_3	$K10011$	0.380540	7.596535	3.574548

Renormalization-07

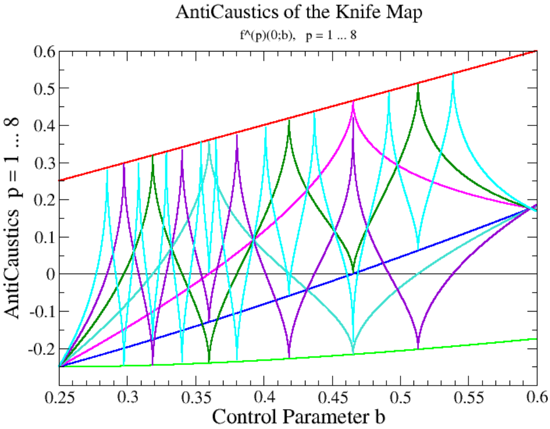
Renormalization for the final period-two explosion.

$$f^{(2)}(1-\epsilon, y) \simeq -\frac{\epsilon}{2} + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)\sqrt{|y|} \tag{1}$$

Hunt for Saddle-Node Bifurcations



Hunt for S. Saddle-Node Bifurcations



Breakpoints

Table: Important parameter values for global stability and unstable periodic orbit behavior.

Global Stability	Unstable Orbits
	0.0
1/4	1/4
	0.5957439420
3/4	
	0.7825988587
	1.0

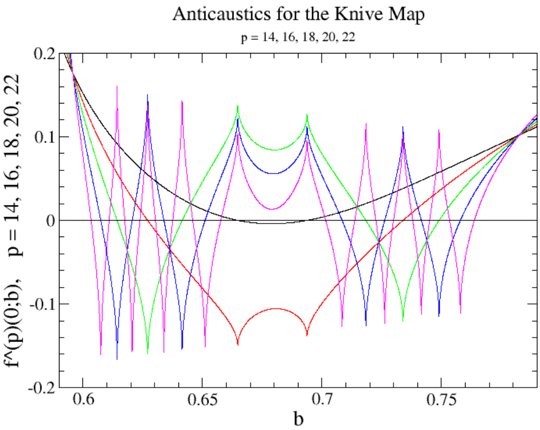
U Sequence

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)[a]

Name	Bifurcation	Name	Bifurcation	Name	Bifurcation
$^0_1$	$1_1[s_1]$	00101 $^0_1$1	$7_3[s_7^3]$	0001 $^0_1$1	$6_4[s_6^3]$
01	$2_1[s_1 \times 2^1]$	001010 $^0_1$1	$8_5[s_8^4]$	000111 $^0_1$1	$8_{11}[s_8^9]$
0111	$4_1[s_1 \times 2^2]$	001 $^0_1$1	$5_2[s_5^2]$	00011 $^0_1$1	$7_7[s_7^7]$
01010111	$8_1[s_1 \times 2^3]$	001110 $^0_1$1	$8_6[s_8^5]$	000110 $^0_1$1	$8_{12}[s_8^{10}]$
0111 $^0_1$1	$6_1[s_6^1]$	00111 $^0_1$1	$7_4[s_7^4]$	000 $^0_1$1	$5_3[s_5^3]$
011111 $^0_1$1	$8_2[s_8^1]$	001111 $^0_1$1	$8_7[s_8^6]$	000010 $^0_1$1	$8_{13}[s_8^{11}]$
01111 $^0_1$1	$7_1[s_7^1]$	0011 $^0_1$1	$6_3[s_6^2]$	00001 $^0_1$1	$7_8[s_7^8]$
011 $^0_1$1	$5_1[s_5^1]$	001101 $^0_1$1	$8_8[s_8^7]$	000011 $^0_1$1	$8_{14}[s_8^{12}]$
01101 $^0_1$1	$7_2[s_7^2]$	00110 $^0_1$1	$7_5[s_7^5]$	0000 $^0_1$1	$6_5[s_6^4]$
011011 $^0_1$1	$8_3[s_8^2]$	00 $^0_1$1	$4_2[s_4^1]$	000001 $^0_1$1	$8_{15}[s_8^{13}]$
0 $^0_1$1	$3_1[s_3]$	00010011	$8_9[s_4^1 \times 2^1]$	00000 $^0_1$1	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	00010 $^0_1$1	$7_6[s_7^6]$	000000 $^0_1$1	$8_{16}[s_8^{14}]$
001011 $^0_1$1	$8_4[s_8^3]$	000101 $^0_1$1	$8_{10}[s_8^8]$		

[a]The notation P_i refers to the ith bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the ith saddle-node bifurcation of period P is denoted s_P^i, and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i.

