A Tale of Two Maps

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Abstract

- The logistic map has been used to enrich our understanding of a large class of highly dissipative dynamical systems, especially those contained in a genus-one torus.
- A different unimodal map can be used to enrich our understanding of highly dissipative dynamical systems contained in tori of genus $g > 1$.
- The two maps are dual in a precise topological sense. The two maps are described and their properties and predictions compared.

Outline

What We Did

1. Studied maps with 2 branches
2. L & R
3. Separated by a Singularity
4. Models for Tearing Mechanism
5. Looked for "Universality"
6. Searched for Scaling

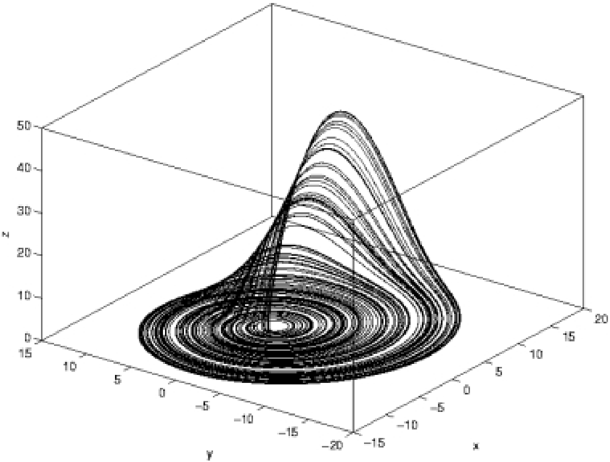
What We Found

1. Simple form: $x' = a - |x|^k$
2. $k = 2 \simeq$ folding; $k = \frac{1}{2} \simeq$ tearing

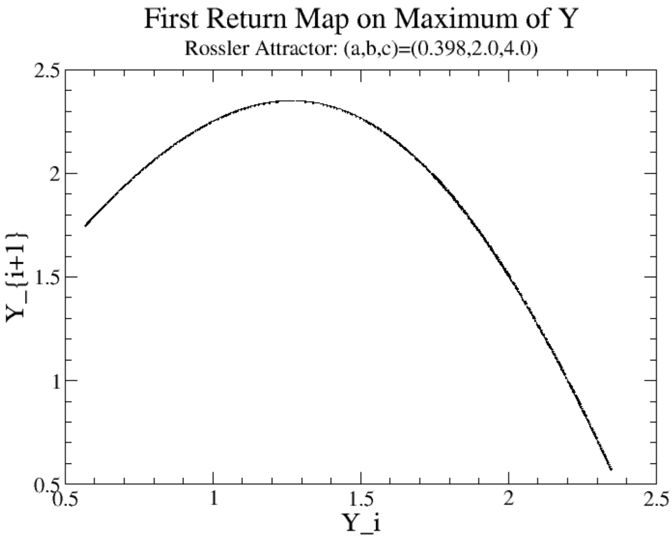
For $0 < k < 1$:

1. Localized global attractor
2. *Either* chaos or stable period 1 fixed point
3. Orbits of periods 1 and 2 organize systematics
4. Explosions
5. Prime and Compound orbits
6. Local and Global focal points

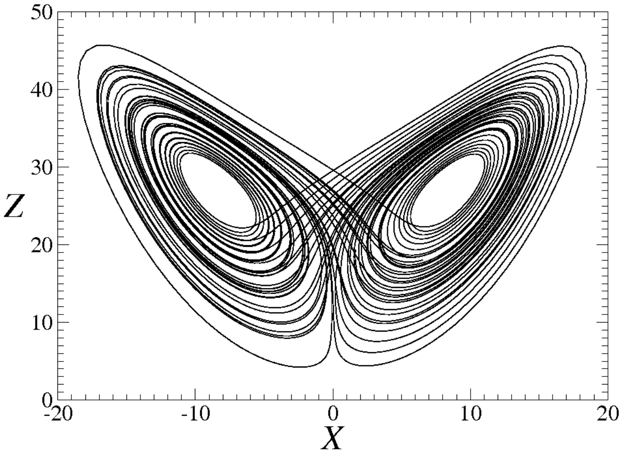
Rössler Attractor



Rössler Attractor



Lorenz Attractor



Lorenz Attractor

Return Map for Lorenz Attractor

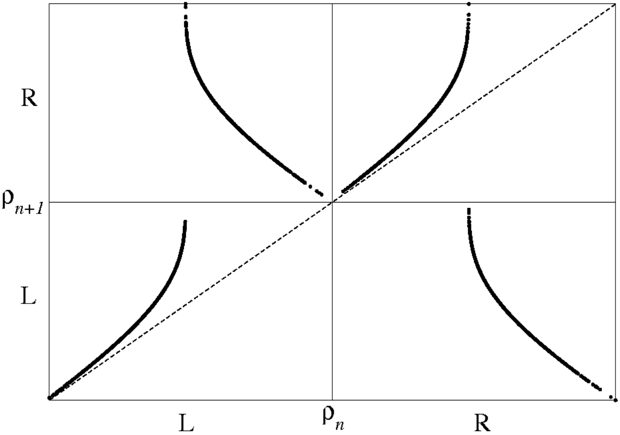
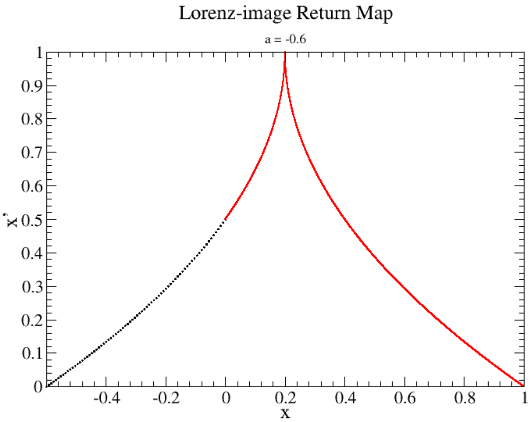


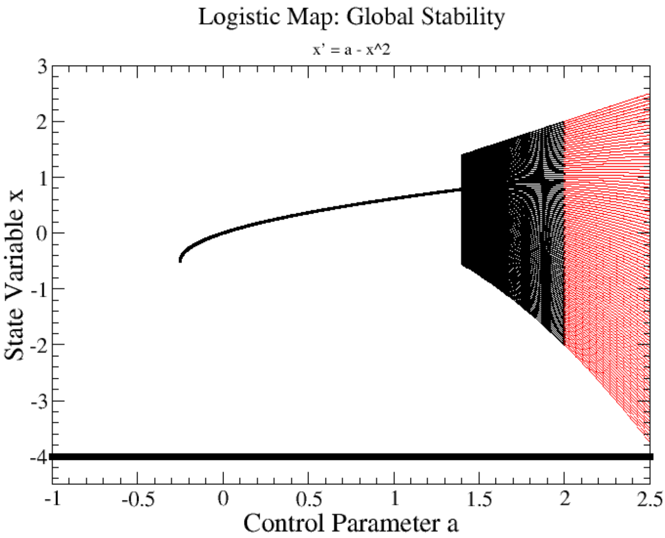
Image of Lorenz Return Map



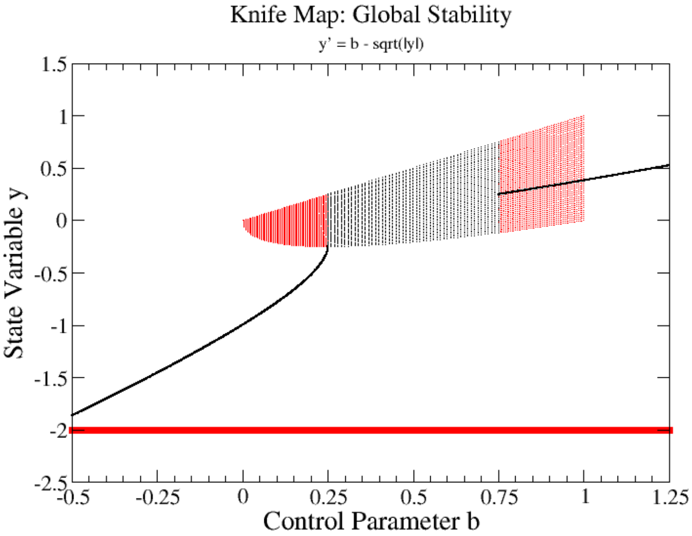
Comparison: Two Maps

Logistic Map	Knife Map
a.k.a. Fold Map	a.k.a. Cusp Map
$x' = f(x;a) = a - (\|x\|)^2$	$y' = f(y;b) = b - (\|y\|)^{1/2}$
Piecewise Concave	**Piecewise Convex**

