Concepts related to nonlinear dynamics

A BRIEF HISTORICAL OUTLINE

The analytic period (before 1880) – characterized by the search for analytic solutions and perturbation methods; searches for *integrals of the motion*, particularly *time independent*, *algebraic* integrals.

Main areas: classical mechanics, celestial mechanics – Newton, Euler, Lagrange, Laplace, Jacobi *et al.* Hydrodynamics – e.g., Rayleigh (who briefly considered limit cycles and bifurcation concept) Kinetic theory of gases – Boltzmann equation: stosszahlansatz (implied concept: complexity arises from the interaction of many particles); the H-function, statistical concept of entropy.

Relevant abstract mathematical concepts: non-Euclidean geometry; set theory; Cantor sets; transfinite numbers; the continuous, space-filling curves of G. Peano; Painlevé transcendentals.

S. Lie (1879–1900): A general principle for obtaining integrals of nonlinear *partial differential equations*, by determing the invariance properties under a *continuous* group (Lie groups); a frequent application is the invariance under some *scaling* ('similarity transformations').

Three historical theorems:

Bruns (1887): the only independent *algebraic integrals* of the motion of the three-body problem (which has 18 integrals) are the *ten 'classic integrals'* (energy, total linear momentum, total angular momentum, and the time-dependent equations for the motion of the center of mass).

Poincaré (1890): if the Hamiltonian of a system, when expressed in terms of action-angle variables (J, θ), is of the form $H(J, \theta, \lambda) = H_0(J) + \lambda H_1(J, \theta)$, where $H_1(J, \theta)$ is periodic in every θ_i $(i = 1, \ldots, N)$, and if the Hessian does not vanish identically, $|\partial^2 H_0/\partial J_i \partial J_k| \not\equiv 0$, then there exists *no analytic, single-valued integral of the motion*, $I(J, \theta, \lambda) = \sum_n \lambda^n I_n(J, \theta)$, which are periodic in θ, other than the Hamiltonian, $H(J, \theta, \lambda)$.

Painlevé (1898): the only independent integrals of the motion of the *N-body problem*, which involve the *velocities algebraically* (regardless how the spatial coordinates enter), are the classic integrals.

Stability of motion – results of A.M. Lyapunov (1892); Lyapunov exponents.

Korteweg and deVries demonstrated the existence of *finite amplitude solitary water waves*.

Poincaré (1880–1910): emphasized the study of the *qualitative, global aspects* of dynamics in *phase space*; developed *topological analysis*; generalized *bifurcation concept*; introduced *mappings in phase space* (difference equations); *surface of section*; introduced *rotation numbers* of maps; *index of a closed curve* in a vector field; initiated the *recursive method of defining dimensions*.

Whittaker: obtained the *adelphic integrals* for coupled harmonic oscillators, where the integrals are *nowhere analytic functions of the frequencies* (1906).

1920–1930

Mathematics: the *theory of dimensions* (Poincaré, Brouwer, Menger, *Hausdorff, et al.*); *fixed point theorems* (Brouwer, Poincaré–Birkhoff); the development of *topology, differential geometry* (*Bäcklund transformations*); Birkhoff studied the *abstract dynamics of analytic one-to-one transformations*, emphasized the various categories of asymptotic sets (*alpha and omega limit sets*, various periodic sets, hyperbolic and elliptic fixed point neighborhoods, *recurrent motions of a discontinuous type*, etc.).

Numerical computations by Størmer, and students (!), of the dynamics of solar particles in the dipole magnetic field of the Earth (a nonintegrable system), during 1907–30.

The Madelstam–Andronov school of applied nonlinear analysis; replacement of nonlinear system by a set of linear segments.

E. Fermi (1923) *attempted* to generalize Poincaré's theorem in order to prove *ergodicity* in some systems.

van der Pol: the extensive *study of limit cycles, relaxation oscillations*, leading to *singular perturbation theory*. Studied the forced van der Pol oscillator with van der Mark (1927); observed *subharmonic generation, hysteresis*, '*noisy*' *regions* in *parameter space*. A *variety of bifurcation phenomena*.

The Andronov–Poincaré bifurcation (1930)
The averaging method of perturbation theory is further refined (Bogoliubov–Krylov–Mitropolsky).

Mathematics: the introduction of the concept of structural stability of equation of motion by Andronov and Pontriagan (1937); gradient dynamics; symbolic dynamics;

embedding concepts; logical foundations (K. Gödel, 1931); computational foundations (A.M. Turing, 1936).

The birth of *mathematical biophysics*; Lotka, Volterra, Fisher, Rashevsky.

