16

Lie Groups and Differential Equations

Contents

Lie group theory was initially developed to facilitate the solution of differential equations. In this guise its many powerful tools and results are not extensively known in the physics community. This Chapter is designed as an antidote to this anemia. Lie's methods are an extension of Galois' methods for algebraic equations to the study of differential equations. The extension is in the spirit of Galois' work: the technical details are not similar. The principle observation — Lie's great insight — is that the simple constant that can by added to any indefinite integral of $dy/dx = g(x)$ is in fact an element of a continuous symmetry group — the group that maps solutions of the differential equation into other solutions. This observation was used — exploited — by Lie to develop an algorithm for determining when a differential equation had an invariance group. If such a group exists, then a first order ODE can be integrated by quadratures, or the order of a higher order ODE can be reduced.

Galois inspired Lie. If the discrete invariance group of an algebraic equation could be exploited to generate algorithms to solve the algebraic equation "by radicals," might it be possible that the continuous invariance group of a differential equation could be exploited to solve the differential equation "by quadratures?" Lie showed emphatically in 1874 that the answer is YES!, and work has hardly slowed down in the field that he pioneered from that time to the present.

But what is the group that leaves the solutions of a differential equation invariant — or maps solutions into solutions? It turns out to be none other than the trivial constant that can be added to any indefinite integral. The additive constant is an element in a translation group.

We outline Lie's methods for first order ordinary differential equations. First, we study the simplest first order equation in one independent variable x and one dependent variable y: $\frac{dy}{dx} = g(x)$. This is treated in Sect. 16.1. In that Section we set up the general formulation in terms of a constraint equation $dy/dx = p$ and a surface equation $F(x, y, p) = 0$. The special forms of the surface and constraint equation are exploited to write down the solution by quadratures.

Lie's methods are presented in Sect. 16.2 in a number of simple, easy to digest steps. Taken altogether, these provide an algorithm for determining whether an ODE possesses a symmetry and, if so, what that symmetry is. Transformation to a set of canonical variables R, S, T is algorithmic. The canonical variable $R(x, y)$ is the new dependent variable (like x), $S(x, y)$ is the new dependent variable (like y), and $T(x, y, p)$ is the new constraint between S and R (like dy/dx). In this new coordinate system the surface and constraint equations assume the desired forms $F(R, -, T) = 0$ and $dS/dR = f(R, -, T)$. The system has been reduced to quadratures, and integration follows immediately.

Despite the simplicity of the algorithm, it is not easy to understand these steps without a roadmap. Such is provided in Sect. 16.3, where a simple example is discussed in detail.

Lie's methods extend in many different directions. Several directions are indicated in Sect. 16.4.

16.1 The Simplest Case

The simplest first order ordinary differential equation to deal with has the form

$$\frac{dy}{dx} = g(x) \tag{16.1}$$

Here x is the independent variable and y is the dependent variable. The solution of this equation is (almost) trivially

$$y = G(x) = \int g(x)\ dx \quad (+\text{ additive constant}) = G(x) + c \tag{16.2}$$

If we write the solution in the form $y - G(x) = 0$, then the surface $y + c - G(x) = 0$ is also a solution of the original equation (16.1). There is a one-parameter group of displacements that maps one solution into another. These displacements can be represented by the Taylor series displacement operator $e^{c\frac{\partial}{\partial y}}$, for

$$e^{c\frac{\partial}{\partial y}}[y - G(x) = 0] = y + c - G(x) = 0 \tag{16.3}$$

In short, the "trivial" additive constant is in fact a one-parameter group of translations that maps solutions (16.2) of (16.1) into other solutions of the original simple equation (16.1). This translation group plays the same role for first order ordinary differential equations that the symmetric group S_n plays for nth degree algebraic equations.

