Nonlinear Dynamics

PHYS 471, 571

Problem Set # 7
Distributed Feb. 26, 2015
Due March 5, 2015

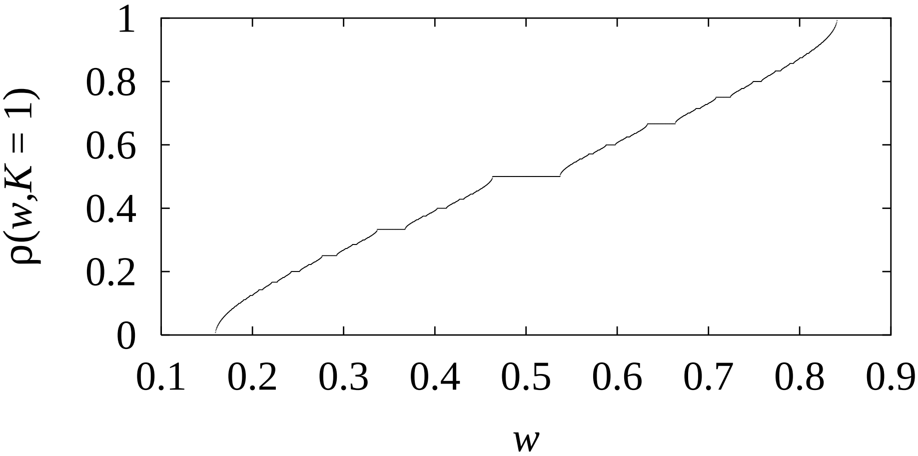
Undergraduates: The *easier* of Problems (1 or 2) and Problems (3 or 4).

Graduates: The *harder* of Problems (1 or 2) and Problems (3 and 4).

All students: Solutions must contain enough words so that I can understand what you think you did, and you will be able to understand what you did in 12 months. No words = No credit!

1. Devil's Staircase: Construct and plot the "Devil's Staircase" for the circle map by plotting $\rho(\omega, K)$ vs. ω for $K = 1$ (Notes, Eq. (7.3)). Use

$$\theta_{n+1} = \theta_n + \omega + \frac{K}{2\pi} \sin 2\pi\theta_n \qquad \text{mod } 1$$



The *winding number* ρ is defined as follows:

$$\rho(\omega, K) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \Delta\theta_j$$

where $\Delta\theta_j = \theta_{j+1} - \theta_j = \omega + \frac{K}{2\pi}\sin 2\pi\theta_j$.

2. Henon's area-preserving quadratic map describing galactic dynamics is

$$\begin{aligned} x_{n+1} &= x_n \cos\alpha - (y_n - x_n^2)\sin\alpha \\ y_{n+1} &= x_n \sin\alpha + (y_n - x_n^2)\cos\alpha \end{aligned}$$

Reproduce any one of the phase portraits shown in Fig. 1 below, or find an exciting new one.

3. Real Chaotic Data: At the bottom of our website there are five data sets labeled 'Scalar Data File # x', with $x = 1, 2, 3, 4, 5$. Each has length 10000 with integer entries. Experimental data are typically taken at equally spaced time intervals and recorded in binary form, and therefore integers. In these files the binary numbers have been transformed to decimal form.

Choose one of these data sets.

a. Identify which one you have chosen.

b. Plot it.

c. Carry out a close returns calculation by recording (i, p) values at which $|d(i) - d(i+p)|$ is "small". Tell me how big "small" is and why you chose that value.

d. Do some values of p occur frequently? Which ones?

e. For one prevalent value of p plot $d(j)$ for $i \leq j \leq i+p$. Does this look like a periodic (an almost periodic) orbit? Is this an orbit of period-1? -2? -3? Explain your logic.

4. Empirical Orthogonal Modes or Singular Value Decomposition: A complicated dynamical system is monitored in 5 output channels. Data are taken at regular intervals in each of these channels. In total 500 measurements are taken. These measurements are collected together into a table. The table is available at the bottom of our website and called 'Weekly Data File'. It consists of five columns of real numbers, each column of length 500.

a. Download this table into computer readable form.

b. Consider this table as a matrix: 500×5. Diagonalize this matrix. (See *Numerical Recipes, Singular Value Decomposition.* Specifically, follow these steps.

c. Data preparation: Normalize each of the 5 data streams by transforming it to zero mean, unit standard deviation.

d. The resulting prepared data is packed into a 500×5 matrix T. Compute the real symmetric 5×5 matrix $T^t T$.

e. Diagonalize it. List the eigenvalues, from large to small.

f. The 5 eigenvectors $u(\beta, \alpha)$ with $1 \leq \alpha, \beta \leq 5$ and are the 5 normal modes. Plot each mode. Do this by fixing the mode number α and plotting $u(\beta, \alpha)$ for the five values of β. Use different symbols to help distinguish the modes. Note that the most important eigenvector ($\simeq 60\%$ of the strength of the data set) is almost "flat": its five components have almost the same value.

