Entropy of Bounding Tori

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Branched manifolds that describe strange attractors in R^3 can be enclosed in, and are organized by, canonical bounding tori. Tori of genus g are labeled by a symbol sequence, or "periodic orbit", of period $g-1$. We show that the number of distinct canonical bounding tori grows exponentially like $N(g) \sim e^{\lambda(g-1)}$, with $\log(2) \leq \lambda \leq \log(3)$. We estimate $\lambda = 1$.

I. INTRODUCTION

Low dimensional strange attractors — those with Lyapunov dimension $d_L < 3$ — can be discretely classified. A doubly discrete classification has been described in [1]. This classification depends ultimately on the existence and rigid organization of an infinity of unstable periodic orbits in a strange attractor [2, 3]. At the lowest level this classifciation depends on a basis set of orbits. This is a set of orbits with positive topological entropy whose presence forces the existence of all the other unstable periodic orbits in the attractor [4–6]. The basis set of orbits for any attractor is discrete, and up to any finite period the basis set of orbits is finite.

At the second level of this organizational hierarchy for strange attractors are branched manifolds [1, 2, 7, 8, 10]. These are obtained from the flow that generates a strange attractor by projecting the flow down along the stable direction. The unstable periodic orbits that exist in the strange attractor exist in 1-1 correspondence with the periodic orbits on the branched manifold, with possibly a small number of exceptions. Information about branched manifolds can be extracted from experimental data [11].

Recently a third level of discreteness in the description and classification of low dimensional strange attractors has been introduced [13, 14]. Branched manifolds can be enclosed in bounding tori. These serve to organize branced manifolds in the same way that branched manifolds organize the periodic orbits in a strange attractor. A bounding torus provides a canonical form for any flow in R^3 that generates a strange attractor. An algorithm for transforming a flow to its canonical form is given in [14]. The bounding tori that enclose every strange attractor that has been studied in R^3 have been described in [13, 14].

Bounding tori are described first by their genus $g \geq 1$. However, genus alone does not uniquely identify a bounding torus when $g > 4$, and in fact the number of distinct bounding tori of genus g grows rapidly with g. It was proposed in [14] that the growth might be exponential, so that an entropy-like limit of the type $\lim_{g\rightarrow\infty} \log[N(g)]/g$ might exist, in analogy with the limiting definition of topological entropy for periodic robits in a strange attractor.

The purpose of the present work is to show that this limit exists, to present hard upper and lower bounds, and to estimate that the "toral entropy" of three-dimensional bounding tori is

$$\lim_{g\rightarrow\infty} \frac{\log[N(g)]}{g} \simeq 1 \tag{1}$$

II. BACKGROUND

A bounding torus of genus $g = 8$ is shown in Fig. 1. This represents a projection of a two dimensional surface in R^3 down onto a plane. The projection consists of the outer boundary of a disk and g interior disks. The interior disks are of two types: circles and even-sided polygons. The flow on the outer boundary is unidirectional; the flow on the n_c interior circles is also unidirectional, and in the same direction as the flow on the exterior boundary. All singularities of the flow lie on the n_p interior polygons: a polygon with $2n$ sides ($n > 1$) has $2n$ singularities, one at each vertex. The genus of the bounding torus is the total number of interior holes: $g = n_c + n_p$. The total number of singularities on the bounding torus (all at the vertices of the interior polygons) is $2(g-1)$.

For the bounding torus shown in Fig. 1 there are $n_c = 5$ interior uniflow circles labeled $A \rightarrow E$ and three interior polygons labeled a, b, c. The global Poincaré section of any flow bounded by this torus has $g-1 = 7$ disconnected components [13, 14]. These are shown as line segments in Fig. 1 and labeled $1 \rightarrow 7$, sequentially in the direction of the flow along the exterior boundary.

There are several ways that bounding tori can be uniquely identified. The labeling algorithms are described in Eq. (2).

$$\begin{array}{ccccccccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 1 & & \\
A & & B & & C & & B & & D & & B & & E & & A & & \\
 & a & & b & & b & & c & & c & & a & & a & & & \\
1 & & 3 & & 1 & & 3 & & 1 & & 3 & & 1 & & 1 & & (2,4,6)
\end{array} \quad (2)$$

The first row lists the components of the global Poincaré section in the order they are encountered traversing the exterior boundary of the projection.