For convenience, we express the derivative dy/dx as a coordinate p. The first order differential equation (16.1) can be written in the form $F(x, y, p) = 0$, where $F(x, y, p) = p - g(x)$ for the particular case at hand. There are two relations among the three variables x, y, p. They are given by the surface equation and the constraint equation:

$$\begin{array}{llcll} \text{Surface Equation :} & F(x,y,p) & = & 0 & \\ \text{Constraint Equation :} & p & = & dy/dx & \text{when } F(x,y,p) = 0 \end{array} \tag{16.4}$$

It is useful to express the action of the three partial derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial p$ on the surface $F(x, y, p)$ defining the ordinary differential equation. It is also useful to express the action of the generator of infinitesimal displacements that maps solutions of this equation into other solutions of this equation, on the three coordinates. These two relations are summarized as follows:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial p} \end{bmatrix} [p - g(x)] = \begin{bmatrix} \star \\ 0 \\ \star \end{bmatrix} \qquad \frac{\partial}{\partial y} \begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{16.5}$$

These two equations will be generalized to the determining equation for the infinitesimal generator of the invariance group and the determining equations for the canonical coordinates.

16.2 First Order Equations

In this section we will summarize Lie's approach to the study of differential equations [14, 18, 70]. We do this for equations of first order ($d^n y/dx^n$, $n = 1$) and first degree (depends on $p^m = (dy/dx)^m$, $m = 1$). The results are independent of degree.

If the equation that defines the first order ODE, $F(x, y, p) = 0$, is not of the form $p - g(x)$, so that $\frac{\partial}{\partial y}F(x, y, p) \neq 0$, then we can attempt to find

a. A one-parameter group that leaves $F(x, y, p) = 0$ unchanged.

b. A new "canonical" coordinate system (R, S, T). In this coordinate system $R = R(x, y)$ is the independent variable, $S = S(x, y)$ is the dependent variable, and $T = T(x, y, p)$ is the new constraint variable. In this canonical coordinate system the surface equation $F(x, y, p) = 0$ is not a function of the new dependent variable: $F(R, -, T) = 0$.

In this new coordinate system the source term for the constraint equation is also independent of the dependent variable: $dS/dR = f(R, -, T)$.

16.2.1 One Parameter Group

We search for a one-parameter group of transformations that leaves the surface equation invariant by changing variables in the (x, y) plane according to

$$\begin{array}{ccccll} x & \rightarrow & \bar{x}(\epsilon) & = & x + \epsilon\xi(x, y) + \mathcal{O}(\epsilon^2) & \bar{x}(\epsilon = 0) = x \\ y & \rightarrow & \bar{y}(\epsilon) & = & y + \epsilon\eta(x, y) + \mathcal{O}(\epsilon^2) & \bar{y}(\epsilon = 0) = y \\ p & \rightarrow & \bar{p}(\epsilon) & = & p + \epsilon\zeta(x, y, p) + \mathcal{O}(\epsilon^2) & \bar{p}(\epsilon = 0) = p \end{array} \quad (16.6)$$

In the simplest case Eq. (16.1), this one-parameter group is $x \rightarrow x$ and $y \rightarrow y + \epsilon$, so that $\xi = 0$, $\eta = 1$, and $\zeta = 0$.

16.2.2 First Prolongation

The function $\zeta(x, y, p)$ is not independent of the functions $\xi(x, y)$ and $\eta(x, y)$. The former is related to the latter pair by the **first prolonga-**

tion formula. Specifically,

$$\bar{p} = \frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}/dx}{d\bar{x}/dx} = \frac{p + \epsilon(\eta_x + \eta_y p)}{1 + \epsilon(\xi_x + \xi_y p)} \longrightarrow p + \epsilon\left[\eta_x + (\eta_y - \xi_x)p - \xi_y p^2\right] \tag{16.7}$$

to first order in ϵ, where $\eta_x = \partial\eta/\partial x$, etc. As a result

$$\zeta(x, y, p) = \eta^{(1)}(x, y, y^{(1)}) = \eta_x + (\eta_y - \xi_x)p - \xi_y p^2 \tag{16.8}$$

16.2.3 Determining Equation

The surface equation must be unchanged under the one-parameter group of transformations, so that

$$F(x, y, p) = 0 \to F\left(\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{p}(\epsilon)\right) \overset{\epsilon \text{ small}}{\longrightarrow} F(x + \epsilon\xi, y + \epsilon\eta, p + \epsilon\zeta)$$

$$= F(x, y, p) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p}\right)F(x, y, p) + \text{h.o.t.} \tag{16.9}$$

These are the leading two terms in the Taylor series expansion

$$F\left(\bar{x}(\epsilon), \bar{y}(\epsilon), \bar{p}(\epsilon)\right) = e^{\epsilon X}F(x, y, p) = 0 \tag{16.10}$$

where the generator of infinitesimal displacements for the one parameter group that leaves the surface equation invariant is

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p} \tag{16.11}$$

The first two terms in Eq. (16.9) and (16.10) are

$$F(x, y, p) = 0 \qquad \text{and} \qquad XF(x, y, p) = 0 \tag{16.12}$$

These are called the **determining equations**. The determining equations (16.12) are generalizations of equations (16.5).