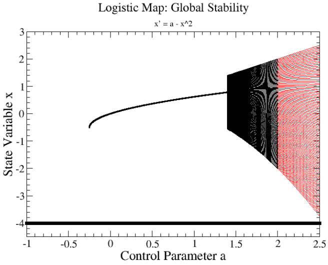
BigView: Logistic Map

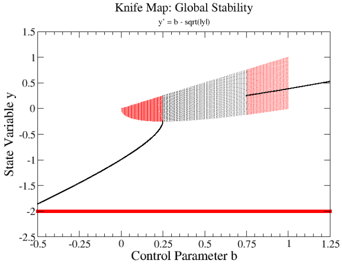


BigView: Knife Map



Stability Regions





Black = stable   Red = unstable

Lorenz Map

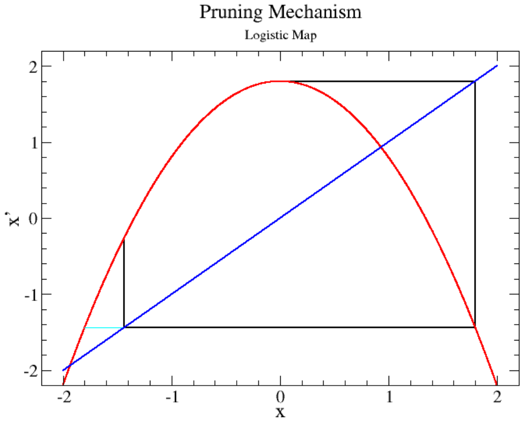
The Lorenz return Map (Fig. 4 in Lorenz 1963) for $(R, \sigma, b) = (28.0, 10.0, 8/3)$ scales to