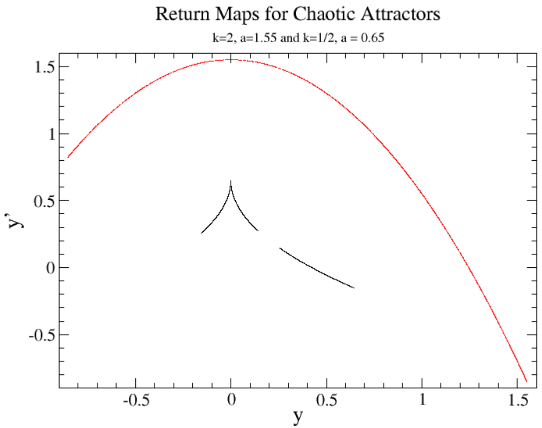
Symbol Exchange Near Endplay



Symbol Exchange Near Endplay

- Symbols 0, 1 created at $b = 0$
- New orbit, (11), created at $b = \frac{3}{4}$
- Symbol pair - 11 -, replaced by - (11) - as $b \rightarrow 1$
- Implosions begin at $b = 0.5957...$, end at midpoin.
- Explosions begin at midpoint, end at $b = 0.7825..$
- Implosions and explosions symmetrically matched

Forcing Diagram - Horseshoe



Return Map Approximations

The Rossler return map is well approximated by the following maps:

$$
\begin{aligned}
x' &= \lambda x(1-x) \\
x' &= a - x^2 \\
x' &= 1 - \mu x^2 \\
x' &= 1 - \left| \frac{x-m}{w} \right|^2
\end{aligned}
$$

Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

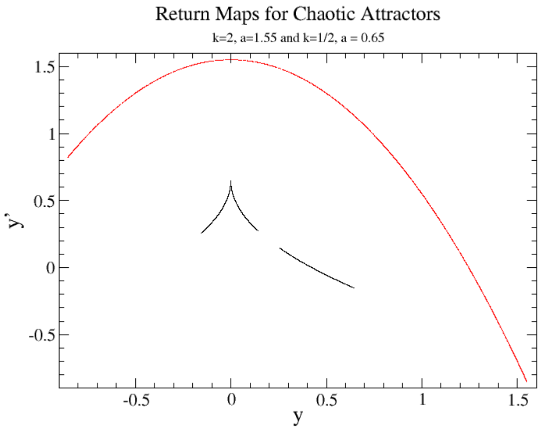
$$
\begin{aligned}
y' &= b - |y|^{1/2} \\
y' &= 1 - \mu|y|^{1/2} \\
y' &= 1 - \left|\frac{y-m}{w}\right|^{1/2}
\end{aligned}
$$

Class of Lopsided Maps

$$x' = f(x; k, a) = 1 - \left| \frac{x-m}{w} \right|^k$$

1. Zero crossings at $x = +1$ and $x = a$, $-1 \leq a \leq 0$
2. Maximum at $m = \frac{1+a}{2}$
3. Half-width $w = \frac{1-a}{2}$
4. $m + w = 1$

