Nonlinear Dynamics

PHYS 471, 571

Problem Set #4
Distributed February 3, 2015
Due February 12, 2015

Do only one of the two problems.
Undergraduates: Problem 1 or Problem 2, a-e.
Graduates: Problem 1 or Problem 2, a-h

All students: Solutions must contain enough words so that I can understand what you think you did, and you will be able to understand what you did in 12 months. No words = No credit!

1. Scaling in State-Variable Space: Feigenbaum has constructed a nonlinear equation to define the value of the scaling parameter α for the logistic map and every map in its universality class. The equation is

$$g(x) = -\alpha g(g(x/\alpha))$$

Assume: $g(x) = g(-x)$ and $g(0) = 1$.

a. Show $\alpha = -1/g(1)$.

b. Set

$$g(x) = 1 + g_1 x^2 + g_2 x^4 + \cdots g_n x^{2n} = \sum_{j=0}^{n} g_j x^{2j}$$

Creep up on a value of α by truncating this equation at $n = 1$ and solving for α, then $n = 2$ and solving, etc. Carry this out as far as you can, subject to the conditions: don't burn yourself out; don't burn out your computer.

c. Plot α_n vs. n. Here α_n is the approximation to α when the series is truncated at the nth term.

2. Flows and Maps: The Rössler equations have been used to model chemical, electronic, vibrating, and laser systems that are nonlinear. The equations are:

$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned} \tag{1}$$

After transients die out, these equations generate a flow like that shown in Fig. 1, for control parameter values $(a, b, c) = (0.398, 2, 4)$. In getting to this chaotic flow a number of different types of behavior are encountered. Some are shown in Fig. 2.

a. What do you think the Feigenbaum scaling constants δ, α are for this flow at the period-doubling accumulation point? **N.B.: THIS IS NOT A REQUEST TO COMPUTE THEM. THIS IS A REQUEST TO MAKE AN EDUCATED GUESS.**

b. Integrate these equations for the parameter values given above and provide a projection of the flow into the x-y plane.

c. Record and plot the (x, z) values every time the flow crosses through the half-plane $y = 0, x < 0$ (this plane is "officially"called the *Poincaré section*). Record the intersections sequentially.

d. Plot x_{i+1} vs. x_i, $1 \leq i \leq N - 1$, where N is the total number of intersections recorded in part **b.**. Your result should look like Fig. 3.

e. Find one unstable orbit of period $p \neq 1$ in this map.

f. Fit a parabola $x' = A + Bx - Cx^2$ to this return map.

g. Find a transformation that takes the return map that you've computed in **f.** to the return map of the form $y' = a - y^2$. How are the control parameters A, B, C and a related? How are the state variables x and y related?

h. What value of a corresponds to the return map that you've computed in **f.**?

i. Use this information to estimate: which periodic orbits are present and which are not in the flow. Also estimate the topological entropy of this flow.

j. Does your fitted return map pass the χ^2 test?

