We construct systems of three autonomous first order differential equations with bounded two-dimensional attracting sets $M$. The flows on $M$ are chaotic and have one dimensional Poincaré sections whose Poincaré maps are diffeomorphic to maps of the interval. The attractors are two dimensional rather than fractal, and when “unzipped” they are topologically equivalent to the templates of suspended horseshoes.

**INTRODUCTION**

The analysis of the dynamics of chaotic systems can be broken roughly into two parts – the quantitative, which emphasizes measures such as Lyapunov exponents, fractal dimensions and power spectra [1], and the topological, where qualitative characterizations such as templates and bounding tori are considered [2]. When we think of the infinite set of unstable periodic orbits in a strange attractor as topological knots, the topology of the attractor can be coded in the set of links the dynamics allows. The primary tool is a template, a construction due to Birman and Williams [3] [4], a branched two-manifold that results when the attractor is collapsed along its stable manifolds. The template supports a semi-flow which contains the same periodic orbits as the original system, with at most two additional extraneous orbits.

Extracting a template from a strange attractor is a relatively straightforward, but inexact process. In the simplest cases, one can guess the form of the template by making a visual inspection of the attractor and its Poincaré sections to try to determine how the stretching and folding is taking place, see Fig. (1). Once a template structure is proposed, the knots and links that are supported by that structure can be determined. One then attempts to verify that the periodic orbits corresponding to these topological knots and links exist in the dynamics of the original system. This verification process is the source of the inexactness: one never knows if one has checked a sufficiently large set of orbits. For more complicated systems, one can extract a set of periodic orbits by the method of close returns or by Pyragas [5] or similar chaos control methods [6], and determine a set of topological invariants such as linking numbers and relative rotation rates for these orbits. One can then construct a template based on symbolic dynamics given to the orbits [7].

Because template construction can sometimes be a difficult iterative process, it is an interesting exercise to construct an explicit chaotic dynamical system whose attractor is one-dimensional. Such an attractor cannot be called “strange” because it does not have a fractal structure. The two-dimensional maps we suspend are diffeomorphic to one-dimensional maps. Because the exact Poincaré map dynamics are known, such continuous chaotic systems might be useful test beds for understanding the behavior of map based and continuous chaos control methods, and other areas where the relation of a Poincaré map to its flow might be useful. Also, these attractors are the infinite dissipation limiting case of families of strange attractors.

**SUSPENDING MAPS TO FLOWS**

The first detailed exposition of the conditions that must hold for an explicit suspension of a map to a flow were given by Channell [8]. Mayer-Kress and Haken [9] suspended a Hénon map to a cylinder $\mathbb{R}^2 \times I$, but the resulting differential equations could not be integrated forward normally: the integrator must reset $t$ whenever $t = 1$ and load in the final conditions at $t = 1$ as the new initial conditions each time. Recently Nicholas [10] described a method for making an exact autonomous suspension of a map in an open half plane to a flow.
The construction embeds a flow on a quotient space \( Q = \mathbb{R}^2 \times [0,1]/\sim \), where 0 and 1 are identified in [0,1], in \( \mathbb{R}^3 \) by wrapping it around the z-axis. More exactly, the flow \( \phi \) on \( Q = \mathbb{R}^2 \times S^1 \) is embedded in \( \mathbb{R}^3 \) by truncating \( \mathbb{R}^2 \) to \((-a,\infty) \times \mathbb{R} \) and constructing \( \mathbb{R}^3 \) as \((0,0,z) \cup \mathcal{F}((-a,\infty) \times \mathbb{R})\), where \( \mathcal{F}((-a,\infty) \times \mathbb{R}) \) is a book-leaf foliation of \( \mathbb{R}^3/(0,0,z) \) rotationally symmetric about the z axis. The method in [10] was demonstrated on two-dimensional maps with strange attractors, a Hénon and a Duffing map. While this method only suspends the dynamics in the right half plane, it produces a system of differential equations that can be integrated in a normal fashion. We use this method to suspend two-dimensional maps with one-dimensional attracting sets, resulting in attractors with no fractal structure, but with verifiable chaotic dynamics.

Consider the difference equations

\[
\begin{align*}
x_{n+1} &= \frac{4}{3} - \frac{5}{4} x_n^3 + \alpha z_n \quad z_{n+1} = -x_n \quad (1) \\
x_{n+1} &= 4 x_n (1 - x_n) + \alpha z_n \quad z_{n+1} = -x_n \quad (2) \\
x_{n+1} &= -13/5 x_n + x_n^3 + \alpha z_n \quad z_{n+1} = -x_n \quad (3)
\end{align*}
\]

where the first and last are analogs of the Hénon and Duffing maps, and the middle a two-dimensional version of the logistic map. For a range of parameter \( \alpha \), these systems are chaotic with fractal attractors. For \( \alpha = 0 \), these maps have one-dimensional attractors in the x, z plane because the points of the iteration are of the form \((x_{n+1},-x_n)\), a clockwise rotation of the functional form of the map \((x_n,z_{n+1})\) by \(\pi/2\). Thus, the exact functional form of the attracting sets are known, and the projection of the dynamics onto the x axis give the dynamics of the original one-dimensional map.