Forcing Diagram - Horseshoe

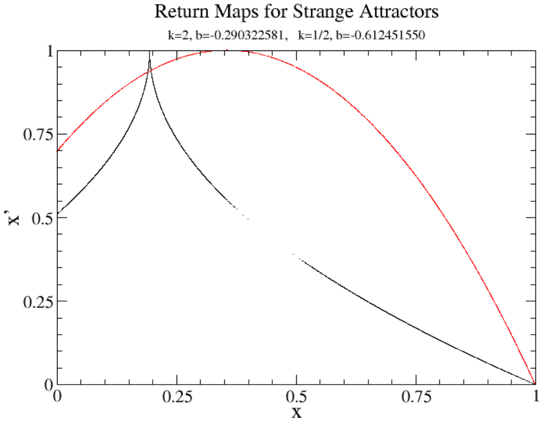


Alice in Stretch & SqueezeLand The Marvels of Topology and Chaos

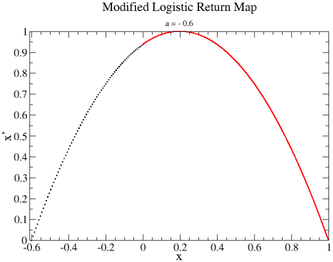
Robert Gilmore

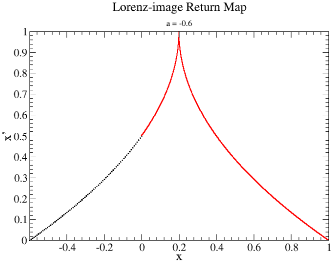
Introduction-01
Introduction-02
Overview-01
Overview-02
Overview-03
Overview-04
Overview-05
Overview-06
Overview-07
Experimental-01

Forcing Diagram - Horseshoe



Map Comparisons





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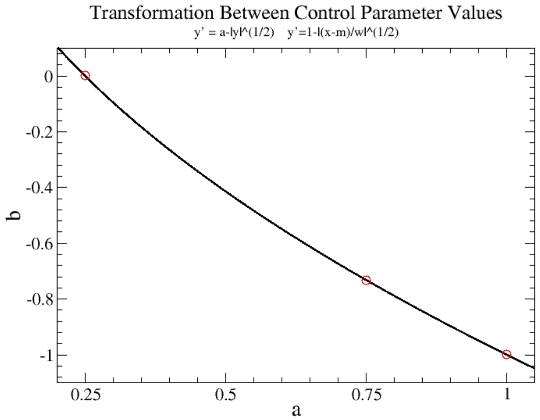

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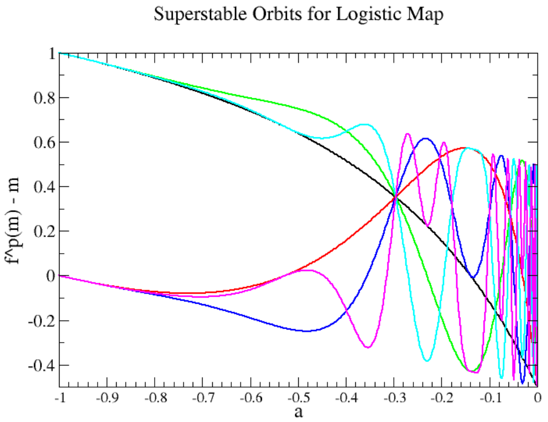

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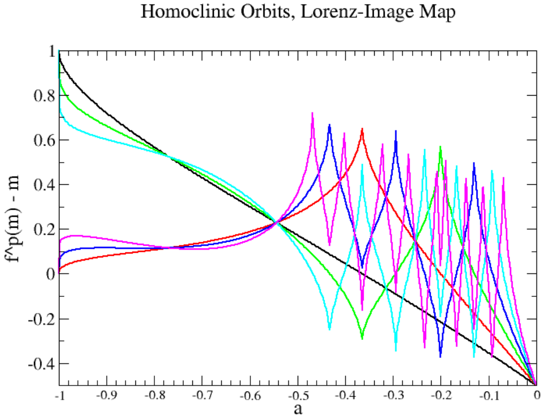