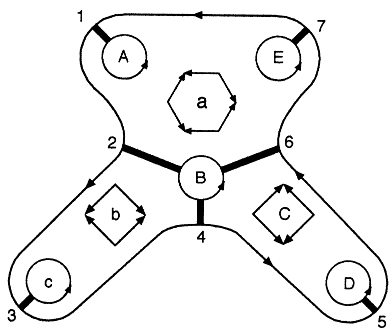

FIG. 1: A canonical bounding torus with genus 8. This is partly described by Young partition (3,2,2).

Below each number i is the capital letter (A, B, C, D, E) that identifies the uniflow circle to which the i^{th} component of the surface of section is attached. In moving from component i to component $i+1$ a hole with singularities is encountered. The sequence ($abbccaa$) that is encountered is shown in the third row of Eq. (2). There is a 1-1 correspondence between the bounding torus and each of the two letter sequences ($ABCBDBE$ and $abbccaa$), up to the usual symmetries (relabeling the holes, changing the starting point). In fact, these two descriptions of a bounding torus are dual to each other. Both sequence strings are in fact infinite, but of finite period $g-1=7$. The last string of integers in Eq. (2) indicates that there is a period-3 orbit around hole B and period-1 orbits around the holes A, C, D, E. A permutation group representation of this bounding torus in terms of permutation group generating cycles is (2,4,6)(1)(3)(5)(7) or more simply (2,4,6). This representation in terms of generating cylces can be used algorithmically to construct the transiton matrix for this bounding torus [13, 14].

Part of the degeneracy associated with enumerating bounding tori of genus g can be lifted by introducing Young partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{n_p})$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_p} \geq 2$ [13, 14]. Each internal polygon with $2\lambda_i$ edges and singularities is visited exactly λ_i times in a tour around the exterior boundary. The partition associated with the torus has n_p rows, one for each interior polygon. For the bounding torus shown in Fig. 1, $\lambda = (3, 2, 2)$. All allowed bounding tori that can be assocaited with this partition are obtained by distributing the $g-1=7$ letters aaa, bb, and cc on the perimeter of a circle subject to the single condition that no interleaving occurs (..$a.b.b.a$.. is allowed but ..$a.b.a.b$.. is not).

The number of bounding tori of genus g can be determined by

1. listing all allowed Young partitions;
2. counting the number of allowed letter distributions (up to cyclic permutation) for each Young partition.

In view of the noninterleaving property, the polygon encounter letter sequences of open and closed parentheses can be represented by three-symbol sequences in which (and) stand for the first and last occurance, respectively, of any given letter, intermediate occurances being indicated by a $*$. The noninterleaving property implies that each $*$ belongs to the the innermost pair of parentheses between which it is imbedded. Thus $aaa \rightarrow (*)$, $aaabb \rightarrow (*)()$, $aabba \rightarrow (*())$, and $baaab \rightarrow ((*))$. This construction guarantees that at each position of the sequence the cumulative number of opening paretheses is not less than the cumulative number of closing parentheses, counting from the left.

The complete set of bounding tori of genus g is obtained by constructing all three-symbol sequences that satisy the requirements (a) that the total number of opening paretheses be equal to that of closing paretheses, (b) that the cumulative number of opening paretheses be always not less than that of closing parentheses, (c) that a $*$ can only appear if the number of opening parentheses preceding it is larger than the number of closing parentheses. Finally, (d) sequences that are related by a cyclic permutation are equivalent.

This algorithm is described more fully in Sect. V.

III. LOWER BOUND ON TORAL ENTROPY

The lower bound on toral entropy is log(2). For tori of genus $g = 2k+1$ the Young partition with the longest column length is $\lambda = 2^k$. In computing $N(g)$ we have found that this class of partition consistently contributes a larger number of bounding tori than partitions with fewer rows. The number of tori associated with partition 2^k provides lower bounds on $N(g)$ when $g = 2k+1$. For $k = 1, 2, 3, 4, 5, 6. \cdots$ $N(2^k) = 1, 1, 2, 3, 6, 34, \cdots$.