Specifically, these equations are used to determine the functions $\xi(x, y)$, $\eta(x, y)$, and $\zeta(x, y, p)$ that define the infinitesimal generator X. These functions are determined by an algorithm based on linear algebra. There are recent versions depending on sophisticated methods of algebraic topology. These methods are elegant improvements of a conceptually simple brute-strength procedure that we summarize briefly. The surface equation $F(x, y, p) = 0$ is solved for p as a function of x and y: $p = p(x, y)$. This expression is substituted into the determining equation $XF(x, y, p(x, y)) = 0$, so that this equation depends only on two independent variables x and y. The generators of the infinitesimal displacements, $\xi(x,y)$ and $\eta(x,y)$, are represented by Laurent expansions, or Taylor series expansions if convergent solutions are sought:

$$\xi(x,y) = \sum_{i,j} \xi_{ij} x^i y^j \qquad 0 \le i,j, \quad i+j \le d_\xi \tag{16.13}$$

and similarly for η. These representations are truncated at finite degrees d_ξ, d_η. The determining equation $XF = 0$ is expanded into the form $\sum C_{ij} x^i y^j = 0$. Each coefficient C_{ij} must separately vanish, by standard linear independence arguments. This gives a set of simultaneous *linear* equations in the expansion amplitudes ξ_{ij}, η_{ij}. In general, there are more equations than unknowns. Since the equations are homogeneous, there are no nontrivial solutions if the rank of this system is equal to the number of unknowns. The number of independent solutions (up to an overall scaling factor) is equal to the corank of this system of equations. This is not larger than one for first order equations but may exceed one for second and higher order equations. This algorithm is effective when $\xi(x,y)$ and $\eta(x,y)$ are polynomials of finite degree.

16.2.4 New Coordinates

If an infinitesimal generator X can be constructed from the determining equations, then it is possible to determine a new system of coordinates R, S, T which "straightens out" the surface equation. This is done by solving the determining equations for canonical coordinates. These are a set of partial differential equations that are analogous to the equations on the right hand side of Eq. (16.5). For convenience, we summarize the determining equations for the infinitesimal generator and for the canonical coordinates, analogs of the two equations in Eq. (16.5), as follows:

$$XF = 0 \qquad\qquad X \begin{bmatrix} R(x,y) \\ S(x,y) \\ T(x,y,p) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{16.14}$$

The three linear partial differential equations on the right determine the new canonical coordinates: the independent variable $R(x,y)$, the dependent variable $S(x,y)$, and the new constraint $T(x,y,p)$ between R and S.

16.2.4.1 Dependent Coordinate

The dependent coordinate S is determined from the differential equation $X(x,y,p)S(x,y) = 1$. We require S to be independent of p, so the condition defining S reduces to

$$\left(\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}\right) S(x,y) = 1 \tag{16.15}$$

The solution is not unique: Any function of x and y that is annihilated by X can be added to the solution. Further, it is not important that $XS = +1$: we could just as well choose a solution satisfying $XS = -1$ or, for that matter, $XS = k \neq 0$, where k is some constant.

16.2.4.2 Invariant Coordinates: Independent Variable

The two invariant coordinates R and T are unchanged under the one-parameter transformation group. These functions obey $XR = 0$ and $XT = 0$, which are explicitly

$$\begin{aligned} \left(\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}\right) R(x,y) &= 0 \\ \left(\xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta(x,y)\frac{\partial}{\partial p}\right) T(x,y,p) &= 0 \end{aligned} \tag{16.16}$$

The solutions are most simply found by the method of characteristics. They obey the differential relations

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{dp}{\zeta(x,y,p)} \tag{16.17}$$

The first equation is used to construct $R(x,y)$.

16.2.4.3 Invariant Coordinates: Constraint Variable

The second equation in (16.17) is used to construct $T(x,y,p)$. It is often possible to construct T so that it is a function of p to the first power. When this is possible, it is the preferred form of the nonunique expression for the invariant cordinate T.