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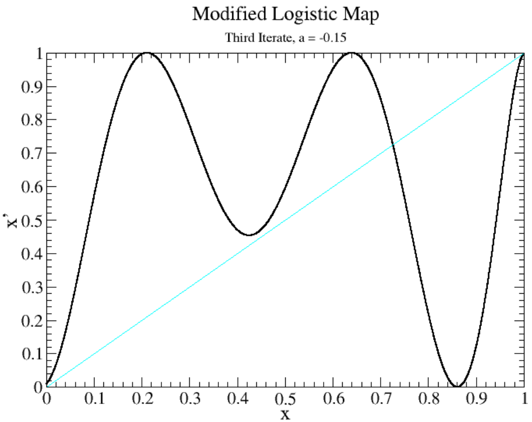
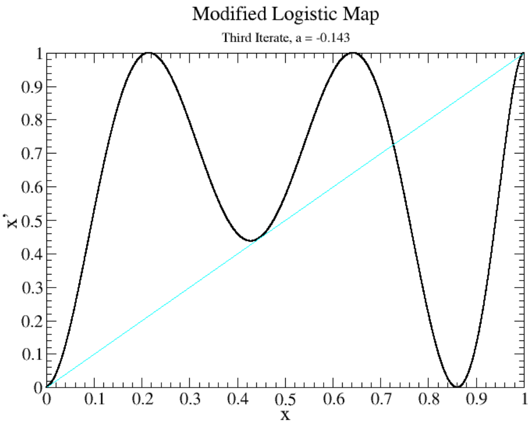
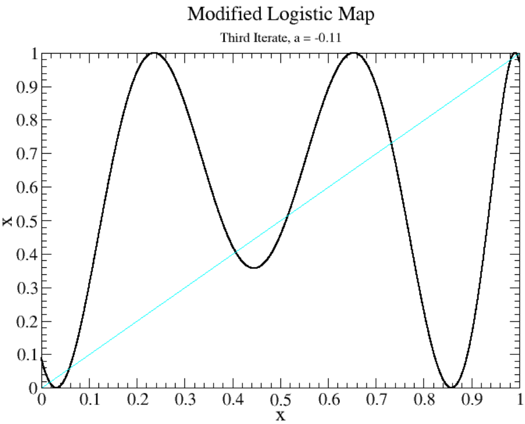


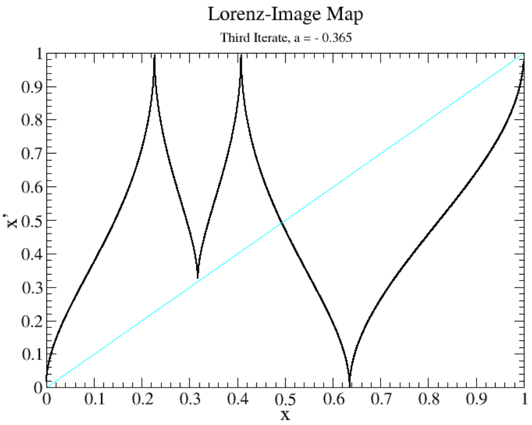
Forcing Diagram - Horseshoe



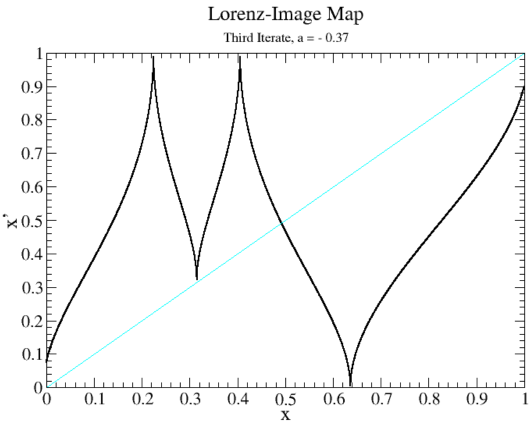
Forcing Diagram - Horseshoe



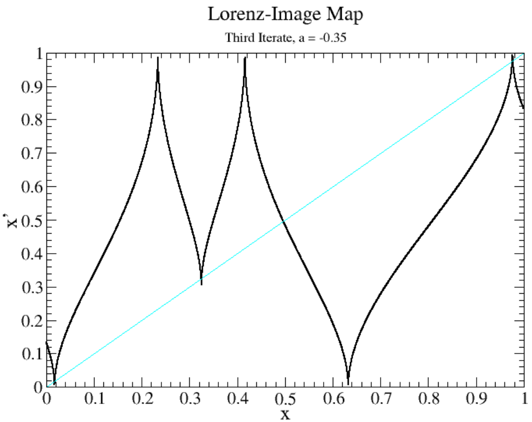
Forcing Diagram - Horseshoe



Forcing Diagram - Horseshoe



Forcing Diagram - Horseshoe



Scaling

- Logistic: SNB Period 3 = scaled version SNB of M.
- Renormalization theory applies.
- U Sequence

- Knife: S-SNB Period 3 = scaled version S-SNB of K.
- Renormalization theory applies.
- U^{-1} Sequence

Summary

1 Question Answered $\Rightarrow$

2 Questions Raised

We must be on the right track !