$$z' = b - z^k$$

$$\begin{aligned} b &\simeq 0.32 \pm \\ k &\simeq 0.43 \pm \end{aligned}$$

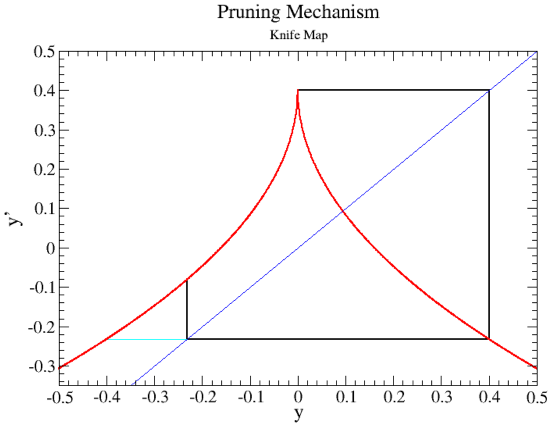
Return Map - Rössler Attractor

Basin Boundaries

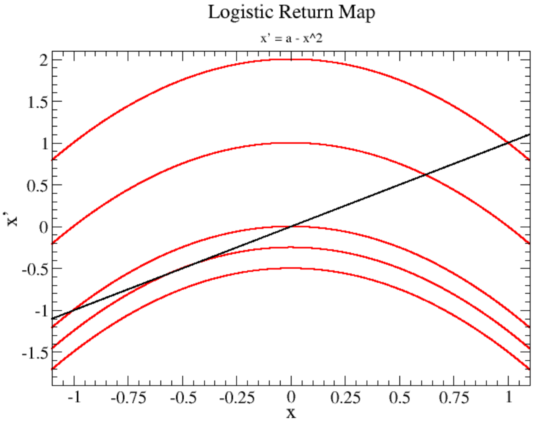


Return Map - Lorenz Image

Basin Boundaries

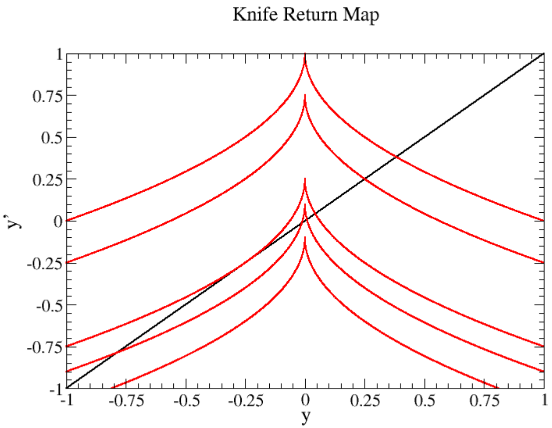


Logistic Map for several values of a



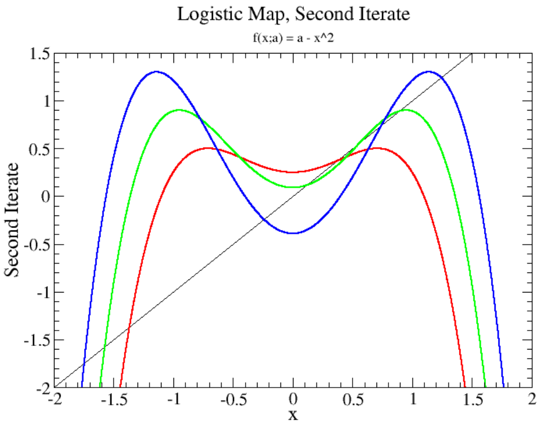
Period-one orbit creation

Knife Return Map for several values of b

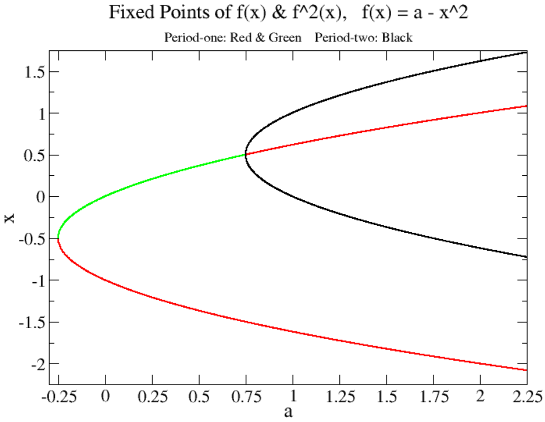


Period-one orbit creation and annihilation

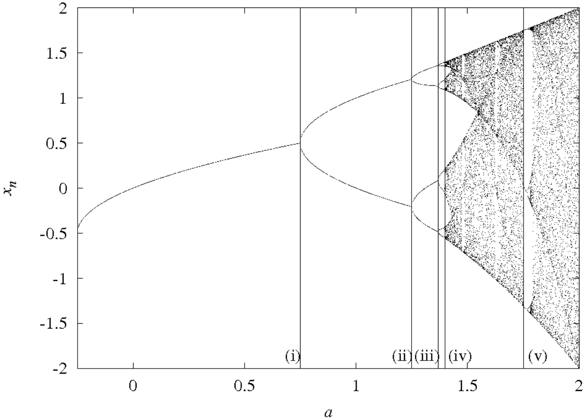
Second Return Map



Period 1 & 2 Orbits - Logistic



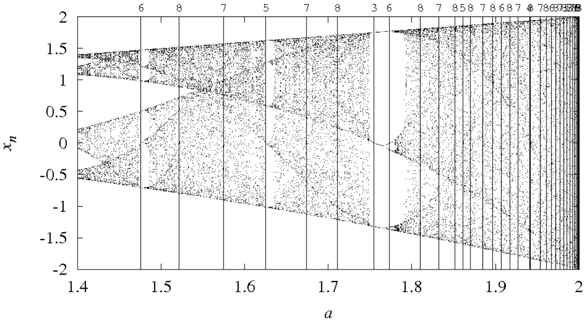
Bifurcation Diagram



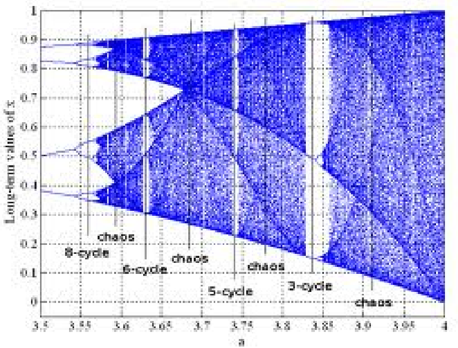
Notice: caustics and focal points

Bifurcation-02

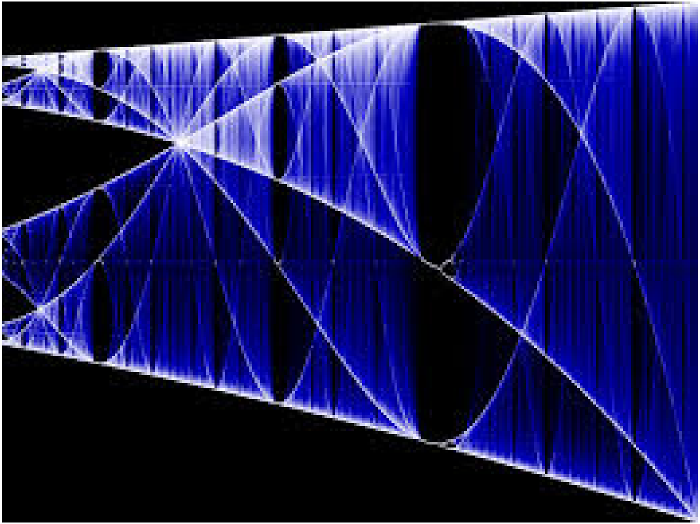
.. Blow Up .... with Caustics



Logistic Caustic 1



Logistic Caustic 2



Caustics

Historically, caustics are regions burned by an anomalously high concentration of sunlight. Dark regions, or high density regions, appear in the bifurcation diagram of the logistic map. They occur at forward images of the critical point, which has zero slope. Their zero-crossings provide information about the parameter values at which orbits become superstable.

For the dual map $y' = b - \sqrt{|y|}$ with a vertical slope, forward images of the critical point are not accompanied by caustics - rather, anticaustics. Zero crossings of anticaustics provide information about the parameter values at which orbit explosions occur.

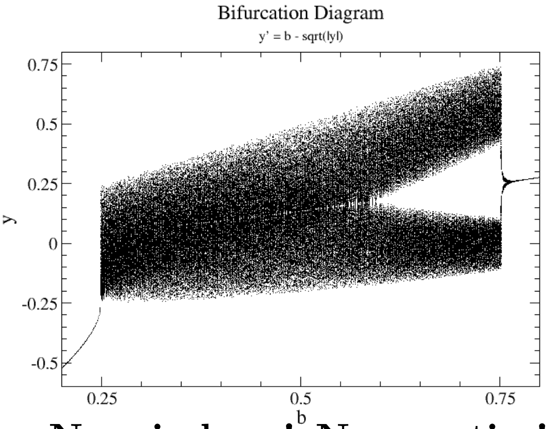
Caustic / Anticaustic Density near Critical Point

$$
\begin{aligned}
z' &= f(z,c) &= c - |z|^p \\
\frac{dz'}{dz} &= \frac{df}{dz} &= -p|z|^{p-1} \\
\rho(z') &= \frac{dz}{dz'}(z') &= \frac{1}{p}(c - |z'|)^{(1/p)-1}
\end{aligned}
$$

For $p = 2$ (fold map) $\rho(z')$ exhibits a van Hove singularity.

For $p < 1$ (cusp map) $\rho(z') \to 0$ as $c - |z'| \to 0$.

Cusp Map - Bifurcation Diagram



No windows! No caustics!

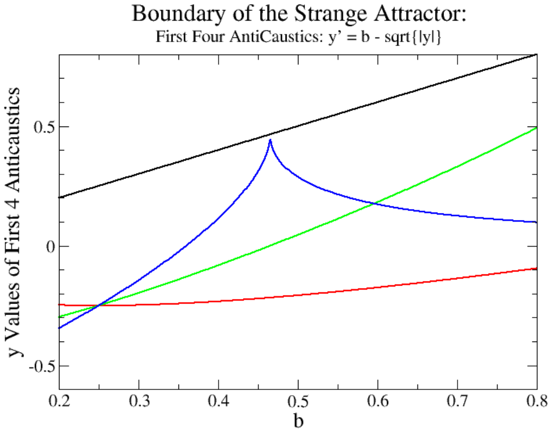
Scaling: Lorenz Map to Cusp Map

In the parameter range $26 \leq R \leq 46$ with $(\sigma, b) = (10.0, 8/3)$ the image of the Lorenz return map scales to the canonical form

$$z' = a - |z|^p$$

where a slowly increases from 0.32 to 0.38 and p slowly decreases from 0.46 to 0.41.

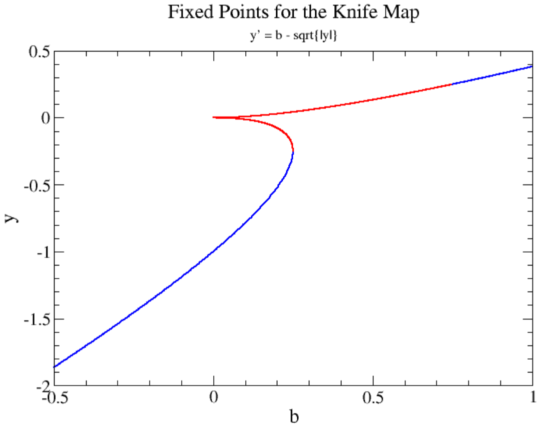
Boundaries of the Strange Attractor



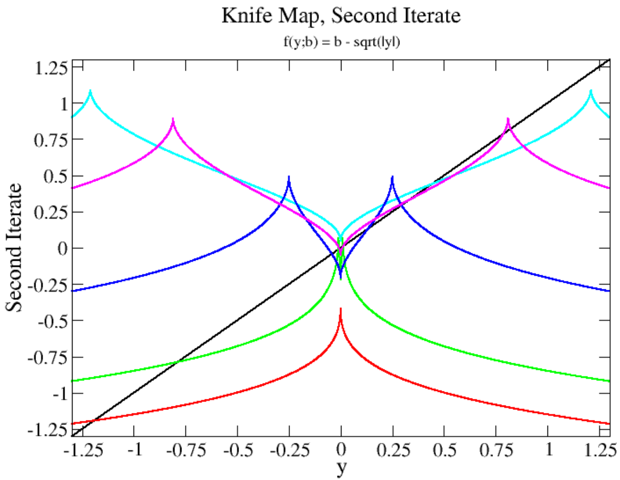
Boundaries of Strange Attractors

- **Logistic Map:**
- The first two caustics bound the strange attractor when both even and odd period orbits are present.
- The third and fourth caustics provide additional boundaries when only even period orbits are present.
- **Lorenz Map:**
- The first two anticaustics bound the strange attractor when both even and odd period orbits are present.
- The third and fourth anticaustics provide additional boundaries when only even period orbits are present.