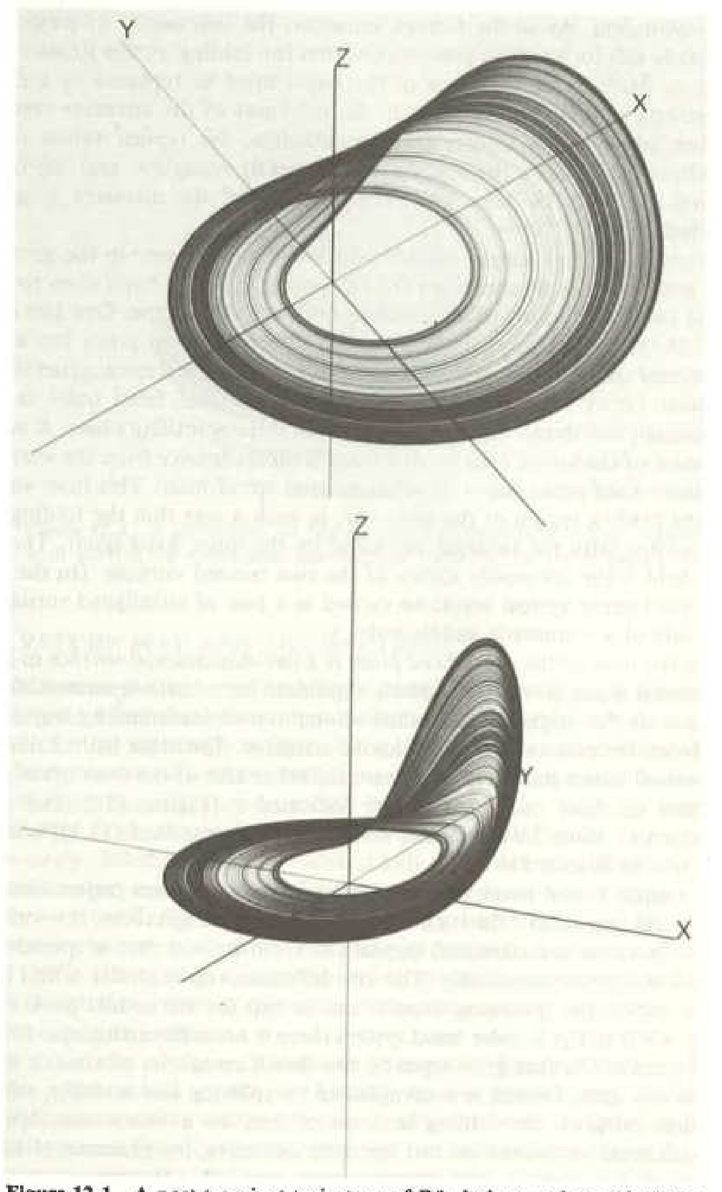

Figure 12.1 **A post-transient trajectory of Rössler's equations (12.1) for the simply folded band attractor. Parameters are $a = 0.398$, $b = 2$, $c = 4$**

Figure 1: Flow generated by the Rössler equations. From: J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos*: London: John Wiley & Sons, Ltd., 1986.