Original Objectives Achieved

There is now a simple, algorithmic procedure for:

- Classifying strange attractors
- Extracting classification information

from experimental signals.

Result

There is now a classification theory for low-dimensional strange attractors.

1. It is topological
2. It has a hierarchy of 4 levels
3. Each is discrete
4. There is rigidity and degrees of freedom
5. It is applicable to R^3 only — for now

The Classification Theory has 4 Levels of Structure

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits

The Classification Theory has 4 Levels of Structure

1. Basis Sets of Orbits
2. Branched Manifolds

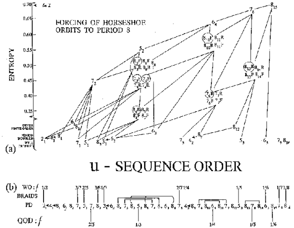
The Classification Theory has 4 Levels of Structure

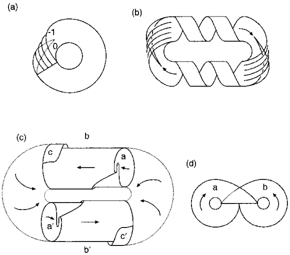
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori

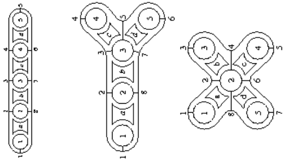
The Classification Theory has 4 Levels of Structure

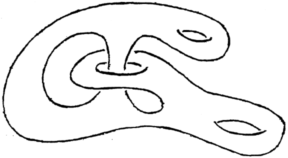
1. Basis Sets of Orbits
2. Branched Manifolds
3. Bounding Tori
4. Extrinsic Embeddings

Four Levels of Structure









Poetic Organization

LINKS OF PERIODIC ORBITS

organize

BOUNDING TORI

organize

BRANCHED MANIFOLDS

organize

LINKS OF PERIODIC ORBITS

Answered Questions

There is a Representation Theory for Strange Attractors

There is a complete set of rerpesentation labels for strange attractors of any genus g.

The labels are complete and discrete.

Representations can become equivalent when immersed in higher dimension.

All representations (embeddings) of a 3-dimensional strange attractor become isotopic (equivalent) in R^5.

The *Universal Representation* of an attractor in R^5 identifies mechanism. No embedding artifacts are left.

The topological index in R^5 that identifies mechanism remains to be discovered.

Some Unexpected Results

- Perestroikas of orbits constrained by branched manifolds
- Routes to Chaos $=$ Paths through orbit forcing diagram
- Perestroikas of branched manifolds constrained by bounding tori
- Global Poincaré section $=$ union of $g-1$ disks
- Systematic methods for cover - image relations
- Existence of topological indices (cover/image)
- Universal image dynamical systems
- NLD version of Cartan's Theorem for Lie Groups
- Topological Continuation – Group Continuuation
- Cauchy-Riemann symmetries
- Quantizing Chaos

We hope to find:

- Robust topological invariants for R^N, $N > 3$
- A Birman-Williams type theorem for higher dimensions
- An algorithm for irreducible embeddings
- Embeddings: better methods and tests
- Analog of χ^2 test for NLD
- Better forcing results: Smale horseshoe, $D^2 \rightarrow D^2$, $n \times D^2 \rightarrow n \times D^2$ (e.g., Lorenz), $D^N \rightarrow D^N$, $N > 2$
- Representation theory: complete
- Singularity Theory: Branched manifolds, splitting points (0 dim.), branch lines (1 dim).
- Singularities as obstructions to isotopy

Thanks

To my colleagues and friends:

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NSF PHY 0754081

Basic Stretch - Fold - Roll Template

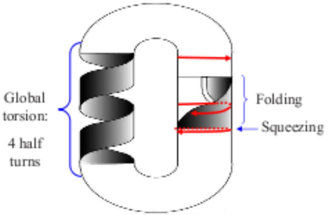


FIG. 1. (Color online) Typical scheme of a template.

Javier Used and Juan Carlos Martin,
Multiple topological structures of chaotic attractors ruling the emission of a driven laser,
Phys. Rev. E**82**, 016218 (2010).

The "S" Folding Mechanism

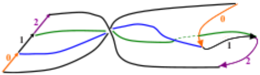
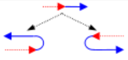
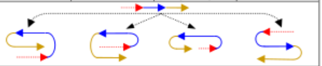


FIG. 2. (Color online) Scheme of a template with three branches and an S folding process.