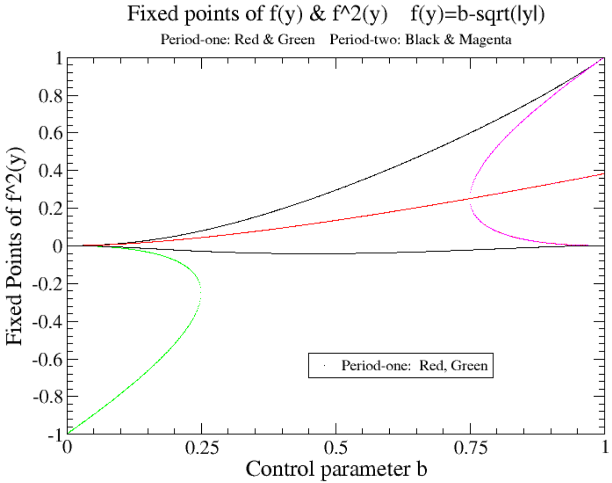
Fixed Points (Cusp Map)



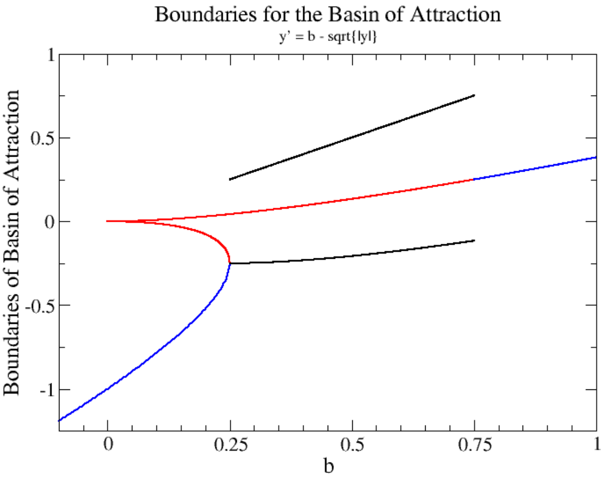
Second Iterates - Cusp Map



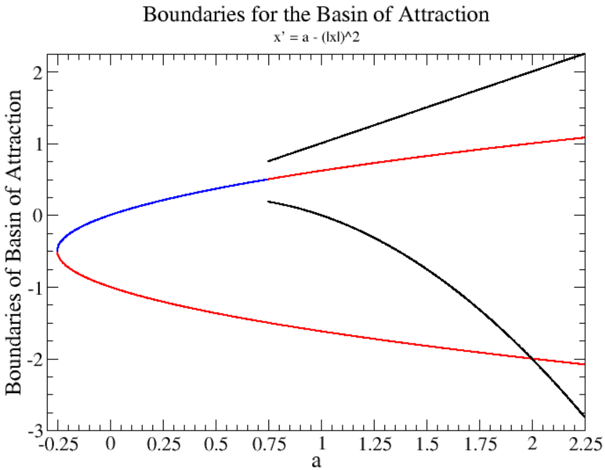
Period-One & Period-Two Orbits



Attractor boundary (Knife)

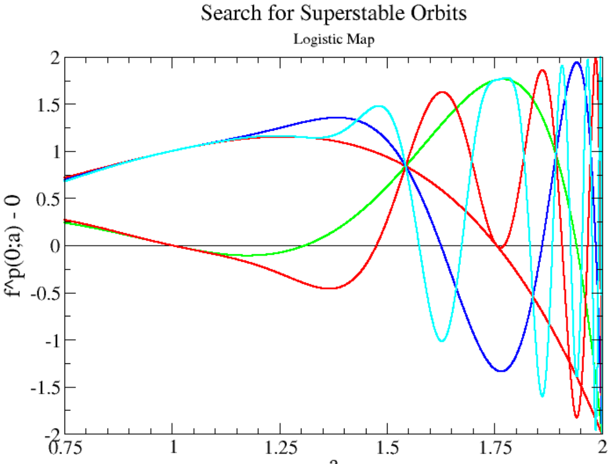


Attractor Boundaries - Logistic



Hunt for Saddle-Node Bifurcations

Caustic Crossings

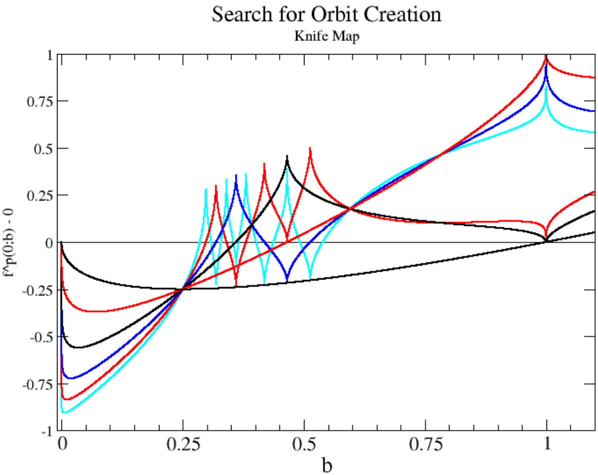


Hunt for Saddle-Node Bifurcations

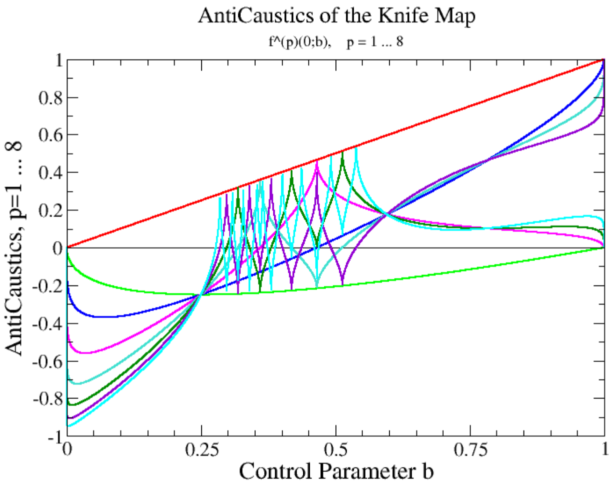
Caustic Fingerprints

- Zero crossings of the p^{th} caustic identify locations of superstable period-p orbits.
- Caustic "focal point" identifies end of the period-halving bifurcations.

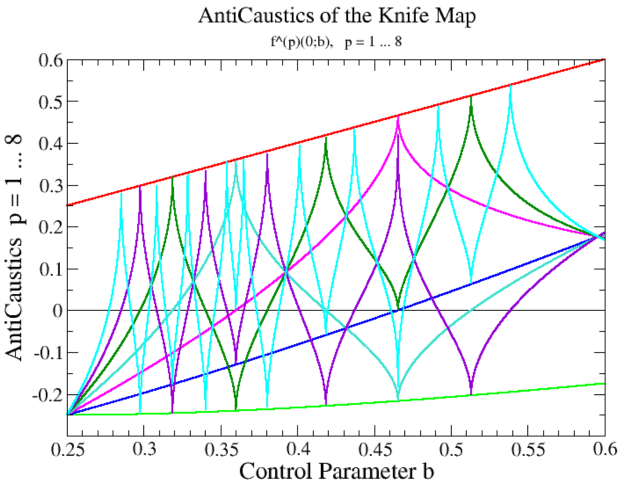
Hunt for Singular SNBs



Anti Caustic Crossings



Anti Caustic Crossings: Expansion



Hunt for Explosions

Caustic Fingerprints

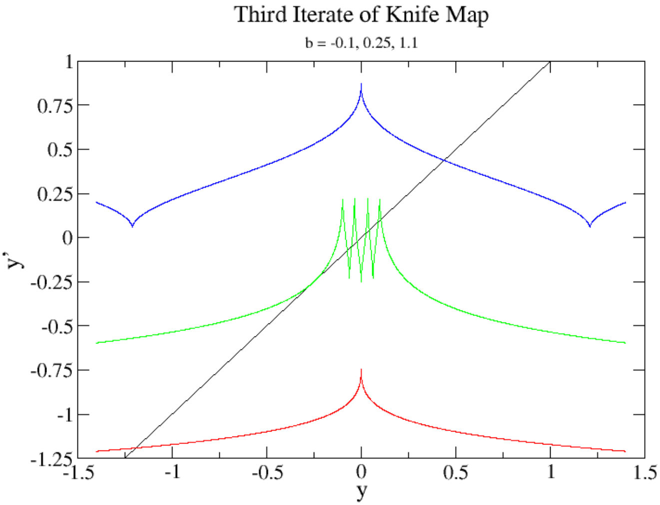
- Zero crossings of the p^{th} anticaustic identify locations of period-p orbit explosions.
- At an explosion two *prime* orbits $K\sigma_2\sigma_3\cdots\sigma_p$ of period p are created, $\sigma_i \in \{0, 1\}$.
- All possible *compound* orbits based on the two prime orbits are simultaneously created.
- Anticaustic focal point at $(b = 0.595743, y = 0.176100)$ identifies end of the period-halving bifurcations as b decreases from $b = 1$.
- Anticaustic focal point at $(b = 1/4, y = -1/4)$ bounds the region $0 < b < 1/4$ where trajectories consisting of all possible symbol sequences exist and no bifurcations occur.

