Mathematical Physics II

PHYS 502

Solutions to Problem Set # 3

1a. Estimate the scaling exponent for the biological data shown in Fig. 1.

1b. Estimate the scaling exponent for the astrophysical data shown in Fig. 2.

Solution: For the first figure (Galileo scaling), for both curves the "rise over run" for the exponents is 2/6, so the scaling exponent is $\frac{1}{3}$.

For the second figure the curve has a "knee" or a break at about $1E+07$. For the dwarfs, $rise/run \simeq \frac{1}{2}/3$ so the scaling exponent is about $\frac{1}{6}$. Above the knee we have points 10^3 at about $1E+13$ and 10^1 at about $1E+06$, so the scaling exponent is about $\frac{2}{7}$.

2. $f_1(\mathbf{x}) = e^{-(\mathbf{x}-\mathbf{a_1})^2/2\sigma^2}$, $f_2(\mathbf{x}) = e^{-(\mathbf{x}-\mathbf{a_2})^2/2\sigma^2}$. Evaluate

$$\int f_1(\mathbf{x})V(\mathbf{x})f_2(\mathbf{x})d\mathbf{x}$$

and approximate the result in terms of the potential and its first and second derivative evaluated as some cleverly chosen point.

Solution: From class notes: $\int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ and $\int_{-\infty}^{+\infty} x^2 e^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$. From previous homework, $f_1 f_2 = e^{-(\mathbf{a_1}-\mathbf{a_2})^2/\sigma^2} e^{-(\mathbf{x}-\mathbf{a}_{\text{av}})^2/\sigma^2}$. This is a gaussian centered at $\mathbf{a}_{\text{av}} = (\mathbf{a_1}+\mathbf{a_2})/\mathbf{2}$ with a prefactor that decreases exponentially with distance between $\mathbf{a_1}$ and $\mathbf{a_2}$. Taylor expand the potential V around $\mathbf{a}_{\text{av}}$ to second order: $V(\mathbf{x}) = V(\mathbf{a}_{\text{av}}) + \Delta x^i V_i + \frac{1}{2}\Delta x^i \Delta x^j V_{ij}(\mathbf{a}_{\text{av}}) + h.o.t.$. The integral of the first (zeroth order) term is $e^{-(\mathbf{a_1}-\mathbf{a_2})^2/\sigma^2} V(\mathbf{a}_{\text{av}})\pi\sigma^2$. The integral of the second term is zero, "by symmetry". The integral of the third term is $e^{-(\mathbf{a_1}-\mathbf{a_2})^2/\sigma^2} V_{ij}(\mathbf{a}_{\text{av}})\delta^{ij}\sqrt{\pi\sigma^2} \times \frac{1}{2}\sigma^2\sqrt{\pi\sigma^2}$. The last term can be cleaned up, as $\delta^{ij}V_{ij} = V_{xx}+V_{yy} = \nabla^2 V$. The result is $e^{-(\mathbf{a_1}-\mathbf{a_2})^2/\sigma^2}\pi\sigma^2\left(V(\mathbf{a}_{\text{av}}) + \frac{1}{2}\sigma^2\nabla^2 V(\mathbf{a}_{\text{av}})\right)$.

3. Convince me that you can approximate the first and second derivatives of a function on a line, $f(x)$, by evaluating the function at discrete points along the line spaced a distance Δ apart, and that

$$\frac{df}{dx}|_{x_i} \simeq \frac{f(x_i+\Delta)-f(x_i-\Delta)}{2\Delta}$$

$$\frac{d^2f}{dx}|_{x_i} \simeq \frac{f(x_i+\Delta)-2f(x_i)+f(x_i-\Delta)}{\Delta^2}$$

For other numerical approximations of derivatives, including ∇ and ∇^2, see the pictures in Chapter 25 of Abramowitz and Stegun.

Solution: The calculus definitions of first and second derivates are

$$\frac{df}{dx}|_x = \lim_{\delta\to 0}\frac{f(x+\delta)-f(x-\delta)}{2\delta} \qquad \frac{d^2f}{dx^2}|_x = \lim_{\delta\to 0}\frac{f(x+\delta)-2f(x)+f(x-\delta)}{\delta^2}$$

As Physicists we drive δ to small nonzero values and 'hope for the best'. Even better is to evaluate these ratios for several values of δ, fit a curve through these values, and evaluate the fit at $\delta = 0$.