**THE EXPLICIT SUSPENSION**

Let \( x_{n+1} = F_1, z_{n+1} = F_2 = -x_n \) be the components of our two-dimensional mappings of the plane. Then the action of the maps can be broken down into two geometrically distinct steps. First the \((x,z)\) plane is deformed by

\[
\begin{bmatrix}
x_{n+1} \\
z_{n+1}
\end{bmatrix} =
\begin{bmatrix}
-F_2 \\
F_1
\end{bmatrix},
\]

then the result is rotated by \(\pi/2\) clockwise

\[
\begin{bmatrix}
x_{n+1} \\
z_{n+1}
\end{bmatrix} =
\begin{bmatrix}
\cos(\pi/2) & \sin(\pi/2) \\
-\sin(\pi/2) & \cos(\pi/2)
\end{bmatrix}
\begin{bmatrix}
-F_2 \\
F_1
\end{bmatrix} =
\begin{bmatrix}
F_1 \\
F_2
\end{bmatrix}.
\]

We suspend the maps to flows by continuous interpolations between initial and final conditions. The interpolation is performed using the basic interpolating functions \( C = \cos(\pi/2 t) \) and \( S = \sin(\pi/2 t) \), which also appear in a rotation matrix \( R = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \). As interpolating diffeomorphisms, these are

\[
\begin{align*}
C^2 &= \cos^2\left(\frac{\pi}{2} t\right) \\
S^2 &= \sin^2\left(\frac{\pi}{2} t\right)
\end{align*}
\]

taking initial to final conditions by \( C^2 x_0 + S^2 x_1 \) as \( t \) varies from 0 to 1. Because of the scaling of the argument, a quarter clockwise rotation and a smooth interpolation between initial and final conditions is accomplished as \( t \) goes from 0 to 1, and this suspends the map to a cylinder \( \mathbb{R}^2 \times I \).

In anticipation of our embedding in \( \mathbb{R}^3 \), we write the suspension to a cylinder \( \mathbb{R}^2 \times \mathbb{I} \) as

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
C & 0 & S \\
0 & 1 & 0 \\
-S & 0 & C
\end{bmatrix}
\begin{bmatrix}
C^2 x_0 - S^2 F_2 \\
0 \\
S^2 x_0 + C^2 F_1
\end{bmatrix} =
\begin{bmatrix}
-C x_0 + S C^2 z_0 + S^3 F_1 \\
0 \\
-S x_0 + C^2 z_0 + C S^2 F_1
\end{bmatrix},
\]

where we have used the fact that \( F_2 = -x \).

In order to embed this in \( \mathbb{R}^3 \), we displace the function in the x direction by an amount (in this case 1) sufficient to include the attractor in the left half plane and to avoid self-intersection, and rotate counterclockwise about the z axis:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
\cos(2\pi t) & -\sin(2\pi t) & 0 \\
\sin(2\pi t) & \cos(2\pi t) & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 + C x_0 + S C^2 z_0 + S^3 F_1 \\
0 \\
-S x_0 + C^2 z_0 + C S^2 F_1
\end{bmatrix} =
\begin{bmatrix}
\cos(2\pi t)(1 + C x_0 + S C^2 z_0 + S^3 F_1) \\
\sin(2\pi t)(1 + C x_0 + S C^2 z_0 + S^3 F_1) \\
-S x_0 + C^2 z_0 + C S^2 F_1
\end{bmatrix}.
\]

To form a differential equation we solve (8) for the initial conditions in terms of the variables \( x \) and \( z \), take
the derivative of (9) with respect to the interpolating parameter \( t \), and substitute the expressions for the initial conditions into the differential equation.

At this point we have non-autonomous differential equations. To make them autonomous, we use the half angle formulas twice to write all interpolation and rotation operators in a form so that their period is \( 2\pi t \) for \( t \in [0,1] \). Thus, let

\[
C = \begin{cases} 
\frac{1}{2} + \frac{1}{4}\sqrt{2+2\cos(2\pi t)} & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{2} - \frac{1}{4}\sqrt{2+2\cos(2\pi t)} & \frac{1}{2} \leq t \leq 1
\end{cases}
\] (10)

\[
S = \begin{cases} 
\frac{1}{2} - \frac{1}{4}\sqrt{2+2\cos(2\pi t)} & 0 \leq t \leq \frac{1}{2} \\
\frac{1}{2} + \frac{1}{4}\sqrt{2+2\cos(2\pi t)} & \frac{1}{2} \leq t \leq 1
\end{cases}
\] (11)