E. Schrödinger's book, *What is Life? The Physical Aspect of the Living Cell.*

Kolmogorov's spectrum for the case of *homogeneous turbulence* in fluids.

The digital computer: the ENIAC, built at the Moore School of Electrical Engineering, the University of Pennsylvania (1943–46).

1945–55

The studies of Cartwright and Littlewood, and of Levinson (around 1950): gave a mathematical proof that the forced van der Pol oscillator has a *family of solutions which is as 'chaotic' as the family of all sequences of coin tosses*; a physical dynamic example of Birkhoff's abstract discontinuous dynamics; the first physical demonstration of the existence of a *curious* 'attractor'.

von Neumann investigated the problem of *self-reproducing automata*; with Ulam, introduced *cellular automata*, whose dynamics is exact (no roundoff errors). He emphasized the *heuristic use of computers*, to discover general dynamic characteristics.

S. Ulam emphasized the interaction between man and computer ('*synergesis*'); looked for the asymptotic properties of certain nonlinear maps; studied the *growth of patterns* in cellular automata.

The quantitative description of membrane currents; Hodgkin and Huxley.

The *Hopf bifurcation*: a local bifurcation from a fixed point to a limit cycle in R^n.

The *Fermi–Pasta–Ulam* computer study of lattice dynamics: the search for relaxation to equilibrium; found *no simple relaxation* (non-Boltzmann, non-Fermi), but nearly-periodic behavior (*simple motion in a 'complex' system*). This is known as the *FPU phenomena*. (Fermi: 'A minor discovery').

The Kolmogorov–Arnold–Moser theorem: proves that, for a special class of solutions of systems whose Hamiltonian satisfies *Poincaré's theorem*, a canonical transformation exists to new action-angle variables, when the Hamiltonian is *weakly* perturbed; this class contains *most solutions, as the perturbation tends to zero*; briefly, most tori which are ergodically covered by solutions only become distorted, but not 'destroyed', by sufficiently small perturbations; these *tori are preserved* in phase space. The preserved tori are known as *KAM surfaces*.

von Neumann's proof of the existence of universal, self-reproducing automata; manuscript completed after his death (1957) by A.W. Burks.

The Turing instability: the instability of a homogeneous system of dynamic cells coupled by diffusion; *morphogenesis of spatial structures.*

Mathematics: Kolmogorov–Sinai concept of *dynamic entropy*; concept of *mixing*; Arnold's '*cat map*'; Smale's '*horseshoe map*' (inspired by strange attractor dynamics of the *forced van der Pol oscillator*).

1960–1970

The computer studies of the *continuum lattice* (*Korteweg–deVries equation*), by Kruskal and Zabusky, inspired by the *FPU phenomena*; rediscovery of solitary waves in nonlinear dispersive media; discovery of '*soliton*' (stability) property in multiple-soliton configurations; nonlinear 'basis' set.

Coherent, periodic oscillations in chemical systems – the Belousov–Zhabotinskii oscillations; *low dimensional attractor in a high dimensional phase space* (around 30 chemical compounds).

Computer study of the Bénard problem by Saltzman; the discovery of sometimes 'erratic' dynamics in solutions of the Navier–Stokes equations.

The Lorenz equations; an ordinary differential equation approximation of the Navier–Stokes equations for the Bénard problem. Solutions bifurcate to '*chaotic dynamics*' – a '*strange attractor*' in an *autonomous system*. Also has *homoclinic orbits*, '*preturbulence*', and stable limit cycles.

The *bifurcation sequence* of *general one-dimensional, single-maxima maps* of the interval into itself (Sharkovsky, 1964). The *logistic map*, developed in biology; *period-two bifurcations*, chaotic regions, *windows of periodicity*.

Inverse cascading (to shorter wavelengths) in two-dimensional hydrodynamics.

The breakup of KAM surfaces: the area preserving map of Hénon–Heiles, motivated by astronomical problem. The estimates of breakup, based on overlap of resonances, by Chirikov.

The further development of the concept of *fractal structures* – sets with *fractional dimensions*, by Mandelbrot.

The introduction of the concept of *topological entropy*.

The *heuristic use of the computer*, by Codd, to simplify von Neumann's self-reproducing automata.

Smale's result; structurally stable systems are not dense (1966).

Catastrophe theory, both elementary and general, as visualized by R. Thom; In part, a study of the structurally stable sets in parameter space where a system is structurally unstable (!); many ethereal and imaginative generalizations are visualized by Thom and others. Roundly criticized by many!

The inverse scattering transformation, due to Gardner, Greene, Kruskal, and Miura: a method for obtaining the *general solution of a particular* ('*integrable*') *class of partial differential equations*; this discovery proves that not all analytic methods have been discovered!