16.2.5 Surface and Constraint Equations

In the new coordinate system there is a constraint equation:

$$\frac{dS}{dR} = \frac{dS(x,y)}{dR(x,y)} = \frac{dS/dx}{dR/dx} = \frac{S_x + S_y p}{R_x + R_y p} \tag{16.18}$$

This derivative is independent of the parameter ϵ of the one-parameter group. Therefore it must be independent of the coordinate S, and depend only on the invariant coordinates R and T. In this new coordinate system the surface and constraint equations are

$$\begin{array}{lrcl} \text{Surface Equation :} & F(R,-,T) & = & 0 \\ \text{Constraint Equation :} & dS/dR & = & f(R,-,T) \end{array} \tag{16.19}$$

These are directly analogous to Eq. (16.1) and $dy/dx = p$ in Sect. 16.1.

16.2.6 Solution in New Coordinates

To integrate the transformed equation, the surface equation is used to determine T as a function of R: $T = T(R)$. This expression is used in the constraint equation, which can then "easily" be integrated to give

$$S = \int f(R,-,T(R))\; dR + c \tag{16.20}$$

The additive parameter c is the image of the parameter ϵ of the one-parameter group of transformations that leaves the original surface equation $F(x,y,p) = 0$ invariant.

16.2.7 Solution in Original Coordinates

The inverse relation $x = x(R,S)$, $y = y(R,S)$ is used to express the solution Eq. (16.20) of the transformed equation in terms of the original coordinates.

16.3 An Example

The algorithm developed in Section 16.2 is, for all practical purposes, impossible to understand without illustrating its workings by a particular example. To illustrate the algorithm, we use it to integrate the equation

$$F(x,y,p) = xp + y - xy^2 = 0 \tag{16.21}$$

Before setting out on this path, we first attempt the following scaling transformation $y \to \alpha y$ and $x \to \beta x$. Under this transformation the equation transforms to $\alpha(xp + y - (\alpha\beta)xy^2) = 0$. The equation is invariant provided $\alpha\beta = 1$. The one-parameter group that leaves the surface constraint $F(x,y,p) = 0$ invariant is $x \to \lambda x$, $y \to \lambda^{-1} y$, $p \to \lambda^{-2} p$. Since there is a one-parameter invariance group for this differential equation, Lie's methods are guaranteed to work. In fact, it is possible to construct the infinitesimal generator $X(x,y,p)$ from this group directly.

The surface $p = y^2 - y/x$ is shown in Fig. 16.1. The value of p clearly depends on both coordinates x and y. The purpose of the change of variables to to find a new coordinate system in which the surface is independent of the new dependent variable $S(x,y)$.

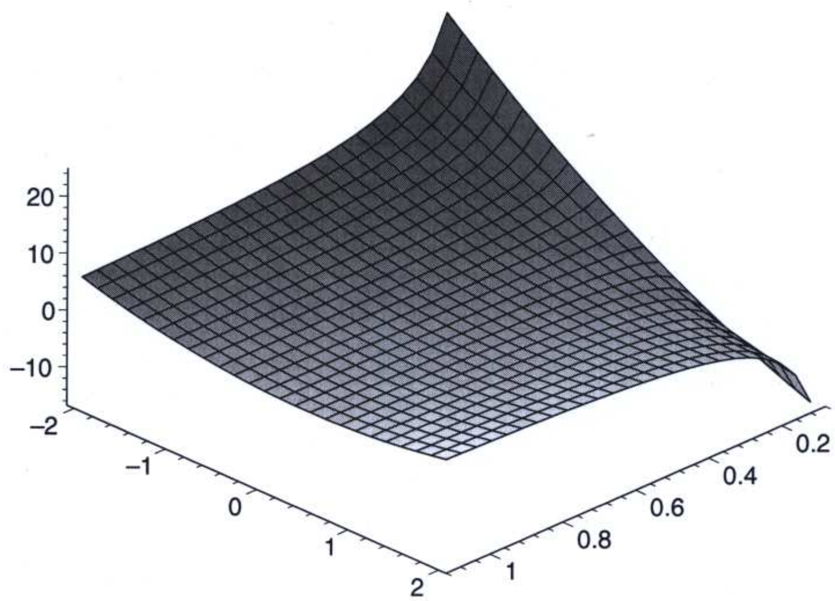

Fig. 16.1. The first order ODE $xp + y - xy^2 = 0$. Here p (vertical) is plotted over the (x,y) plane for $0.1 \le x \le 1.1$ and $-2 \le y \le +2$. The shape of the surface depends on both coordinates x and y.

