Homework #4

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1. Let $f_1(\boldsymbol{x}) = \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{a}_1)^2}{2\,\sigma^2}\right)$ and $f_2(\boldsymbol{x}) = \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{a}_2)^2}{2\,\sigma^2}\right)$.

$$\mathcal{I} = \int_{\mathbb{R}^n} \nabla f_1(\boldsymbol{x}) \cdot \nabla f_2(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

$$\mathcal{I} = \frac{1}{\sigma^4}\int_{\mathbb{R}^n} (\boldsymbol{x}-\boldsymbol{a}_1)\cdot(\boldsymbol{x}-\boldsymbol{a}_2)\exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{a}_1)^2}{2\,\sigma^2} - \frac{(\boldsymbol{x}-\boldsymbol{a}_2)^2}{2\,\sigma^2}\right)\mathrm{d}\boldsymbol{x}$$

$$\mathcal{I} = \frac{1}{\sigma^4}\exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\int_{\mathbb{R}^n}\left[\left(\boldsymbol{x}-\frac{\boldsymbol{a}_1+\boldsymbol{a}_2}{2}\right)^2 - \frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4}\right]\exp\left(-\frac{\left(\boldsymbol{x}-\frac{\boldsymbol{a}_1+\boldsymbol{a}_2}{2}\right)^2}{\sigma^2}\right)\mathrm{d}\boldsymbol{x}$$

$$\mathcal{I} = = \frac{1}{\sigma^4}\exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\left\{\int\left(\boldsymbol{x}-\frac{\boldsymbol{a}_1+\boldsymbol{a}_2}{2}\right)^2\exp\left(-\frac{\left(\boldsymbol{x}-\frac{\boldsymbol{a}_1+\boldsymbol{a}_2}{2}\right)^2}{\sigma^2}\right)\mathrm{d}\boldsymbol{x} - \frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4}\int\exp\left(-\frac{\left(\boldsymbol{x}-\frac{\boldsymbol{a}_1+\boldsymbol{a}_2}{2}\right)^2}{\sigma^2}\right)\mathrm{d}\boldsymbol{x}\right\}$$

We'vd computed these integrals in class, so now it becomes a matter of using known integrals. The second integral is straightforward. It is the equivalent of n such one-dimensional integrals, where n is the number of dimension in the problem. Computing the first integral, however, is equivalent to computing one integral like the second-moment integral and $n-1$ integrals of the zeroth-moment. Therefore,

$$\mathcal{I}_n = = \frac{1}{\sigma^4}\exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\left\{\frac{n\,\pi^{n/2}\,\sigma^{n+2}}{2} - \frac{\pi^{n/2}}{4}(\boldsymbol{a}_1-\boldsymbol{a}_2)^2\sigma^n\right\}$$

$$\mathcal{I}_n = \exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\left\{\frac{n\,\pi^{n/2}\,\sigma^{n-2}}{2} - \frac{\pi^{n/2}}{4}(\boldsymbol{a}_1-\boldsymbol{a}_2)^2\sigma^{n-4}\right\}$$

For one-dimension

$$\mathcal{I}_1 = \frac{\sqrt{\pi}}{2}\exp\left(-\frac{(a_1-a_2)^2}{4\sigma^2}\right)\left\{\frac{1}{\sigma} - \frac{(a_1-a_2)^2}{2\,\sigma^3}\right\}$$

For two-dimensions,

$$\mathcal{I}_2 = \frac{\pi}{2}\exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\left\{2 - \frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{2\,\sigma^2}\right\}$$

For three-dimensions,

$$\mathcal{I}_3 = \frac{\pi^{3/2}}{2}\exp\left(-\frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{4\sigma^2}\right)\left\{3\,\sigma - \frac{(\boldsymbol{a}_1-\boldsymbol{a}_2)^2}{2\,\sigma}\right\}$$

And so on.

2.b. The total probability of transitioning from O to B can be found by summing over paths:

$$\Pr(O\to B) = \frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\left(\frac{1}{4}+\frac{1}{4}\times 1\right) = \frac{7}{24}$$

c. It is impossible to transition from O to B in a single step; there are no such paths.

Three of the paths allow a transition from O to B, with probability

$$\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{1}{4} = \frac{11}{48}$$

One of the paths results in transition from O to B, with probability $1/16$.

There are no paths that result in transition from O to B in three or more steps.

d. The transition matrix is given by

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The square of the transition matrix expresses the number of ways that one can transition from O to B in two steps.

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

g. The matrix $(I-M)^{-1} = I + M + M^2 + \cdots$ and gives the total number of ways to transition between two states on the off-diagonal elements. In this case, this is equal to

$$(I-M)^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

h. The generating function for the transition from O to B is given by

$$f(t) = 3\,t^2 + t^3$$

3. We repeat the above problem with the probability matrix:

$$P = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The probability of transitioning from one state to another in two steps is given by the matrix P^2.

$$P^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{16} & \frac{11}{48} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Again the total probability of transitioning between two states, in any number of steps is given by the off-diagonal elements of the matrix $(I-P)^{-1}$.

$$(I-P)^{-1}=\begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{16} & \frac{7}{24} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

And the generating function for the transition between from state O to B is given by

$$f(t)=\frac{11}{48}t^2+\frac{1}{16}t^3$$

4. Adding the loop changes the matrix in two places, but drastically changes the solution to the problem.

$$P=\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Finding the matrix $(I-t\,P)^{-1}$, reveals that the generating function for the transition from O to B is

$$f(t)=\frac{11\,t^2+t^3}{48-t^3}=\frac{\frac{11}{48}t^2+\frac{1}{48}t^3}{1-\frac{1}{48}t^3}$$

$$f(t)=\frac{11}{48}t^2+\frac{1}{48}t^3+\frac{1}{48}t^3\left(\frac{11}{48}t^2+\frac{1}{48}t^3\right)+\frac{1}{2304}t^6\left(\frac{11}{48}t^2+\frac{1}{48}t^3\right)+\cdots$$

The probability of passing through the loop is $\frac{1}{48}$ and requires three steps to complete. Once the process returns to O, the process repeats itself the same as before. Therefore, each completion of the loop introduces a factor of $\frac{1}{48}t^3$.

5. The contour plot appears below.