The proof of the existence of *Lyapunov exponents* for systems of ordinary differential equations (Oseledec, 1967).

1970–1980

The concept of '*Synergetics*' becomes more diversified, expanded: Zabusky; Haken, *et al.*

Self-organization of matter; Biological evolution; Eigen (1971), Smale (1975).

The Newhouse, Ruelle, Takens theorem – roughly, 'most' systems which are nearly the same as a system whose dynamics consists of three or more periodic components, will have a strange attractor. This suggests that the bifurcation sequence to chaos is from a fixed point, to periodic, then doubly periodic, and then 'turbulence' (a strange attractor). This theorem was preceeded by the *Ruelle–Takens theorem* (1971).

Solitons found in the discrete *Toda lattice.*

Computer and Poincaré map used to test for integrability: Ford predicts the Toda lattice is integrable.

Toda lattice is proved to be integrable – but no use is made of all those integrals of the motion, even when they are known explicitly! Why not? Better yet, *how*? Are they 'macroscopically controllable'?

'Direct method' of obtaining soliton solutions – another analytic method, by Hirota.

Bennett's introduction of *logical reversibility* in computations (1973).

Strange attractor in the two-dimensional map of Hénon's; explicit example of Birkhoff's dynamics.

Possible mechanics for organization of *memory and learning*; Cooper (1973).

Ruelle–Takens introduce the '*strange attractor*' characterization and definition (variously modified later).

The cellular automata game of '*Life*' is invented by J.H. Conway.

Qualitative '*universal*' *features* of the bifurcation patterns of *many* one-dimensional maps is discovered by Metropolis, Stein, and Stein.

Solitons found in many partial differential equations; generalizations of the inverse scattering transformation (Zakharov–Shabat, Ablowitz–Kaup–Newell–Segur).

The *logistic map is 'discovered'* by many people, thanks to the article by R. May (1976).

Quantitative 'universal' features are discovered in the bifurcation sequence of the logistic and similar maps, by Feigenbaum; importance of *renormalization concepts.*

The dynamo problem: advances are made in the self-consistent theory of geomagnetic dynamics; simplified models immitate the chaotic flip-flop of the Earth's magnetic field (Lorenz equations).

Experimental determination of bifurcation sequences in hydrodynamic systems (Gollub, Swinney, Ahlers *et al.*) spatial patterns, intermittent spatial patterns; bifurcation sequences differ from theoretical 'generic' predictions.

Protein molecules: possible soliton energy transmission (Davydov); experimentally determined 'fractal dimension' (Stapleton *et al.*).

The *semi-periodic dynamics* of the logistic map – similarity with weather 'periodicity'.

The *homoclinic bifurcation in the Lorenz system* – 'perturbulence'.

Conjecture on the relationship between the *capacity* of an attractor and the spectrum of the *Lyapunov exponents* (Kaplan and Yorke, 1979).

The possible relationship between the *Painlevé property* and integrability.

Many mathematical models of biological systems; Eigen and Schuster's *hypercycle*; Generalized Lotka–Volterra systems.

1980–

Theorems concerning attractors in *infinite dimensional systems.*

Conjectured criteria concerning the *breakup of KAM surfaces* in the standard map. The possible use of *embedding concepts* in chaotic dynamics – Takens.

Experiments on the bifurcations in homogeneous chemical oscillations – *embedding dimension* of attractor.

The study of the *KAM breakup* in the standard map using *renormalization methods* – Kadanoff–Greene–MacKay.

Studies of '*soliton*' *interactions* in higher dimensions; 'Resonances'.

Nonlinear (3D) instability ('hard' loss of stability) of Poiseuille flow in the *Navier–Stokes equations.*

Cellular atuomata studies – spatial patterns and growth; self-reproduction which is simpler than 'universal' type, but not trivial; statistical characteristics of dynamics by Wolfram.

'non-universal' behavior of bifurcations in solid state devices, etc.; experimental dimensions of chaos.

Generalization of Hirota's *direct method* – analytical extensions of soliton solutions.

Reversible cellular automata: conservative logic; *Digital Information Mechanics* (Fredkin, 1982) has the maximum number of constants of the motion. Is it *a basic description of nature*?; *Can quantum phenomena be described* in a cellular automata scheme?

Many types of chemical oscillations; biological oscillations.

Nondiffusive behavior in the chaotic region of standard map – '*sticky island*' effect.

Topological character of the homoclinic bifurcation in the Lorenz equations; *fractal basin boundaries.*

Spatial order vs. temporal chaos; spatial pattern competition leading to chaos; space–time 'entropies'.

Neural network dynamics –

Where and what is quantum 'chaos'?