g. Use the eigenvectors $u(\beta, \alpha)$, $\alpha = 1, 2, 3, 4, 5$, to compute the duals to each of the 5 normal modes. This is done by constructing the inner product $v(i, \alpha) = \sum_{\beta=1}^{5} T(i, \beta) u(\beta, \alpha)$. Plot each (plot $v(i, \alpha)$) vs. the time index i ($1 \leq i \leq 500$) for each of the 5 values of the normal mode index α. The components $v(i, \alpha)$ are surrogates for the time evolution of normal mode α. They are constructed from the initial data set through the Singular Value Decomposition.

h. Information statement, don't solve anything. At this stage someone who analyzes data asks the question: Can I construct a set of 5 (maybe 4, or even 3) ordinary differential equations that more or less generate the time-dependent behavior exhibited by these 5 (or 4, or 3, if we neglect the modes with smallest eigenvalues) time series. If Y, then the data are generated by a low-dimensional deterministic process that may or may not be chaotic (chaos if you can extract a couple of unstable periodic orbits). If N, then plan B must be enacted.

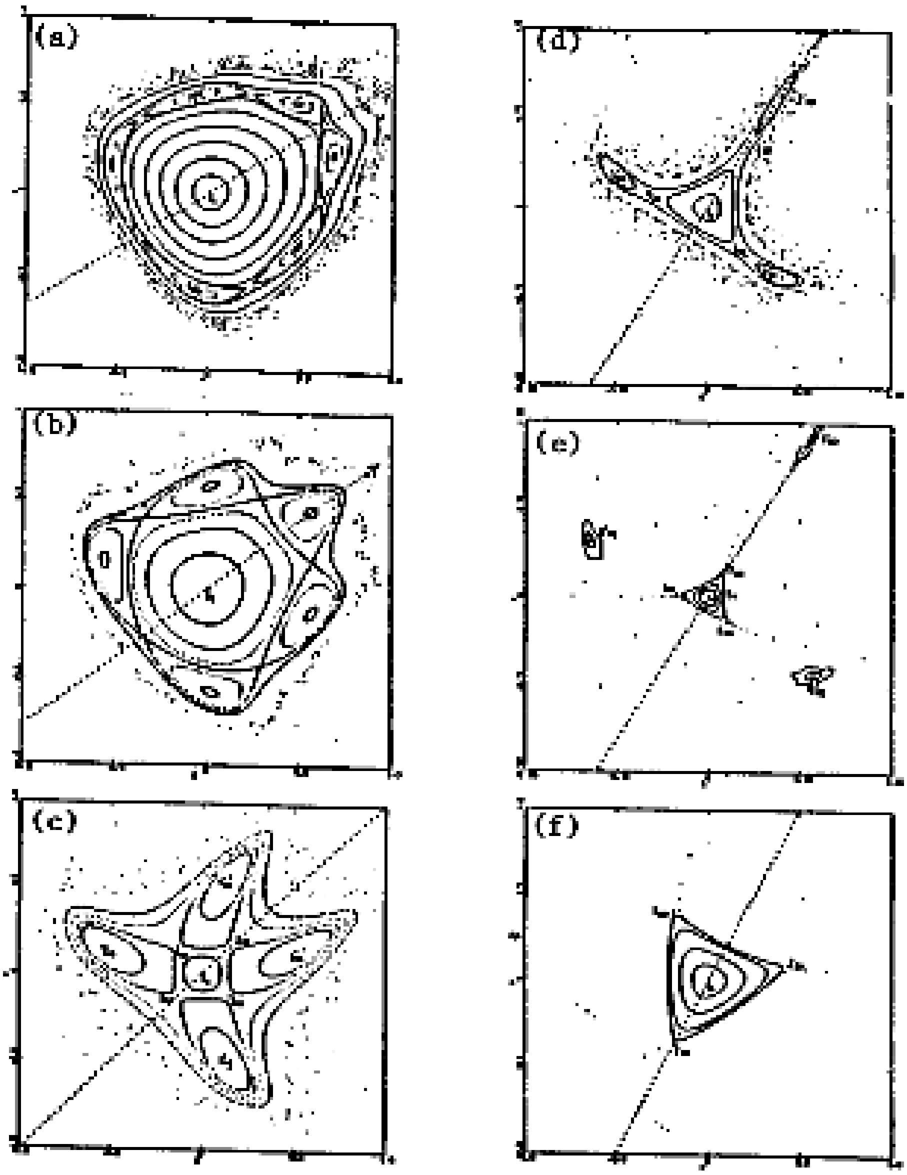


Figure 1: Some dynamics genrated by the quadratic Henon map for selected control parameter values. (a) $\alpha = 1.16$; (b) $\alpha = 1.33$; (c) $\alpha = 1.58$; (d) $\alpha = 2.00$; (e) $\alpha = 2.04$; (f) $\alpha = 2.21$.