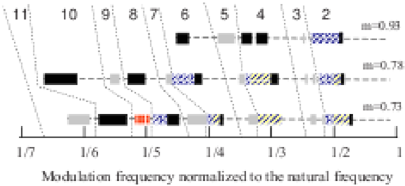
2 Branches & 3 Branches

TABLE I. (Color online) Folding processes characteristic of the different species of templates treated in this work.

Species	Horseshoe	Reverse horseshoe	Out-to-in spiral	In-to-out spiral	Staple	S
Code in Fig. 1				Not found here		
Insertion matrix	(0 1)	(1 0)	(0 2 1)	(1 2 0)	(0 2 1) or (1 2 0)	(2 1 0)
Sketch of the folding process						

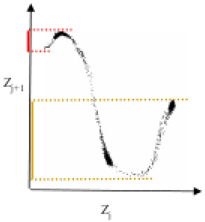
Spectrum of Behaviors

in Resonance Regions

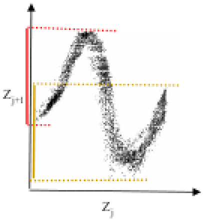


Constraints:

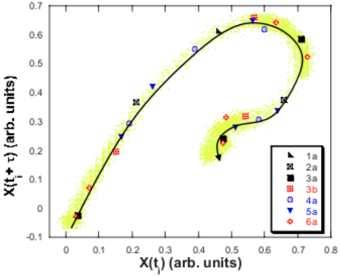
No

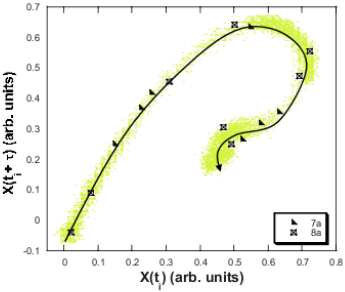


Yes

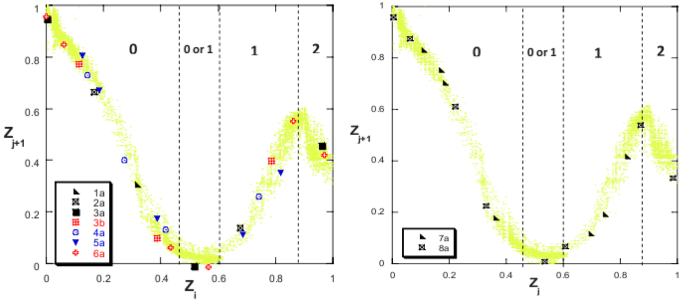


Poincaré Sections & Periodic Orbits





"Generating" Partition



Symbol Sets - Periodic Orbits

20 Equally-Spaced Planes

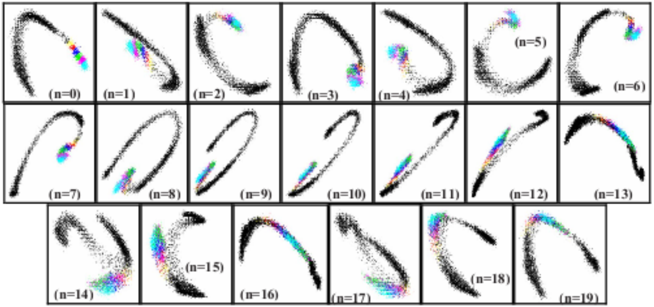


Table of Linking Numbers

Linking Numbers for Certain Orbits

TABLE II. Linking numbers between the UPOs extracted from the time series corresponding to pump modulation frequency f =4.25 KHz and modulation index m=0.73.

	$\bar{0}$	$\overline{10}$	3a	$\overline{100}$	$\overline{1000}$	$\overline{10010}$	6a	$\overline{1001010}$	8a
$\bar{0}$	0								
$\overline{10}$	9	9							
3a	14	28	28						
$\overline{100}$	14	28	42	28					
$\overline{1000}$	18	37	56	56	55				
$\overline{10010}$	23	*	70	*	92	92			
6a	28	56	*	*	112	*	139		
$\overline{1001010}$	32	*	98	*	129	*	*	194	
8a	37	74	111	111	148	*	*	*	259