Caustic & Anticaustic Crossings

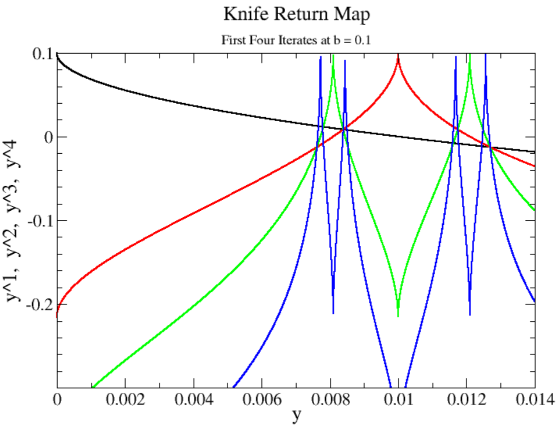
For the logistic map there is one point at which caustics focus, at $(x = 0.839286, a = 1.543689)$. To the left of this point the caustics are separated into those with p even (lower) and p odd (upper). This crossing marks the point at which the last noisy period-halving bifurcation occurs. To the left there are only even period windows. Odd period windows begin immediately to the right of the caustic focal point.

For the dual map there are two anticaustic focal points. The one at the right at $(b_2 = 0.595743, y = 0.176100)$ marks the point at which only unstable even period orbits can be found (to its right). The one on the left at $(b_1 = 1/4, y = -1/4)$ separates the control parameter region in which trajectories of all possible periods and symbol sequences exist $(b < b_1)$ from the region $b_1 \leq b \leq 1$ in which orbits are systematically removed by explosions.

Rite of Passage-01

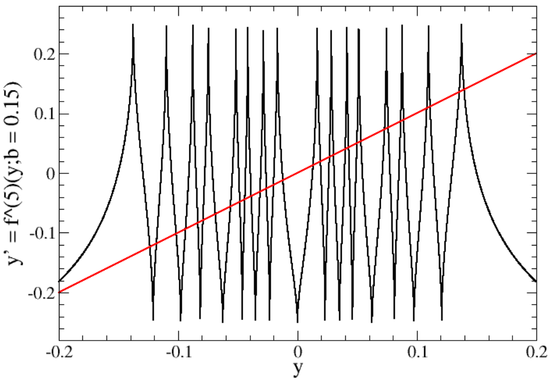


Knife Map Iterates



Structural Stability: $0 < b < \frac{1}{4}$

Knife Map, fifth iterate at b=0.15



Explosions-02

Table : Values $M^{(p)}$ of y where the pth iterate $f^{(p)}(y;b)$ has maxima. These locations are determined by a simple recursion relation (last line) where the indices $s_p = \pm 1$ are incoherent.

p	Number Max.	Coordinate Values
1	1	0
2	2	$\pm b^2$
3	4	$\pm(b \pm b^2)^2$
$\cdots$	$\cdots$	$\cdots$
$p+1$	2^p	$M^{(p+1)} = s_p(b + M^{(p)})^2$

Explosions-03

As $p \to \infty$, with all $s_j = +1$, the abscissa of the rightmost point goes to a limit. The quadratic equation for this limit gives:

$$y_r(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b}\right)$$

At $b = \frac{1}{4}$ the bounding box is a rectangle — beyond that the diagonal fails to intersect all the zig - zags. Orbits begin to get pruned away in singular saddle node bifurcations.

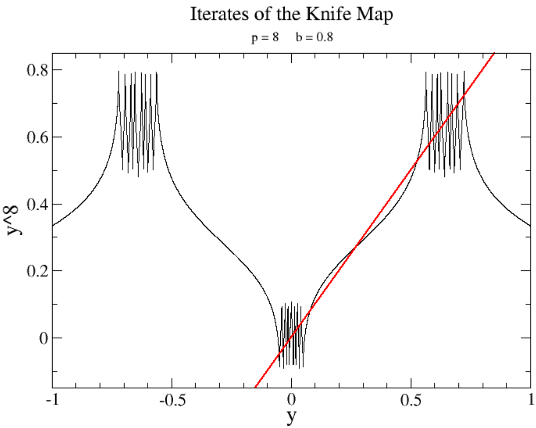
Extent of No-Explosion Range

- All local maxima (near $y = 0$) occur at $y_{loc.\ Max.} = b$.
- All local minima (near $y = 0$) occur at $y_{loc.\ min.} = b - \sqrt{b}$.
- As long as the rightmost peak occurs at $y_r(b) < b$ there are $2^p \pm 1$ fixed points for $f^p(y; b)$ near $y = 0$.
- Singular bifurcations begin when

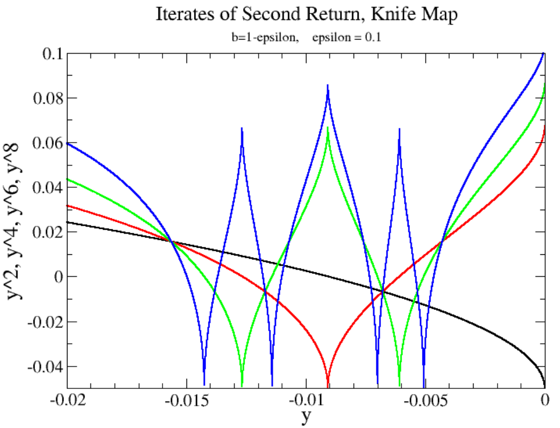
$$y_r(b) = \left(\frac{1}{2} - b - \sqrt{\frac{1}{4} - b} \right) > b$$

- No explosions: $0 < b < \frac{1}{4}$.

