A Topological Test for Embeddings

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A new test for embedding time series data from low-dimensional chaotic systems into three-dimensional phase spaces is proposed. It is topological, depending on specific invariant structures within a data set, rather than statistical averages over the entire attractor. We illustrate this test by applying it to electrolytic data. Although it can only be used to test for three-dimensional embeddings, it succeeds where the classical tests, depending on fractal dimension, largest Lyapunov exponent estimates, and false neighbor estimates, often fail.

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The first step in analyzing chaotic data generated by a physical system is the search for a suitable embedding. The classical tests for embeddings depend on estimating geometric or dynamical quantities. Geometric quantities include fractal dimensions and fractions of false near neighbors. Dynamical quantities include Lyapunov exponents and dimensions, and the ability to predict behavior short times into the future.

Once an embedding has been determined the reconstructed attractor can be analyzed. Analyses are of three types: geometric, dynamical, and topological. Topological analyses concentrate on determining the spectrum and organization of the unstable periodic orbits in the attractor. In three dimensions the organization of the unstable periodic orbits can be used to determine the mechanism responsible for producing chaotic behavior. This type of analysis is restricted to three dimensions because knot organization is determined by a topological index, the Gauss linking number that can be computed for pairs of orbits, only in three dimensions.

In this note we rectify the asymmetry that exists between tests for embeddings — there are two types — and analyses after an embedding has been constructed — there are three types. We do this by introducing a new test for embedding. This can only be used to test whether a mapping into a three-dimensional space is an embedding. It cannot be used in higher dimensions. However, this test succeeds where the classical tests often fail. The new test proceeds as follows. A set of unstable periodic orbits is located in the data. A mapping of the data into $\mathbb{R}^3$ depending on one or more parameters is proposed. The unstable orbits are mapped into $\mathbb{R}^3$ under this mapping. The minimum distance between each pair of orbits is computed as a function of the mapping parameter. Where orbits cross the minimum distance goes to zero and an embedding is not possible. This is so because crossing points do not have a unique future: they are on two different trajectories. A plot of minimum distance as a fucntion of mapping parameter will show values of the parameter for which an embedding does not exist and values where an embedding is possible.

We illustrate this new test on experimental data taken in an electrolytic experiment [1, 2]. It has already been shown that a three dimensional differential embedding is possible [3]. Such an mapping is equivalent a time delay mapping

$$x(i) \rightarrow [x(i), x(i-\tau_1), x(i-\tau_2)] \tag{1}$$

with minimum delay: $\tau_1 = 1$ and $\tau_2 = 2$. It is widely appreciated that each derivative or integral of a scalar time series costs about an order of magnitude in the signal to noise ratio. It is therefore worth investigating differential-delay mappings as possible embeddings. These have the form $\tau_1 = 1$ and $\tau_2 = \tau + 1$ with . We use the topological test for embeddings to determine ranges of values of the single delay parameter τ for which this mapping might be an embedding.

The first step is to locate a set of segments of the trajectory that behave like unstable periodic orbits. We used a simple version of the recurrence test [4] on the unfiltered scalar time series data $x(i)$, sampled at 5000 Hz. The statistic $D = \sum_{k=0}^{K} |x(i+k) - x(i+p+k)|$ was computed over the data set i as a function of p in the range $50 < p < 1100$. For this statistic to be small, $x(j)$ must be near $x(j+p)$ for $j = i, i+1, ...i+K$. After p data samples the trajectory returns to its original neighborhood for the next K samples (at least). The time series from i to $i+p-1$ is a candidate for an unstable periodic orbit. For this data set $32 \leq x(i) \leq 56$ for all i. We used $K = 20$ and investigated all 20 data segments with $D < 0.5$. We looked at each segment and chose 9 segments that described distinct orbits. The period one, two, and three orbits appeared more than once in this list. (Period is defined by the number of intersections with a Poincaré section [3].) Multiple copies of the same orbit were represented by samples of different lengths $p-1$. The lengths differed by about 2%. This indicates that the data are not stationary. Such nonstationarity presents severe impediments to tests for embeddings based on fractal dimension estimates.