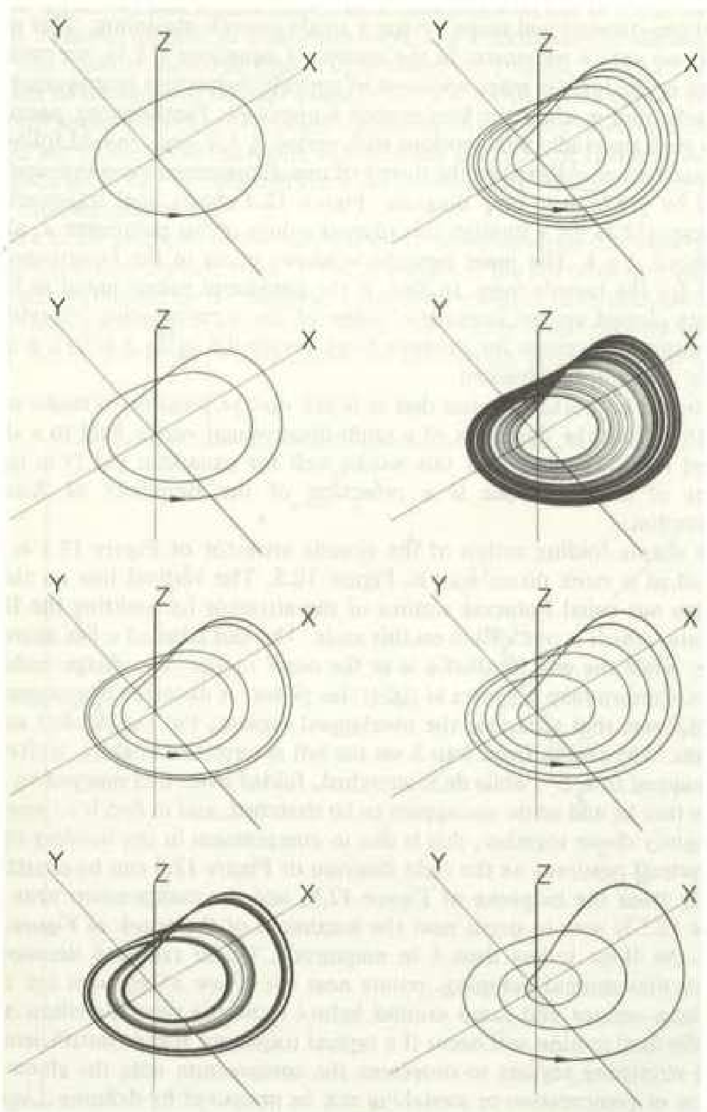

Figure 12.4 Final trajectories of equations (12.1) for different values of the parameter a. Left row, top to bottom: limit cycle, $a = 0.3$; period 2 limit cycle, $a = 0.35$; period 4, $a = 0.375$; four-band chaotic attractor, $a = 0.386$. Right row, top to bottom: period 6, $a = 0.3909$; single-band chaos, $a = 0.398$; period 5, $a = 0.4$; period 3, $a = 0.411$. In all cases $b = 2$, $c = 4$

Figure 2: Flow generated by the Rössler equations for some values of a in the range $0.3 \leq a \leq 0.411$. From: J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos*: London: John Wiley & Sons, Ltd., 1986.

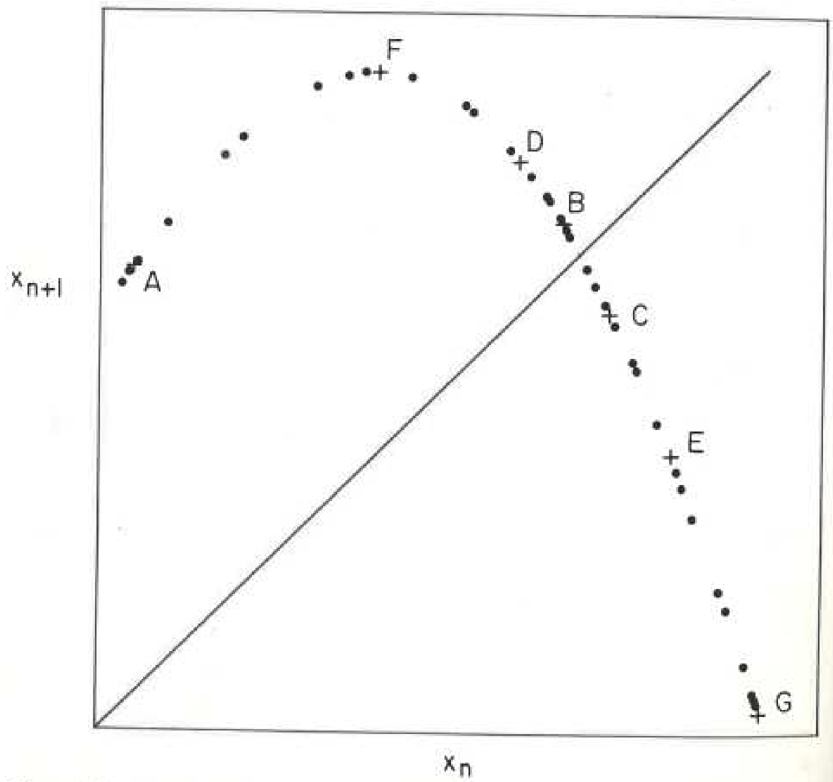

Figure 12.3 One-dimensional mapping points taken from a trajectory of the differential equations (12.1). A sequence of seven successive points is identified by the crosses labelled A to G

Figure 3: Return map on the Poincaré section for the Rössler attractor. From: J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos*: London: John Wiley & Sons, Ltd., 1986.