End Play - Near $b = 1$

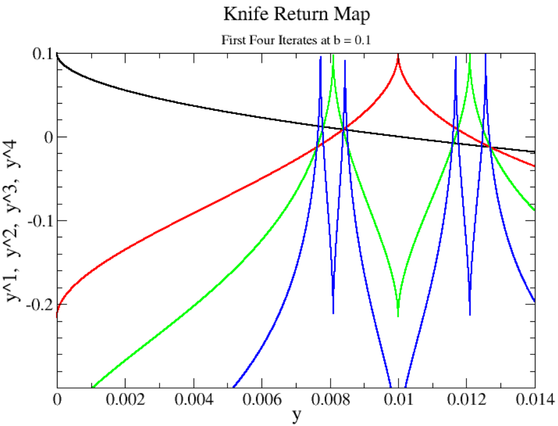


Iterates Near $b = 1$

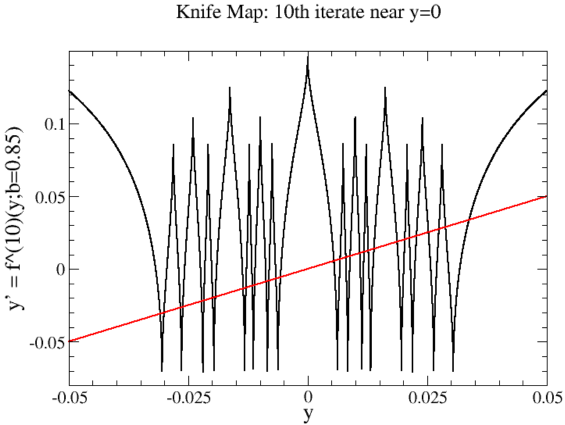


implosion1

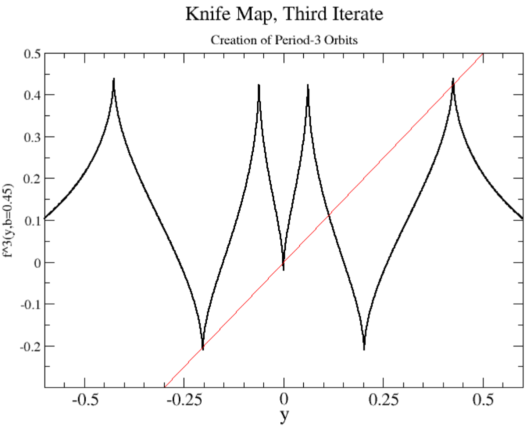
Note Scaling Relations



Structural Stability: $\frac{3}{4} < b < 1$



Period Three Singular SNB



Renormalization-02

Local expression near $y = 0$ for the period-three explosion:

$$h(y;b) = f^{(3)}(y;b) = b - \sqrt{|b - \sqrt{|b - \sqrt{|y|}|}|}$$

$$h(b_3+\epsilon; y) \rightarrow \left(b_3 - \sqrt{\sqrt{b_3} - b_3}\right) +$$

$$\left(1 + \frac{2\sqrt{b_3} - 1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\epsilon + \left(\frac{1}{4\sqrt{\sqrt{b_3} - b_3}\sqrt{b_3}}\right)\sqrt{|y|}$$

Renormalization-03

Renormalization for the period-three explosion.

$$y' = h(y; b_3 + \epsilon) \rightarrow \Delta(b - b_3) + \alpha\sqrt{|y|} =$$

$$1.286974759(b - b_3) + 0.7869747590\sqrt{|y|}$$

$$z' = (\Delta/\alpha^2)(b_3 - b) - \sqrt{|z|}$$

Renormalization Algorithm: K10*

1. Write down the symbol sequence for the primary period-p orbit: $K10* = K\sigma_1\sigma_2\cdots\sigma_{p-1}$.
2. Make the identification
   $\sigma = +1 \rightarrow s = +1, \qquad \sigma = 0 \rightarrow s = -1.$
3. Construct $f^{(p)}(b;y) \rightarrow$

$$b - \sqrt{s_{p-1}(b - \cdots \sqrt{s_2(b - \sqrt{s_1(b - \sqrt{y})})} \cdots)}$$

4. Taylor expand this function to terms linear in b and $\sqrt{y}$ and determine the value of b for which the constant term vanishes.

Equations: K10*

For the saddle node pair $5_2 = K1001$ this algorithm gives

$$b - \sqrt{(+1)(b - \sqrt{(-1)(b - \sqrt{(-1)(b - \sqrt{(+1)(b - \sqrt{y})})})})}$$

The constant term vanishes for $b = 0.418656$, and for this value of b

$$y' = \Delta(b - b_{5_2}) + \alpha\sqrt{|y|} = -3.231180\Delta b - 1.983690\sqrt{|y|}$$

Results: K10* to Period 6

$$y' = \Delta(b - b_c) + \alpha\sqrt{|y|} \qquad y', y \simeq 0$$

Orbit	Symbolics	b_c	Δ	α
3_1	$K10$	0.465571	1.286974	0.786974
4_2	$K100$	0.360157	2.624703	1.180563
5_3	$K1000$	0.318897	4.647225	1.664335
5_2	$K1001$	0.418656	−3.231180	−1.983690
5_1	$K1011$	0.513175	2.628970	1.509712
6_5	$K10000$	0.297846	7.481728	2.233184
6_4	$K10001$	0.340328	−8.535145	−3.639587
6_3	$K10011$	0.380540	7.596535	3.574548

Renormalization-07

Renormalization for the final period-two explosion.

$$f^{(2)}(1-\epsilon, y) \simeq -\frac{\epsilon}{2} + \left(\frac{1}{2} + \frac{\epsilon}{4}\right)\sqrt{|y|} \tag{1}$$

Breakpoints

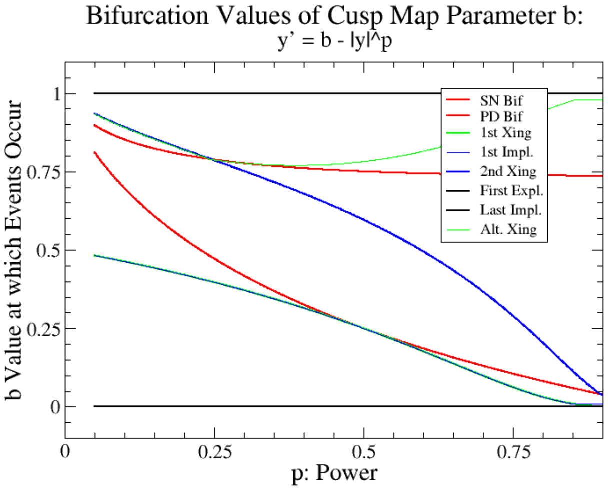
Table : Important parameter values for global stability and unstable periodic orbit behavior.

Global Stability	Unstable Orbits
	0.0
1/4	1/4
	0.5957439420
3/4	
	0.7825988587
	1.0

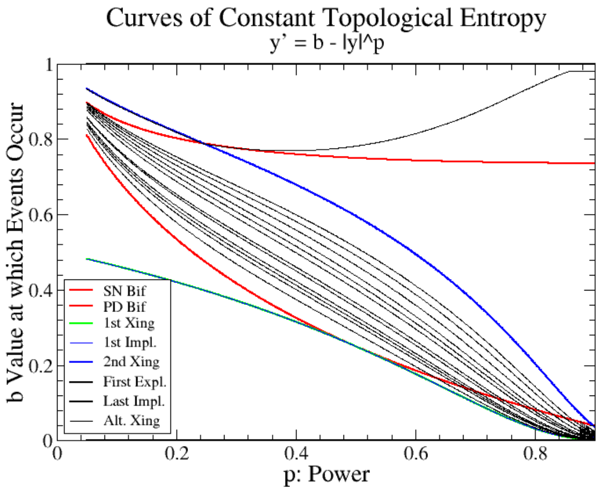
Parameter Ranges

Left	Right	What goes on
$-\infty$	0	Stable period one orbit
0	1/4	Explosion at $b = 0$ creates all orbits based on symbols 0, 1. No further bifurcations in this range
1/4	0.575974	Explosions remove all odd- and most even-period orbits
0.575943	0.782598	Explosions remove even period orbits
0.782598	1	No further bifurcations in this range
1	$+\infty$	Stable period one orbit. Explosion at $b = 1$ removes remaining even-period orbits.