Then, using the identity \( \cos(2\pi t) = \frac{x}{\sqrt{x^2+y^2}} \), and the fact that \( 0 \leq t \leq 1/2 \Rightarrow y \geq 0 \), we obtain

\[
C = \begin{cases} 
\frac{1}{2} + \frac{1}{4}\sqrt{2+\frac{2y}{\sqrt{x^2+y^2}}} & y \geq 0 \\
\frac{1}{2} - \frac{1}{4}\sqrt{2+\frac{2y}{\sqrt{x^2+y^2}}} & y < 0
\end{cases}
\] (12)

\[
S = \begin{cases} 
\frac{1}{2} - \frac{1}{4}\sqrt{2+\frac{2y}{\sqrt{x^2+y^2}}} & y \geq 0 \\
\frac{1}{2} + \frac{1}{4}\sqrt{2+\frac{2y}{\sqrt{x^2+y^2}}} & y < 0
\end{cases}
\] (13)

Substitution of these into the differential equation results in autonomous differential equations that can be numerically integrated in a normal fashion: one does not have to reset \( t \) to zero and substitute the final conditions in as new initial conditions to integrate the solution forward past \( t = 1 \) as in the usual case of a suspension to a cylinder\[9\].

The suspension of these one-dimensional attracting sets results in a two-dimensional attracting set, which is the unstable manifold of the system. This manifold, though, is a branched two manifold because the stretching and folding of the original map attractor is two to one. The branches join smoothly at the \( y = 0 \) plane in a one-dimensional set. At the critical point of the map there is a non-differentiable fold. To form a template from this stable manifold, we “unzip” it by flowing the critical point(s) at the Poincaré section backward in time to the previous Poincaré section and removing this piece of orbit. What remains is a branched two-manifold, a template in the usual sense of the word, except for the open boundaries where the orbit of the critical point(s) were removed. If we close this set, we have a specific parameterized template. In the case of the logistic suspension, the template is (topologically) exactly the template of the suspended Smale horseshoe.

**BIFURCATIONS**

The period doubling bifurcation sequence of the suspension of the logistic map is shown in figure (6). Because we know the exact bifurcation sequence of the logistic map, we know the related bifurcation sequence of its suspension. We can use this knowledge to determine the explicit expressions for the parameterized surfaces comprising the template through its bifurcation sequence.
FIG. 6: From top to bottom, left to right, periods 1, 2, 4, 8, chaos and period 3.

The basins of attraction of these maps are especially simple, as are the basins of the suspended attractors. As an example, consider the Poincaré section of the logistic suspension at $y = 0$. For the original map $x_{n+1} = 4x_n(1 - x_n), z_{n+1} = -x_n$ under iteration, points with $x$ coordinate in $[0, 1]$ stay in that interval for all time, while points outside it go to $-\infty$. The basin of the map in the $z$ direction is unbounded, as no matter what the value of $z$ in the point $(x, 0, z)$, it will have the value $-x$ after one iterate. Thus, all points in the vertical strip $x \in [0, 1], y = 0, z$ land on the attractor after one fundamental time unit. For the suspension, however, the interpolation includes a rotation and is valid only for the right half plane, so there exist points with $z < -1$ in the basin of attraction of the map that map into the $z$ axis over the course of a cycle, and therefore are not in the basin of attraction of the suspended system.

STABLE MANIFOLDS

Because the differential equations for these systems are derived from explicit expressions for the solutions, the formulas for the stable manifolds can be expressed explicitly as parameterized surfaces in parameters $s$ and $t$, with $s$ parameterizing the $z$ coordinate and $t$ the interpolation and rotation parameter as usual. Thus, for any initial $x_0 \in [0, 1], 0 \leq t \leq 1$ and appropriate range of $s$ (avoiding $z$ values out of the basin of attraction),

$$
\begin{align*}
\cos(2\pi t)(1 + x_0 \cos \frac{\pi}{2} t + s \sin \frac{\pi}{2} t \cos^2 \frac{\pi}{2} t + \sin^3 \frac{\pi}{2} t F_1) \\
\sin(2\pi t)(1 + x_0 \cos \frac{\pi}{2} t + s \sin \frac{\pi}{2} t \cos^2 \frac{\pi}{2} t + \sin^3 \frac{\pi}{2} t F_1) \\
- x_0 \sin \frac{\pi}{2} t + s \cos^3 \frac{\pi}{2} t + \cos \frac{\pi}{2} t \sin^2 \frac{\pi}{2} t F_1
\end{align*}
$$
gives a portion of the stable manifold that collapses, at $t = 1$, onto the orbit with initial condition $(F_1(x_0), 0, -x_0)$.

FIG. 7: A cut through the attractor showing the portions of the stable manifolds of three orbits of the logistic suspension.

SUMMARY

We have constructed systems of autonomous differential equations for continuous chaotic systems in $\mathbb{R}^3$ with non-fractal attractors that are, up to an unzipping, topologically equivalent to their template representations. The stable and unstable manifolds can be explicitly given as parameterized surfaces, and the solution curves can be explicitly given as parameterized curves.