Mathematical Physics II

PHYS 502

Solutions to Midterm Exam, February 12, 2016

1. Scaling: Tests of the number of iterations that the Page Rank algorithm undergoes before convergence gave the first two lines below.

Test	n=# links	# Iterations
1.	161×10^6	45
2.	322×10^6	52
3.	10^9	

The first two lines suggest that the number of iterations obeys a logarithmic scaling law depending on the number of links. How many iterations do you expect to have to make to converge for a network with a billion links?

Solution: The number of operatins, N_{op}, grows no more quickly than the log of the number of links, N_l. We adopt a growth model of the form $N_{op} \simeq a \log N_l + b \log(\log(N_l))$ or, more simply $N_{op} \simeq a \log N_l + b$. The first two lines can be written

$$\begin{array}{rcl} a\log(161 \times 10^6) + b & = & 45 \\ a\log(2 \times 161 \times 10^6) + b & = & 52 \end{array} \Rightarrow a = \frac{7}{\log 2}$$

Then for $N_l = 10^9$, $N_{op} \simeq a \log\left(\frac{1000}{161} \times 161 \times 10^6\right) + b = 45 + a\log\left(\frac{1000}{161}\right) = 45 + 7\frac{\log(1000/161)}{\log(2)} \simeq 64$.

The scaling form $\log + \log(\log)$ yields the same estimate.

2. Masses: It has been discovered that three bodies of masses m_1, m_2, m_3 undergo a three-body interaction. Construct three invariant functions of the three masses with the property that each function has the dimensions of mass. Choose the functions $f_i(m_1, m_2, m_3)$ with the property that

$$\begin{array}{ccc} f_1 & \xrightarrow{L} & m_1 \\ f_2 & \xrightarrow{L} & m_2 \\ f_3 & \xrightarrow{L} & m_3 \end{array}$$

Here L is the limit $m_1 \gg m_2 \gg m_3$.

Solution: Newton's method of constructing symmetric functions by computing the coefficients of the monomials x^k in algebraic expressions of the form $(x-m_1)(x-m_2)(x-m_3)$ yields three symmetric functions $g_1 = m_1+m_2+m_3$, $g_2 = m_1m_2+m_2m_3+m_3m_1$, $g_3 = m_1m_2m_3$ with dimensions $[g_i] = M^i$. Invariant functions with the dimensions of mass are ratios $f_i = g_i/g_{i-1}$, with $g_0 = 1$. These ratios, and their limits, are

$$
\begin{array}{lllllll}
f_1 & = & g_1/1 & = & m_1+m_2+m_3 & \to & m_1 \\
f_2 & = & g_2/g_1 & = & \dfrac{m_1m_2+m_2m_3+m_3m_1}{m_1+m_2+m_3} & \to & m_2 \\
f_3 & = & g_3/g_2 & = & \dfrac{m_1m_2m_3}{m_1m_2+m_2m_3+m_3m_1} & \to & m_3
\end{array}
$$

3. Experimental: A point source of light (*e.g.*, a star) is processed by an optical device and its image on a focal plane is a gaussian distribution $psf(\mathbf{x}) = \frac{\sqrt{det|g_{ij}|}}{(2\pi)^{2/2}} exp\left(-\frac{1}{2}g_{ij}x^ix^j\right)$. Here psf stands for 'point spread function'. The point spread function has a covariance matrix given by $\langle x^ix^j\rangle = g^{ij}$. An extended source with unknown intensity distribution $S(\mathbf{x})$ is imaged by this device. The image is also a gaussian distribution with covariance matrix $\langle x^ix^j\rangle = I^{ij}$ and total integrated intensity I_0.

a. Show that the image is a convolution of the source with the point spread function of the optical device.

b. Compute the intensity distribution of the source.

c. What is the covariance matrix of the source?

Solution: The intensity of the image at the point $\mathbf{x}$ on the image screen is the intensity of the source at the point $\mathbf{y}$ at the input of the imaging device, multiplied by the response of the point spread function in the transfer from $\mathbf{y}$ to $\mathbf{x}$: $I(\mathbf{x}) = \int S(\mathbf{y}) \times psf(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$, or $I = S * psf$ (a convolution).

The convolution-transformation theorem tells us that the fourier transform of a convolution is the product of the Fourier transforms of the two functions in the convolution: $I(\mathbf{k}) = S(\mathbf{k})psf(\mathbf{k})$. Thus $S(\mathbf{k}) = I(\mathbf{k})/psf(\mathbf{k})$. Since both $I(\mathbf{x})$ and $psf(\mathbf{x})$ are gaussians, both $I(\mathbf{k})$ and $psf(\mathbf{k})$ are gaussians and their quotient is a gaussian and the covariance matrix of the quotient is the difference of the covariance matrices of I and psf:

$$I^{ij} = g^{ij} + S^{ij} \to S^{ij} = I^{ij} - g^{ij}$$

The intensity distribution of the source is $I_0 \frac{\sqrt{det|S_{ij}|}}{(2\pi)^{\frac{2}{2}}} exp^{-\frac{1}{2}S_{ij}x^ix^j}$.

4. Quantum Gravity: It is all the rage these days to construct a relativistically (c) correct quantum ($\hbar$) theory of gravity (G).

a. Construct an expression for the characteristic size in such a theory.

b. Use the measured values of $c = 3\times 10^8$ m/sec. $\hbar = 1\times 10^{-34}$kg.m^2/sec. and $G = 7\times 10^{-11}$m^3/kg. sec.^{2}, precise to about 0 decimal points, to estimate this size.

Solution: $[c] = LT^{-1}, [\hbar] = ML^2T^{-1}, [G] = M^{-1}L^3T^{-2}$. Anything independent of the mass we must involve $G\hbar$ whose dimensions are $[G\hbar] = L^5T^{-3}$. Divide by c^3 to get rid of the dimension and find $\left[G\hbar/c^3\right] = L^2$. Then

$$a_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^3}} \to \sqrt{7\times 10^{-11}\frac{m^3}{kg.sec.^2}10^{-34}\frac{kg.m^2}{sec.}/(27*10^{24}\frac{m^3}{sec^3})} \simeq 1.5{\times}10^{-35}m$$