The determining equation Eq. (16.14) is

$$\xi(p - y^2) + \eta(1 - 2xy) + \left[\eta_x + (\eta_y - \xi_x)p - \xi_y p^2\right] x = 0 \qquad (16.22)$$

The functions $\xi(x,y)$ and $\eta(x,y)$ describing the generators of infinitesimal displacements are determined following the algorithm outlined in Sect. 16.2.3. First, we use the surface equation $F(x,y,p) = 0$ to find an expression for p: $p(x,y) = -y/x + y^2$. This is substituted into the determining equation $XF(x,y,p) = 0$ to provide a functional relation between x and y:

$$\xi\left(-\frac{y}{x}\right)+\eta(1-2xy)+\eta_x x+(\eta_y-\xi_x)(xy^2-y)-\xi_y\left(xy^4-2y^3+\frac{y^2}{x}\right)=0 \tag{16.23}$$

We first attempt zeroth degree expressions for ξ and η: $\xi=\xi_{00}$, $\eta=\eta_{00}$. When these are substituted into Eq. (16.23) we obtain three equations for the two unknowns. The coefficients of the monomials y/x, 1, and xy depend on the unknown parameters ξ_{00},η_{00} as follows:

$$\begin{array}{c} \text{Monomial} \\ y/x \\ x^0y^0=1 \\ xy \end{array} \quad \begin{array}{cc} \xi_{00} & \eta_{00} \\ \end{array} \left[\begin{array}{rr} -1 & 0 \\ 0 & 1 \\ 0 & -2 \end{array}\right] \left[\begin{array}{c} \xi_{00} \\ \eta_{00} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \tag{16.24}$$

This system of three simultaneous linear equations in two unknowns has rank two, therefore no nontrivial solutions.

We therefore increase the degree of $\xi(x,y)$ and $\eta(x,y)$ to one and repeat the process. The relation Eq. (16.23) between x and y is now

$$(\xi_{00}+\xi_{10}x+\xi_{01}y)\left(-\frac{y}{x}\right)+(\eta_{00}+\eta_{10}x+\eta_{01}y)(1-2xy)+\eta_{10}x+$$

$$(\eta_{01}-\xi_{10})(xy^2-y)-\xi_{01}\left(xy^4-2y^3+\frac{y^2}{x}\right)=0 \tag{16.25}$$

This results in the following set of 10 equations for 6 unknowns:

$$\begin{array}{c} \text{Monomial} \\ y^2/x \\ y/x \\ 1 \\ x \\ y \\ xy \\ x^2y \\ xy^2 \\ xy^3 \\ xy^4 \end{array} \quad \begin{array}{c} \begin{array}{cccccc} \xi_{00} & \xi_{10} & \xi_{01} & \eta_{00} & \eta_{10} & \eta_{01} \end{array} \\ \left[\begin{array}{rrrrrr} 0 & 0 & -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & +2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \end{array}\right] \end{array} \left[\begin{array}{c} \xi_{00} \\ \xi_{10} \\ \xi_{01} \\ \eta_{00} \\ \eta_{10} \\ \eta_{01} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \tag{16.26}$$

This set of equations has rank 5, so there is one nontrivial solution. From the first four equations we determine $\xi_{01}=\xi_{00}=\eta_{00}=\eta_{10}=0$, and from the coefficient of xy^2 we learn $-\xi_{10}-\eta_{01}=0$ so that, up to some overall scaling factor we can take $\xi(x,y)=x$ and $\eta(x,y)=-y$. Since we have found one nontrivial solution for an infinitesimal generator of a one-parameter group of a first order equation, we can stop searching for additional solutions to the determining equation (for second order equations there may be additional solutions).