Generic Cusp Map $y' = b - |y|^p$



Generic Cusp Map $y' = b - |y|^p$



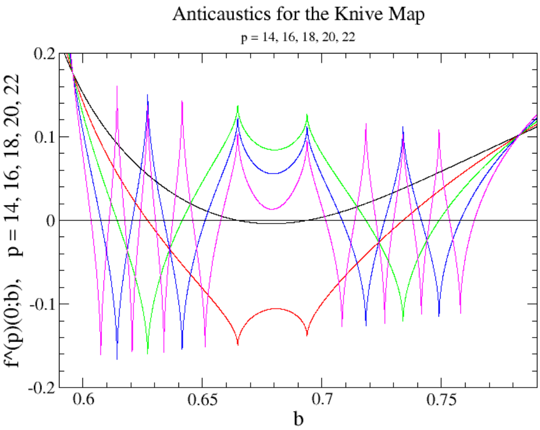
U Sequence

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)[a]

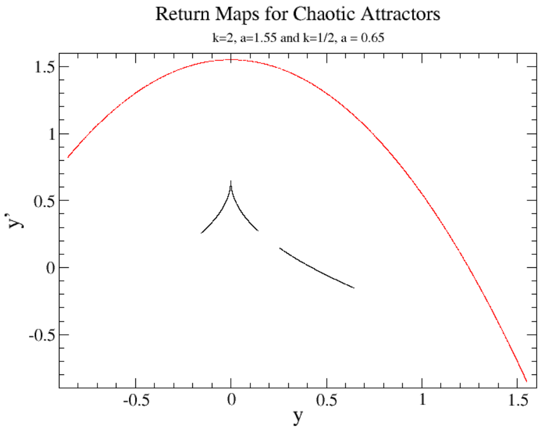
Name	Bifurcation	Name	Bifurcation	Name	Bifurcation
$^{0}_{1}$	$1_1[s_1]$	00101 $^{0}_{1}1$	$7_3[s_7^3]$	0001 $^{0}_{1}1$	$6_4[s_6^3]$
01	$2_1[s_1 \times 2^1]$	001010 $^{0}_{1}1$	$8_5[s_8^4]$	000111 $^{0}_{1}1$	$8_{11}[s_8^9]$
0111	$4_1[s_1 \times 2^2]$	001 $^{0}_{1}1$	$5_2[s_5^2]$	00011 $^{0}_{1}1$	$7_7[s_7^7]$
01010111	$8_1[s_1 \times 2^3]$	001110 $^{0}_{1}1$	$8_6[s_8^5]$	000110 $^{0}_{1}1$	$8_{12}[s_8^{10}]$
0111 $^{0}_{1}1$	$6_1[s_6^1]$	00111 $^{0}_{1}1$	$7_4[s_7^4]$	000 $^{0}_{1}1$	$5_3[s_5^3]$
011111 $^{0}_{1}1$	$8_2[s_8^1]$	001111 $^{0}_{1}1$	$8_7[s_8^6]$	000010 $^{0}_{1}1$	$8_{13}[s_8^{11}]$
01111 $^{0}_{1}1$	$7_1[s_7^1]$	0011 $^{0}_{1}1$	$6_3[s_6^2]$	00001 $^{0}_{1}1$	$7_8[s_7^8]$
011 $^{0}_{1}1$	$5_1[s_5^1]$	001101 $^{0}_{1}1$	$8_8[s_8^7]$	000011 $^{0}_{1}1$	$8_{14}[s_8^{12}]$
01101 $^{0}_{1}1$	$7_2[s_7^2]$	00110 $^{0}_{1}1$	$7_5[s_7^5]$	0000 $^{0}_{1}1$	$6_5[s_6^4]$
011011 $^{0}_{1}1$	$8_3[s_8^2]$	00 $^{0}_{1}1$	$4_2[s_4^1]$	000001 $^{0}_{1}1$	$8_{15}[s_8^{13}]$
0 $^{0}_{1}1$	$3_1[s_3]$	00010011	$8_9[s_4^1 \times 2^1]$	00000 $^{0}_{1}1$	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	00010 $^{0}_{1}1$	$7_6[s_7^6]$	000000 $^{0}_{1}1$	$8_{16}[s_8^{14}]$
001011 $^{0}_{1}1$	$8_4[s_8^3]$	000101 $^{0}_{1}1$	$8_{10}[s_8^8]$		

[a] The notation P_i refers to the ith bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the ith saddle-node bifurcation of period P is denoted s_P^i, and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i.

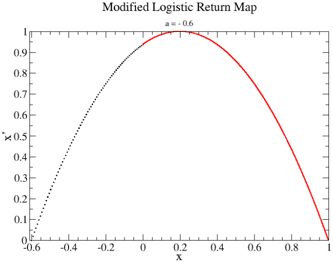
Symbol Exchange Near Endplay

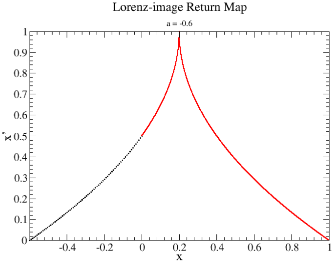


Forcing Diagram - Horseshoe

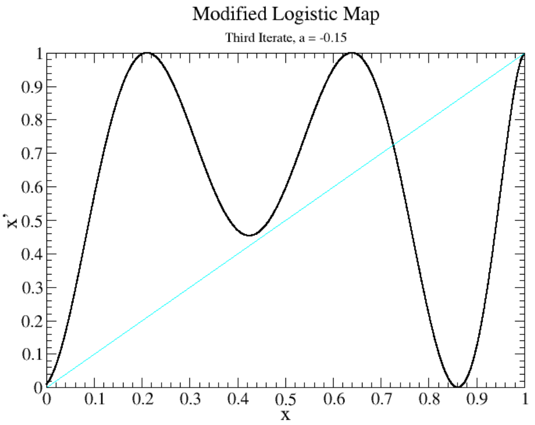


Map Comparisons

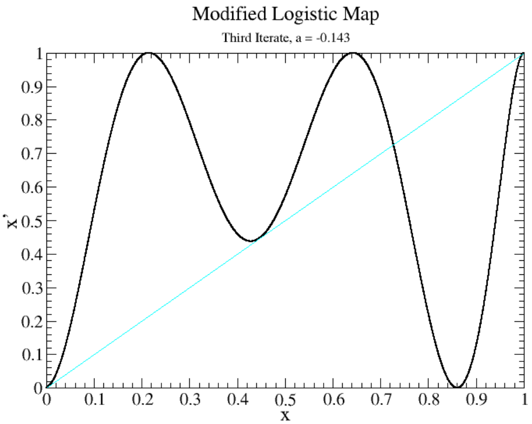




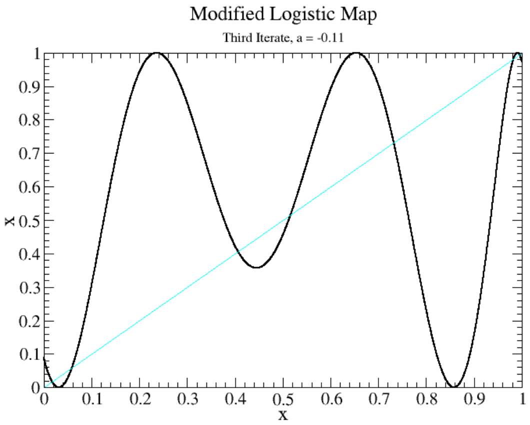
Forcing Diagram - Horseshoe



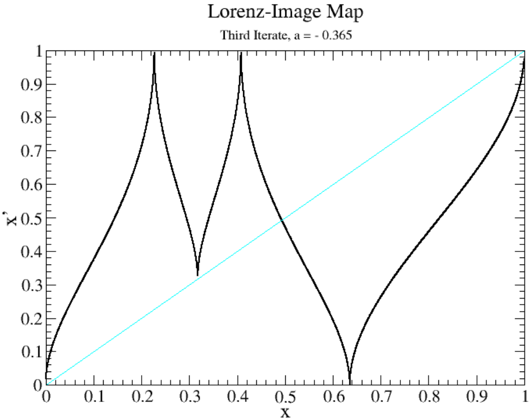
Forcing Diagram - Horseshoe



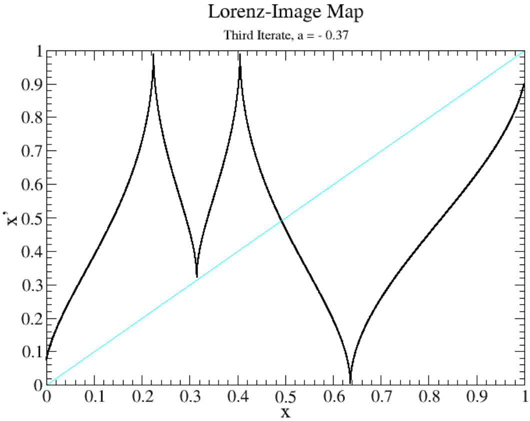
Forcing Diagram - Horseshoe



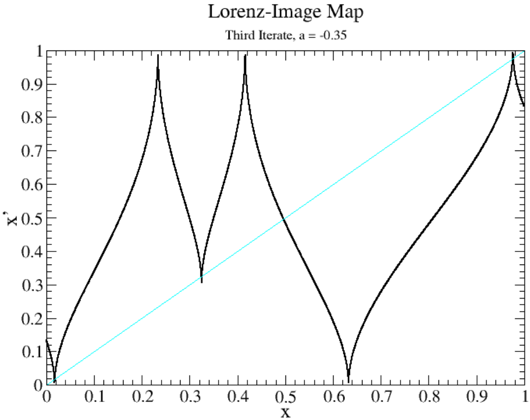
Forcing Diagram - Horseshoe



Forcing Diagram - Horseshoe



Forcing Diagram - Horseshoe



Scaling

- Logistic: SNB Period 3 = scaled version SNB of M.
- Renormalization theory applies.
- U Sequence