In total, we found 9 segments of the chaotic trajectory that described orbits with periods from 1 to 7, with two distinct orbits of periods 5 and 6. A Fourier series was constructed for each using $[p/10]$ Fourier sine and cosine coefficients. An FFT was not used so that we did not have to interpolate the data. We checked that the Fourier representation was an excellent representation of the orbit. The first derivative was constructed in the Fourier representation. The nine orbits in the x-dx/dt p

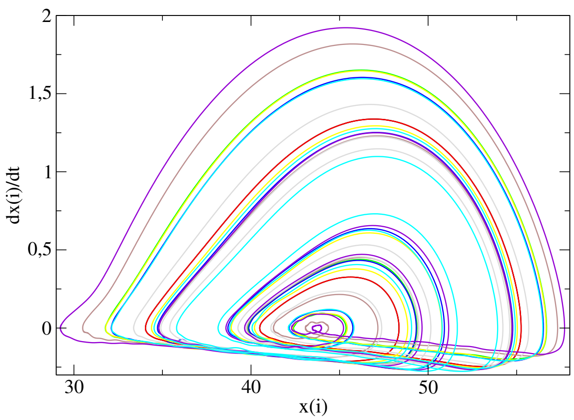

FIG. 1: Nine periodic orbits found in chaotic data from an electrolysis experiment are shown in the x-dx/dt plane.

Each orbit was then mapped into $\mathbb{R}^3$ under

$$x(i) \rightarrow [x(i), \frac{dx}{dt}(i), x(i-\tau)] \tag{2}$$

The distance between every point on one orbit and every point on another orbit was computed and the minimum distance saved. This minimum distance was computed as a function of the delay, τ. Distances were computed with a diagonal metric $g_{ij} = g(i)\delta_{ij}$, with $g(1) = g(3) = (2.0/21.0)^2$ and $g(2) = (2.0/1.88)^2$ to scale all components into an interval of length 2. The minimum distance function for all $9 \times 8/4 = 36$ orbit pairs is shown in Fig. 2. For regions where the minimum distance function is bounded away from zero an embedding is possible. This figure suggests that an embedding is possible in the ranges $1 \leq \tau < 50$ and $55 < \tau < 65$, and $80 < \tau < 100$.

The intersections of some orbits as the delay τ changes could be masked by the discrete sampling size. In such

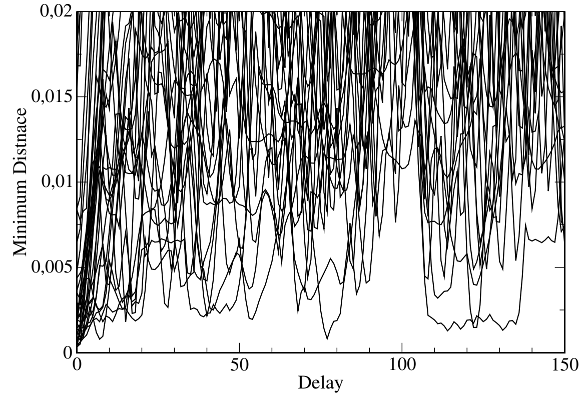

FIG. 2: Minimum distance between all pairs of orbits as a function of the delay τ.

cases the minimim distiance approaches zero rapidly on one side of the intersection and departs rapidly from zero on the other, without ever reaching zero because the index τ is integer. At such suggestive "knife point" signatures, it is a simple matter to interpolate between the integer values of τ using the Fourier representation of the orbits. We have done this for the period-one, -two, and -three orbits, and have found intersections at $\tau \simeq 11$, $\tau \simeq 84$ and $\tau =\simeq 135$ (c.f., Fig. 3). The linking numbers of the orbits can also be computed on either side of a possible crossing to see if the orbits actually undergo an intersection [8]. Since these orbits are in the attractor, they are surrounded by both the attractor and other periodic orbits. If they intersect at τ_{xing} the mapping of the attractor is not an embedding for values of τ in the ne

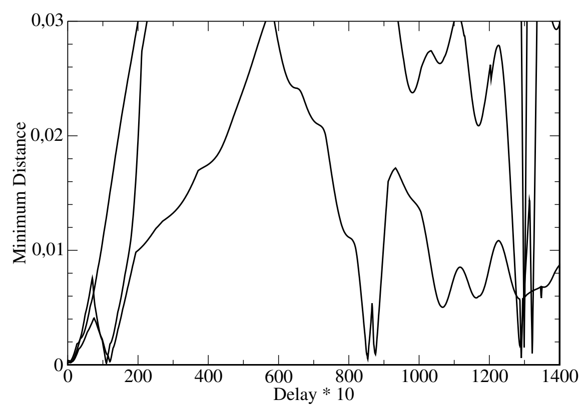

FIG. 3: Minimum distance between the three lowest period oribts. These orbits were interpolated by a factor of 10 in the Fourier representation, allowing the delay to assume nonintegral values ($10 \times \tau$ is an integer).

