Mathematical Physics II

PHYS 502

Problem Set # 2
Distributed January 8, 2016
Due January 15, 2016

In the first two problems you will learn (hopefully) how to use numerical computations to suggest analytic results to you.

1. Fixed BC: Tridiagonal matrices of the form $K = 2\delta_{ij} - \delta_{i,i-1} - \delta_{i,i+1}$ always occur in problems involving coupled harmonic oscillators in one dimension with fixed boundary conditions, as well as lots of other places. In the 5×5 case they have the form shown on the left in the equation below:

$$K_{\text{fixed}} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad K_{\text{free}} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

a. Choose a convenient value for n (the size of the matrix you will have to diagonalize).

b. Create a matrix with the structure shown on the left above.

c. Diagonalize this matrix.

d. Sort the output from "bottom to top", that is, from smallest to largest eigenvalue.

e. Plot the lowest couple of eigenvectors (if you don't understand what that means, ask in class).

f. Do the plots suggest that the components of the eigenvectors have some analytic form in terms of functions you are aware of? (The answer is yes.)

g. Guess this form.

h. Test this guess by doing some simple paper-and-pencil calculations.

i. When you get this right you will find an analytic expression for the eigenvalues also.

j. Plot you analytic expression for the eigenvalues and compare with a plot of the numerically computed eigenvalues.

k. Announce EURIKA! when everything works as suggested.

2. Free BC: For free boundary conditions (*e.g.*, an organ pipe with open ends) the matrices that arise have +1 instead of +2 at the corners (top left and bottom right). In the 5×5 case they have the form shown on the right in the equation above.

a. Choose a convenient value for n (the size of the matrix you will have to diagonalize).

b. Create a matrix with the structure shown on the right above.

c. Diagonalize this matrix.

d. Sort the output from "bottom to top".

e. Plot the lowest couple of eigenvectors.

f. Do the plots suggest that the components of the eigenvectors have some analytic form in terms of functions you are aware of? (The answer is still yes.)

g. Guess this form.

h. Test this guess by doing some simple paper-and-pencil calculations.

i. When you get this right you will find an analytic expression for the eigenvalues also.

j. Plot you analytic expression for the eigenvalues and compare with a plot of the numerically computed eigenvalues.

k. Announce EURIKA! when everything works as suggested.

3. Feynman's Favorite Slick Trick: is "completing the square". Here are two gaussian functions: $f_1 = exp\left[-(x-a_1)^2 - (y-b_1)^2\right]$ and $f_2 = exp\left[-(x-a_2)^2 - (y-b_2)^2\right]$. Write $f_1 \times f_2$ as a gaussian f_3 multiplied by a prefactor, where f_3 has the same form as f_1 and f_2.