Mathematical Physics II

PHYS 502

Problem Set # 4
Distributed January 22, 2016
Due January 29, 2016

1 (again). $f_1(\mathbf{x}) = e^{-(\mathbf{x}-\mathbf{a_1})^2/2\sigma^2}, f_2(\mathbf{x}) = e^{-(\mathbf{x}-\mathbf{a_2})^2/2\sigma^2}$. Evaluate

$$\int \nabla f_1(\mathbf{x}) \cdot \nabla f_2(\mathbf{x}) d\mathbf{x} \ .$$

2. Network Transitions: Look at Fig. 16.5 pg. 624 in Hobson and Riley. Use the transition probabilities as given.

a. One of the important nodes is unmarked. Identify it and label it A_5.

b. Compute the probability of the transition $O \to B$ in: 1 step; 2 steps; 3 steps; 4 or more steps. Use Feynmanesque logic.

c. What is the probability of the transition $O \to B$.

d. Represent the transitions in this network by a *Markov* transiton matrix. This matrix is "binary": it has only 0 and 1 as matrix elements: 0 if a one-step transition does not occur and 1 if it does occur with nonzero probability. Call this matrix M. It should be a 7×7 matrix with rows and columns 1-1 with the nodes $O, A_1 - A_5, B$. The matrix element $M_{ij} = 0$ if $i \to j$ does not occur in one step; $M_{ij} = 1$ if $i \to j$ can occur in one step.

e. Show that the transition $i \to j$ can occur in 2 steps if $\sum_k M_{ik} M_{kj} = n > 0$. Convince me that n is the number of ways you can get from i to j in two steps.

f. Generalize.

g. Show that the number of ways of getting from i to j in any number of steps is $M_{ij} + {M^2}_{ij} + {M^3}_{ij} + {M^4}_{ij} + \cdots = \left(\frac{I}{I-M}\right)_{ij}$. Compute this matrix for this example.

h. It is easier to keep count of the number of ways to get from i to j by introducing a *generating function*, for example $tM_{ij} + t^2{M^2}_{ij} + t^3{M^3}_{ij} + t^4{M^4}_{ij} + \cdots = \frac{I}{I-tM}$. Then in the Taylor expansion of the rhs the power of t in any matrix element indicates the number of steps and its coefficient is the number of ways to get from i to j. Compute the matrix element of the generating function for the transition $O \to B$.

i. Would Feynman agree with you?

3. Network Probabilities: Repeat the problem above, but replace the Markov Transition matrix M by the Markov Probability matrix P. This is also a 7×7 matrix describing one-step transitions, but now P_{ij} is the probability of the transition from i to j in one step. Keep in mind that you will need to interpret statements like "..indicates the number of steps and its coefficient is the number of ways..." as " ..indicates the number of steps and its coefficient is the probability...". Where necessary, make these new interpretations explicit.

4. Loops: Modify the network by connection node A_5 to O and assuming $P_{5O} = 1/3$ and $P_{5B} = 1/3$. The matrix element $\left(\frac{I}{I_7 - tP}\right)_{OB}$ is now a fraction involving the variable t. Find this fraction. Do a Taylor series expansion of this fraction. Interpret your results. Does Feynman smile on you?

5. Quadratic Functions: Make a contour plot of the potential

$$V(x,y) = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$