Nonlinear Dynamics

PHYS 471, 571

Problem Set # 4
Distributed Jan. 31, 2013
Due February 12, 2013

Undergraduates: Problems 1, 3, 4 and 5.
Graduates: Problems 2, 3, 4 and 5.
All students: Solutions must contain enough words so that I can understand what you think you did, and you will be able to understand what you did in 12 months. No words = No credit!

1. Saddle-Node Bifurcation: Determine the value of the control parameter a at which the two period-one orbits are formed (hint: $a_1 = -1/4$).

a. Plot the return map for $a = a_1 - \epsilon, a = a_1, a = a_1 + \epsilon$. Describe how the two period-one orbits are created by pointing out the changes in the return map as a passes through the value a_1.

b. Plot the distance between the two period one orbits as a increases above a_1. Describe the nature of the singularity as $a \downarrow a_1$.

c. Plot the distance between the imaginary solutions as a decreases below a_1. Describe the nature of the singularity as $a \uparrow a_1$.

2. Period-Three Saddle-Node Bifurcation: Find the value of a at which the period-three orbit is created. If you can't compute it analytically, guess where the third caustic crosses the value of the critical point and work backward (slowly) from there. Plot the return map for $a = a_3 - \epsilon, a = a_3, a = a_3 + \epsilon$. Describe how the two period-three orbits are created by pointing out the changes in the return map as a passes through the value a_3.

3. Pitchfork Bifurcation: Find the value of a at which the stable period-one orbit becomes unstable (where $f'(y_1) = -1$).

a. Plot both the first return map and the second return map for $a = a_2 - \epsilon, a = a_2, a = a_2 + \epsilon$.

b. Plot the distance (both positive and negative) between the two values $y_\pm(a)$ and the unstable period one fixed point as a function of $a - a_2$.

c. Describe the nature of the singularity as $a \downarrow a_2$. (If it's not obvious why this type of bifurcation is called a pitchfork bifurcation, it might be useful to do the calculation over.)

4. Lyapunov Exponent: Construct and plot the Lyapunov exponent for the map $y' = a - y^2$. Estimate the Lyapunov exponent at $a = 2$. Say something useful about the (negative) spikey structure of this plot.

5. Symbolic Dynamics: Write out the symbolic name of the periodic orbits $5_2, 4_2, 5_3$ (you can find them in the handout Chapter 5, Table 5.2).

a. Relate each point on the trajectory of 5_2 (00101) with a rational fraction.

b. Show that these rational fractions map to each other under the tent map.

c. Choose one of these rational fractions to identify the orbit 00101.

d. Construct the corresponding rational fraction for 4_2 (0011) and 5_3 (00011).

e. Explain how you use these rational fractions to determine the order in which these periodic orbits are created in the logistic map "on the road to chaos."