- Knife: S-SNB Period 3 = scaled version S-SNB of K.
- Renormalization theory applies.
- U^{-1} Sequence

Topological Organization

The knife and logistic maps are suspensions of flows. Corresponding orbits (identical names) in each suspension are organized identically.

Identical lifts of the logistic and knife maps lead to identical covering orbit organization.

The mysteries of orbit organization in flows with $g > 1$ are the same for stretch-and-fold and for tear-and-squeeze mechanicms.

U Sequence

Table 2.1 Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)[a]

Name	Bifurcation	Name	Bifurcation	Name	Bifurcation
${}^{0}_{1}$	$1_1[s_1]$	$00101\,{}^{0}_{1}1$	$7_3[s_7^3]$	$0001\,{}^{0}_{1}1$	$6_4[s_6^3]$
01	$2_1[s_1 \times 2^1]$	$001010\,{}^{0}_{1}1$	$8_5[s_8^4]$	$000111\,{}^{0}_{1}1$	$8_{11}[s_8^9]$
0111	$4_1[s_1 \times 2^2]$	$001\,{}^{0}_{1}1$	$5_2[s_5^2]$	$00011\,{}^{0}_{1}1$	$7_7[s_7^7]$
01010111	$8_1[s_1 \times 2^3]$	$001110\,{}^{0}_{1}1$	$8_6[s_8^5]$	$000110\,{}^{0}_{1}1$	$8_{12}[s_8^{10}]$
$0111\,{}^{0}_{1}1$	$6_1[s_6^1]$	$00111\,{}^{0}_{1}1$	$7_4[s_7^4]$	$000\,{}^{0}_{1}1$	$5_3[s_5^3]$
$011111\,{}^{0}_{1}1$	$8_2[s_8^1]$	$001111\,{}^{0}_{1}1$	$8_7[s_8^6]$	$000010\,{}^{0}_{1}1$	$8_{13}[s_8^{11}]$
$01111\,{}^{0}_{1}1$	$7_1[s_7^1]$	$0011\,{}^{0}_{1}1$	$6_3[s_6^2]$	$00001\,{}^{0}_{1}1$	$7_8[s_7^8]$
$011\,{}^{0}_{1}1$	$5_1[s_5^1]$	$001101\,{}^{0}_{1}1$	$8_8[s_8^7]$	$000011\,{}^{0}_{1}1$	$8_{14}[s_8^{12}]$
$01101\,{}^{0}_{1}1$	$7_2[s_7^2]$	$00110\,{}^{0}_{1}1$	$7_5[s_7^5]$	$0000\,{}^{0}_{1}1$	$6_5[s_6^4]$
$011011\,{}^{0}_{1}1$	$8_3[s_8^2]$	$00\,{}^{0}_{1}1$	$4_2[s_4^1]$	$000001\,{}^{0}_{1}1$	$8_{15}[s_8^{13}]$
$0\,{}^{0}_{1}1$	$3_1[s_3]$	00010011	$8_9[s_4^1 \times 2^1]$	$00000\,{}^{0}_{1}1$	$7_9[s_7^9]$
001011	$6_2[s_3 \times 2^1]$	$00010\,{}^{0}_{1}1$	$7_6[s_7^6]$	$000000\,{}^{0}_{1}1$	$8_{16}[s_8^{14}]$
$001011\,{}^{0}_{1}1$	$8_4[s_8^3]$	$000101\,{}^{0}_{1}1$	$8_{10}[s_8^8]$		

[a] The notation P_i refers to the ith bifurcation of period P. We also give inside brackets an alternative classification that distinguishes between saddle-node and period-doubling bifurcations. In this scheme, the ith saddle-node bifurcation of period P is denoted s_P^i, and $s_P^i \times 2^k$ is the orbit of period $P \times 2^k$ belonging to the period-doubling cascade originating from s_P^i.

Renormalization-08

Table : Sequence of bifurcations in the logistic map up to period 8 (from top and to bottom and left to right)[a]

Name	Bifurcation	Name	Bifurcation	Name
$^{0}_{1}$	$1_1[s_1]$	00101 $^{0}_{1}$1	$7_3[s_7^3]$	0001 $^{0}_{1}$1
01	$2_1[s_1 \times 2^1]$	001010 $^{0}_{1}$1	$8_5[s_8^4]$	000111 $^{0}_{1}$1
0111	$4_1[s_1 \times 2^2]$	001 $^{0}_{1}$1	$5_2[s_5^2]$	00011 $^{0}_{1}$1
01010111	$8_1[s_1 \times 2^3]$	001110 $^{0}_{1}$1	$8_6[s_8^5]$	000110 $^{0}_{1}$1
0111 $^{0}_{1}$1	$6_1[s_6^1]$	00111 $^{0}_{1}$1	$7_4[s_7^4]$	000 $^{0}_{1}$1
011111 $^{0}_{1}$1	$8_2[s_8^1]$	001111 $^{0}_{1}$1	$8_7[s_8^6]$	000010 $^{0}_{1}$1
01111 $^{0}_{1}$1	$7_1[s_7^1]$	0011 $^{0}_{1}$1	$6_3[s_6^2]$	00001 $^{0}_{1}$1
011 $^{0}_{1}$1	$5_1[s_5^1]$	001101 $^{0}_{1}$1	$8_8[s_8^7]$	000011 $^{0}_{1}$1
01101 $^{0}_{1}$1	$7_2[s_7^2]$	00110 $^{0}_{1}$1	$7_5[s_7^5]$	0000 $^{0}_{1}$1
011011 $^{0}_{1}$1	$8_3[s_8^2]$	00 $^{0}_{1}$1	$4_2[s_4^1]$	000001 $^{0}_{1}$1
0 $^{0}_{1}$1	$3_1[s_3]$	00010011	$8_9[s_4^1 \times 2^1]$	00000 $^{0}_{1}$1

Return Map Approximations

The Rossler return map is well approximated by the following maps:

$$
\begin{aligned}
x' &= \lambda x(1-x) \\
x' &= a - x^2 \\
x' &= 1 - \mu x^2 \\
x' &= 1 - \left| \frac{x-m}{w} \right|^2
\end{aligned}
$$

Image of Lorenz Return Map

The image of the Lorenz return map is well approximated by the following maps:

$$
\begin{aligned}
y' &= b - |y|^{1/2} \\
y' &= 1 - \mu |y|^{1/2} \\
y' &= 1 - \left| \frac{y-m}{w} \right|^{1/2}
\end{aligned}
$$

Symbol Exchange Near Endplay

- Symbols 0, 1 created at $b = 0$
- New orbit, (11), created at $b = \frac{3}{4}$
- Symbol pair - 11 -, replaced by - (11) - as $b \to 1$
- Implosions begin at $b = 0.5957...$, end at midpoint
- Explosions begin at midpoint, end at $b = 0.7825..$
- Implosions and explosions symmetrically matched