4. Can this integral be evaluated analytically? If so, what is it?

$$f = 2z\ e^{z^2}\ \log z + \frac{e^{z^2}}{z} + \frac{\log z - 2}{\left[(\log z)^2+z\right]^2} + \frac{(2/z)\log z + 1/z + 1}{(\log z)^2+z}$$

Solution:

$$e^{z^2}\ \log z - \frac{\log z}{(\log z)^2+z} + \log\left[(\log z)^2+z\right]$$

Can this integral be evaluated in closed form:

Solution: Maple (or other) echoes out the input. This is the signal that the integral cannot be evaluated in closed form.

$$f = 2z\ e^{z^2}\ \log z + \frac{e^{z^2}}{z} + \frac{\log z + 2}{\left[(\log z)^2+z\right]^2} + \frac{(2/z)\log z + 1/z + 1}{(\log z)^2+z}$$

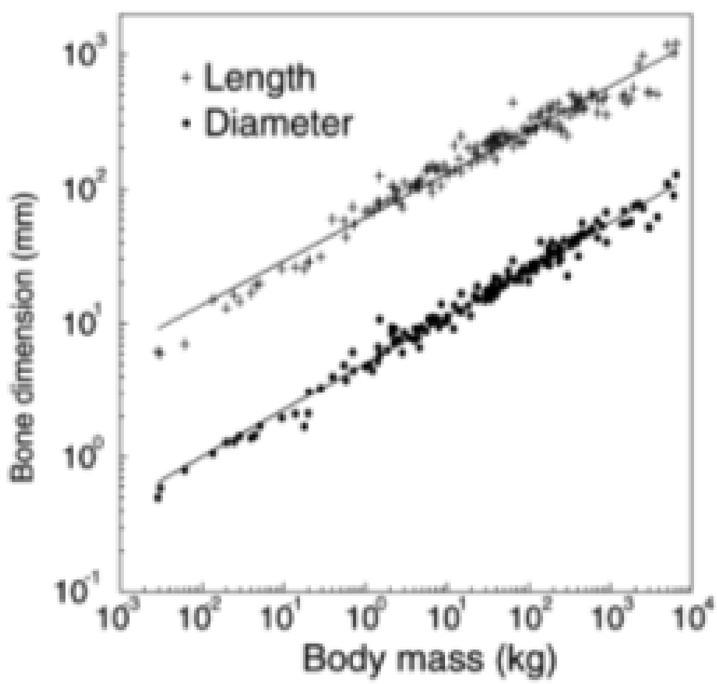
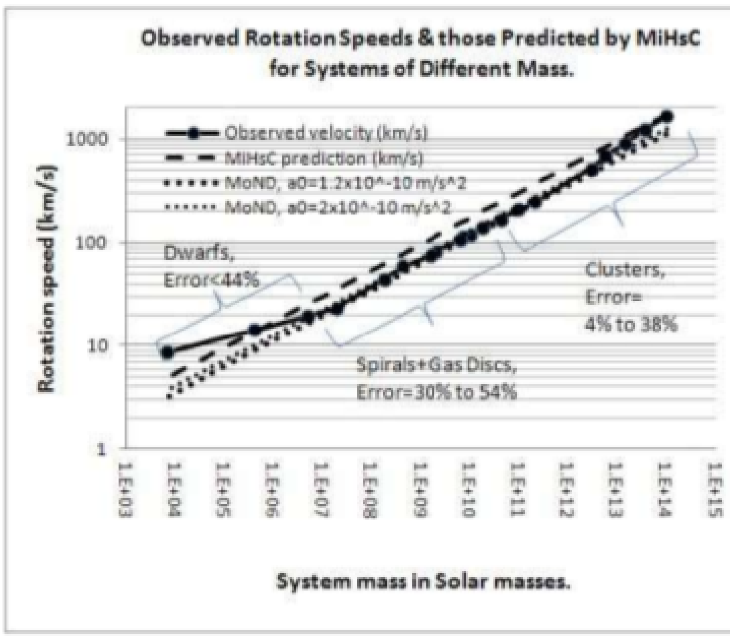


Figure 1: (Two data sets that were used to test Galilean scaling in biological creatures.



A comparison of the observed rotation speeds in km/s (black dots) with the predictions of MoND (dotted) and MiHsC (dashed) for galaxies and galaxy clusters of increasing baryonic mass (in Solar masses). Credit: M.E. McCulloch

Figure 2: (Several data sets used to discern information about "dark matter".