With this solution $\xi(x,y) = x$ and $\eta(x,y) = -y$ the prolongation formula Eq. (16.8) gives $\zeta = -2p$, so that the generator of infinitesimal displacements is

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2p\frac{\partial}{\partial p} \tag{16.27}$$

The infinitesimal generator is now used to determine the new set of coordinates. We first determine the dependent coordinate $S(x,y)$ by attempting to solve

$$\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) S(x,y) = 1 \tag{16.28}$$

It is useful first to seek a solution $S(x,y)$ depending only on the single variable y. Such a solution can be found if the equation $-ydS(y)/dy = 1$ can be solved. The solution, up to an additive constant, is $-\ln(y)$. We will adopt this solution, neglecting the negative sign: $S(x,y) = \ln(y)$.

The invariant coordinates are determined using the method of characteristics:

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{-2p} \tag{16.29}$$

The first equation for the new independent variable simplifies to $ydx = -xdy$ or $d(xy) = 0$, from which we conclude that $R(x,y) = xy$ is an invariant coordinate that obeys Eq. (16.14). The invariant coordinate involving p is determined by setting $-dp/2p$ equal to either of the other two differentials. We set it equal to dx/x to avoid having the second invariant coordinate dependent on y. The equation is $dx/x = -dp/2p$ and the solution is $(1/x)d(x^2p) = 0$, so that $T(x,y,p) = x^2p$. The forward and backward transformations between the two coordinate systems are

$$\begin{pmatrix} R \\ S \\ T \end{pmatrix} = \begin{pmatrix} xy \\ \ln(y) \\ x^2p \end{pmatrix} \qquad \begin{pmatrix} x \\ y \\ p \end{pmatrix} = \begin{pmatrix} Re^{-S} \\ e^S \\ Te^{2S}/R^2 \end{pmatrix} \tag{16.30}$$

In the new coordinate system the surface equation transforms to

$$F(x,y,p) = xp + y - xy^2 = 0 \longrightarrow e^S\left[\frac{T}{R} + 1 - R\right] = 0 \tag{16.31}$$

The expression within the bracket is the transformed surface equation.

It is independent of S. This surface $T = T(R, S)$ is plotted in Fig. 16.2. It has the desired form: a ruled surface whose shape (height) is independent of the dependent variable S. Such a surface is sometimes called a "cylinder."

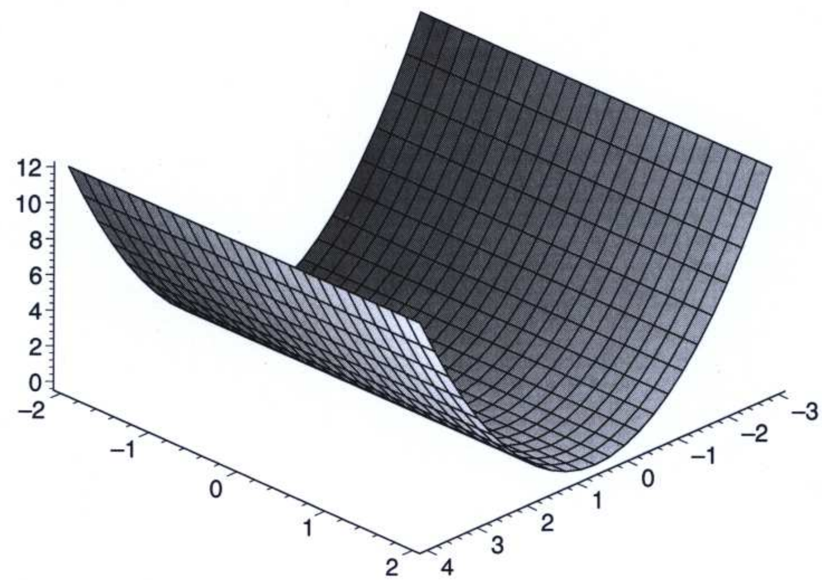

Fig. 16.2. The surface $xp+y-xy^2 = 0$ transforms to the surface $T/R+1-R = 0$ in canonical coordinates. Here T (vertical) is plotted over the (R, S) plane for $-3 \leq R \leq +4$ and $-2 \leq S \leq +2$. The function is a simple ruled surface, independent of S.