We have compared these results of the topological test for embeddings with traditional tests for embeddings. In all the tests described below each component of each vector time series studied was scaled into the interval $[-1, +1]$. We first investigated the false near neighbor test [5] using embeddings of the form $x(i) \rightarrow [x(i), x(i-1), x(i-1-\tau), x(i-1-\tau-40), x(i-1-\tau-2\times 40)]$. Fig. 4 shows the fraction of false near neighbors in going from $2 \rightarrow 3$ dimensions, from $3 \rightarrow 4$, and from $4 \rightarrow 5$ dimensions. The fraction of false near neighbors is sufficiently small only in the case $n = 2rightarrown + 1 = 3$ for very small values of the delay τ. This is incorrect since chaotic dynamics requires $n \geq 3$. In the other two cases $n = 3rightarrown+1 = 4$ and $n = 4rightarrown+1 = 5$ the fraction of false near neighbors is so large that embeddings for any value of τ are ruled out (icorrectly).

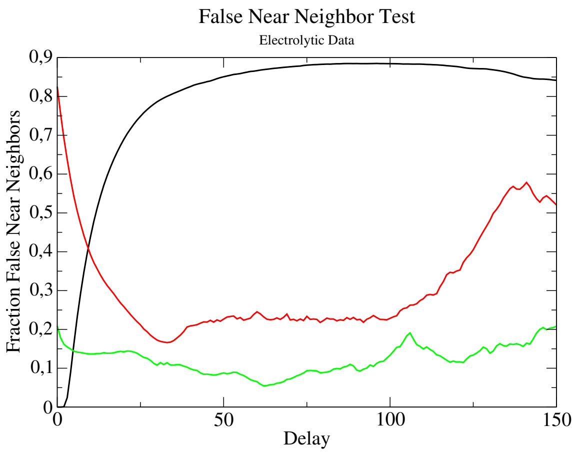

FIG. 4: Fraction of false near neighbors. These results suggest that no embedding into dimension $n = 3$ or $n = 4$ is possible.

We also performed correlation dimension estimates and largest Lyapunov exponent estimates on the three dimensional mappings $x(i) \rightarrow [x(i), x(i-1), x(i-1-\tau)]$ using the algorithm proposed in [6]. The results are shown in Fig. 5 and Fig. 6. Since the correlation dimension estimate D_2 should be larger than 2 if the mapping is an embedding, these results suggest that no three dimensional mapping of the type used is an embedding. In fact, it is well-known that small values of the delay can cause the estimate of the correlation dimension to be anomalously large [7]. There is nothing in the behavior of the largest Lyapunov exponent to suggest that the mapping is an embedding for some values of the delay and not for others.

In this work we have proposed a new topological test for embeddings into $\mathbb{R}^3$. This test has been used to determine ranges of the time delay for which a differential-delay mapping of experimental data into $\mathbb{R}^3$ cannot be an embedding and ranges for which it might be an embedding. For parameter ranges not ruled out, further topological studies, such as those carried out in [3], should be used to confirm or reject whether the mapping is an

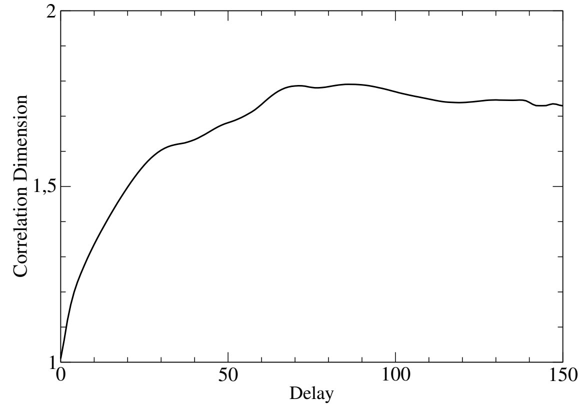

FIG. 5: Correlation dimension D_2 estimate as a function of the mapping parameter τ. In principle this should be larger than 2 if the mapping is an embedding.

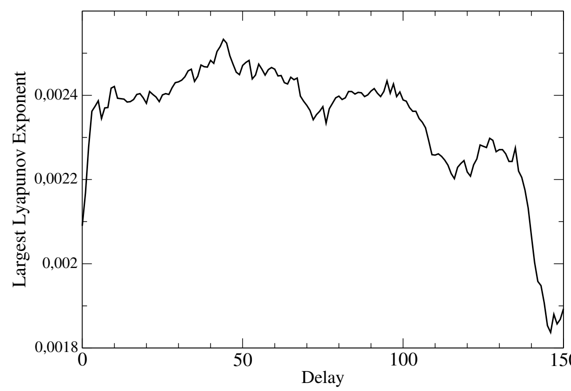

FIG. 6: Largest Lyapunov exponent estimate as a function of delay τ. Nothing about this plot suggests that the mapping is an embedding for some values of τ and not for others.

embedding. This new test depends on invariant structures (unstable periodic orbits) in chaotic data that can be extracted and studied in detail. It does not depend on statistical averages. We have compared this new topological test with older tests that do depend on some sort of statistical averaging. In general, the traditional tests have failed to indicate values of the delay τ for which embeddings can and cannot be possible.

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