For partitions of type 2^k the three-symbol sequences reduce into two-symbol sequences that consist of k opening and k closing paretheses. Each matching pair of parentheses can be interpreted as a handshake between two people within a cycle. The noninterleaving property implies nonintersection of handshakes. The solution of the handshake problem — how many different ways can $2k$ people seated around a table shake hands without crossing handshakes, up to rotations, is

$$N(2^k) = \frac{1}{2k}\sum_{d|k}\Phi\left(\frac{k}{d}\right)\binom{2d}{d} - \frac{1}{2(k+1)}\binom{2k}{k}$$

$$+\frac{1}{k+1}\binom{k-1}{\frac{1}{2}(k-1)} \qquad (3)$$

The last term contributes only if $k-1$ is divisible by 2. The sum is over those positive integers d that divide k, indicated by the symbol $d|k$ below the summation sign. The function $\Phi(n)$ counts the number of integers, j, $1 \leq j \leq n$ for which j and n have no common divisors. The sum over d is dominated by $d = k$. Keeping only this term in the sum, and neglecting the last term, we find

$$N(g) > N(2^k) > \binom{2k}{k} \times \left(\frac{1}{2k} - \frac{1}{2(k+1)}\right) \qquad (4)$$

The logarithmic limit is easily taken using Stirling's approximation, and we find

$$\lim_{k\to\infty} \frac{\log[N(2^k)]}{2k} = \log(2)$$

IV. UPPER BOUND ON TORAL ENTROPY

The upper bound on toral entropy is log(3). A crude upper bound to $N(g-1)$ is obtained by noting that a word of length $g-1$ can be formed with the three-symbol alphabet (, *,) in 3^{g-1} ways. This bound ignores the requirements (a)-(d) specified above.

A more refined upper bound is constructed by requiring that in any sequence the cumulative number of open parentheses must not be less than the cumulative number of closed parentheses. This is equivalent, under the association $(\to M_S = +1, * \to M_S = 0,) \to M_S = -1$, to counting the number of ways $g-1$ particles of spin $S = 1$ can be combined to total angular momentum $J_{Tot} = 0$: $N(g-1, S = 1, J_{Tot} = 0)$. For this counting problem it is known that

$$\lim_{g\to\infty} \frac{\log[N(g-1, S=1, J_{Tot}=0)]}{g-1} = \log(3) \qquad (5)$$

V. ESTIMATES FOR THE TORAL ENTROPY

The algorithm for building (and counting) the complete set of inequivalent three-symbol sequences of length g, that respect requirements (a)-(d), proceeds as follows: An overcomplete list is generated from the complete set of sequences for genus $g-1$ by applying to each one of them the following operations: (1) Inserting a $*$ at each legal position (i.e., lengthening a cycle). (2) Replacing a $*$ by the sequence () (i.e., imbedding a two-cycle). (3) Replacing a $*$ by the sequence)((i.e, splitting a cycle into two cycles). In fact, operation (3) is only capable of generating sequences that have not already been generated by operations (1) and (2) if applied to a genus $g-1$ sequence with a maximum ($\lfloor \frac{g-2}{2} \rfloor$) number of cycles.

The list thus created contains repetitions that have to be eliminated. Furthermore, sequences on the list that are equivalent by cyclic permutations to other sequences need to be discarded.

TABLE I: Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	$\log[N(g)]/(g-1)$	g	$N(g)$	$\log[N(g)]/(g-1)$
3	1	0.000000	12	145	0.452430
4	1	0.000000	13	368	0.492340
5	2	0.173287	14	870	0.520653
6	2	0.138629	15	2211	0.550086
7	5	0.268240	16	5549	0.574758
8	6	0.255966	17	14290	0.597957
9	15	0.338506	18	36824	0.618465
10	28	0.370245	19	96347	0.637540
11	67	0.420469	20	252927	0.654782

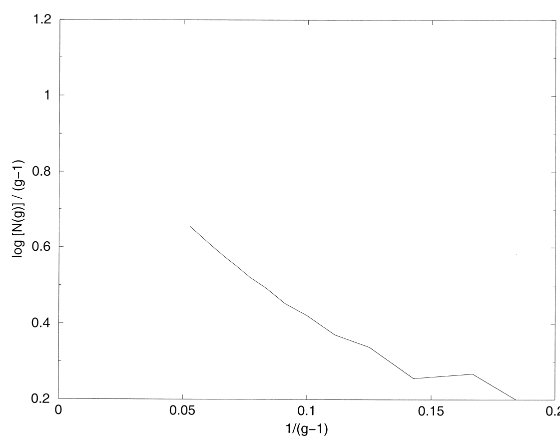

FIG. 2: The ratio $\log[N(g)]/(g-1)$ appears to converge at 1.0.

VI. CONCLUSION

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