The new constraint equation is

$$\frac{dS}{dR} = \frac{d(\ln y)}{d(xy)} = \frac{p/y}{y+xp} = \frac{Te^S/R^2}{e^S + (Re^{-S})(Te^{2S}/R^2)} \tag{16.32}$$

The surface and constraint equations are

$$\begin{array}{lrcl} \text{Surface equation :} & T/R+1-R & = & 0 \\ \text{Constraint equation :} & dS/dR & = & (T/R)/(T+R) \end{array} \tag{16.33}$$

The surface equation is solved for T as a function of R: $T(R) = R^2 - R$. This expression is substituted into the constraint equation to give a first order differential equation in quadratures:

$$\frac{dS}{dR} = \frac{1}{R} - \frac{1}{R^2} \Longrightarrow S = \ln(R) + \frac{1}{R} + c \tag{16.34}$$

The parameter c is the parameter of the translation group that leaves invariant the transformed equation.

The inverse transformation, Eq. (16.30), from (R, S) to (x, y) is finally used to rewrite the solution in terms of the original set of variables:

$$y = \frac{-1}{x(c + \ln x)} \tag{16.35}$$

Remarks: The operator $x\frac{d}{dx}$ is the infinitesimal generator for scaling transformations, since $e^{\lambda x \frac{d}{dx}} x = e^{\lambda} x$. As a result, the infinitesimal generator X has the following effect on the coordinates (x, y, p):

$$EXP\left(\lambda\left\{x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2p\frac{\partial}{\partial p}\right\}\right)\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} e^{\lambda}x \\ e^{-\lambda}y \\ e^{-2\lambda}p \end{bmatrix} \tag{16.36}$$

From this scaling behavior, it is easy to see that $\ln(y)$ is linear in the Lie translation group parameter: $\ln(e^{-\lambda}y) = \ln(y) - \lambda$. The invariant operators come right out of the scaling transformations: xy and x^2p are unchanged by the scaling transformation. None of these operators is unique. The operator $\ln(xy^2)$ is linear and x^3yp is invariant. We have just chosen the most convenient (simplest) solutions to the equations defining the new coordinates.

16.4 Additional Insights

Lie's theory of infinitesimal transformation groups has been extended in many different directions, all of which are powerful and beautiful. It is barely possible to scratch the surface here. Instead, we content ourselves by indicating some of the directions in which it can be extended. These directions are simple consequences of the analyses presented in the previous two sections.

16.4.1 Other Equations, Same Symmetry

Many differential equations can share the same invariance group. The most general first order ordinary differential equation invariant under the scaling group Eq. (16.36) has the form $F(R, -, T) = 0$ or more simply $F(xy, x^2p) = 0$. The most general first order equation of first degree with this symmetry has the form $x^2p = h(xy)$ or $dy/dx = x^{-2}h(xy)$. For the equation studied in Sect. 16.3, $h(z) = -z + z^2$. For the Riccati equation $dy/dx + y^2 - 2/x^2 = 0$, $h(z) = z^2 - 2$.

16.4.2 Higher Degree Equations

These methods work equally well with first order equations of higher degree. For example, the first order, second degree equation $y'^2 + y^4 - x^{-4} = 0$ has canonical form $R^4 + T^2 = 1$. The original equation has two solution branches $p = \pm\sqrt{x^{-4} - y^4}$, corresponding to the two solution branches in the canonical coordinate system $T = \pm\sqrt{1 - R^4}$.

16.4.3 Other Symmetries

The methods described in Sect. 16.2 and illustrated by example in Sect. 16.3 apply to any first order ordinary differential equation with a one parameter group. Table 16.1 provides a list of symmetries that may be encountered for ordinary differential equations. For each symmetry the functions $\xi(x, y)$ and $\eta(x, y)$ are tabulated, as well as the first prolongation $\zeta(x, y, p) = \eta^{(1)}(x, y, p)$. We also present the canonical coordinates (R, S, T). Since the constraint equation dS/dR depends only on the change of variables, it also can be tabulated, and has been. The simplest case, Eq. (16.1), is present in the first line of this table. The equation studied in Sect. 16.3 is present in the eighth line of this table.

The Lie symmetries leaving the equation invariant can be determined from this table in one of two ways. We can use the generator of infinitesimal displacements to compute them, as in Eq. (16.36). Or we can look at the transformations effected by $S \to S' = S + c$, $R' = R$. In the latter case we find $\ln(y) \to \ln(y) + c = \ln(e^c y) = \ln(\bar{y}(c))$ and since $xy = \bar{x}(c)\bar{y}(c)$, the transformation is $\bar{x}(c) = e^{-c}x$ and $\bar{y}(c) = e^{+c}y$.

16.4.4 Second Order Equations

Second order equations can be studied by simple extensions of the methods used to study first order equations. The infinitesimal generator for displacements now involves derivatives with respect to $y^{(2)}$ and is given by

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y^{(1)}} + \eta^{(2)}\frac{\partial}{\partial y^{(2)}} \tag{16.37}$$

Table 16.1. *Infinitesimal generators ξ, η, ζ, canonical coordinates R, S, T, and constraint equation dS/dR for some Lie symmetries.*

Infinitesimal Generators			Canonical Coordinates			Constraint
$\xi(x,y)$	$\eta(x,y)$	$\zeta(x,y,p)$	$R(x,y)$	$S(x,y)$	$T(x,y,p)$	dS/dR
0	1	0	x	y	p	T
1	0	0	y	x	p	$1/T$
$1/a$	$-1/b$	0	$ax+by$	$bx-ay$	p	$(b-aT)/(a+bT)$
x	0	$-p$	y	$\ln x$	xp	$1/T$
0	y	p	x	$\ln y$	p/y	T
x/a	y/b	$(1/b-1/a)p$	y^b/x^a	$b\ln y$	$p/x^{(a/b-1)}$	$(bT/R)/(bT-aR^{(1/b)})$
x	y	0	y/x	$\ln y$	p	$(T/R)/(T-R)$
x	$-y$	$-2p$	xy	$\ln y$	x^2p	$(T/R)/(T+R)$
$2x$	y	$-p$	y^2/x	$\ln y$	yp	$(T/R)/(2T-R)$
x	$2y$	p	y/x^2	$\ln y$	p/x	$(T/R)/(T-2R)$
y	0	$-p^2$	y	x/y	$x-y/p$	$-T/R^2$
0	x	1	x	y/x	$xp-y$	T/R^2
$-y$	x	$1+p^2$	$\sqrt{x^2+y^2}$	$\tan^{-1}(y/x)$	$(y-xp)/(x+yp)$	$-T/R$
1	y/x	$(px-y)/x^2$	y/x	x	$(xp-y)/x^2$	$1/T$
a	x	1	x^2-2ay	x/a	$x-ap$	$1/(2aT)$
a	y	p	$x-a\ln y$	x/a	p/y	$(1/a)/(1-aT)$
x	b	$-p$	e^y/x^b	y/b	$e^y * p^b$	$(bR)^{-1}/[1-b(R/T)^{(1/b)}]$
y	b	$-p^2$	y^2-2bx	y/b	$y-b/p$	$1/(2bT)$
0	$e^{f(x)}$	$f'e^f$	x	y/e^f	$p-yf'$	$T/e^{f(R)}$
x^2	xy	$y-xp$	y/x	$1/x$	$xp-y$	$1/T$
xy	y^2	$yp-xp^2$	y/x	$1/y$	$y/p-x$	$1/(TR^2)$
xy	0	$-yp-xp^2$	y	$(\ln x)/y$	$y/(xp)-\ln x$	T/R^2
0	xy	$y+xp$	x	$(\ln y)/x$	$xp/y-\ln y$	T/R^2
$g(y)$	0	$-g'p^2$	y	x/g	$1/p-xg'/g$	$T/g(R)$
0	$f(x)$	f'	x	y/f	$fp-f'y$	$T/f^2(R)$
$f(x)$	0	$-f'p$	y	$F\ (F'f=1)$	pf	$1/T$
0	$g(y)$	$g'p$	x	$G\ (G'g=1)$	p/g	T
x^{k+1}	kx^ky	$x^k(k^2y/x-p)$	y/x^k	$1/x^k$	$xp-ky$	$-k/T$
kxy^k	y^{k+1}	$y^k(p-k^2xp^2/y)$	x/y^k	$1/y^k$	$y/p-kx$	$-k/T$

The second prolongation can be determined from the first in a straightforward computation

$$\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{d}{d\bar{x}}\left(\frac{d\bar{y}}{d\bar{x}}\right) = \frac{d}{d\bar{x}}(p+\epsilon\eta^{(1)}) = \frac{D^{(1)}(p+\epsilon\eta^{(1)})}{D^{(0)}(x+\epsilon\xi)} = \frac{y^{(2)}+\epsilon D^{(1)}\eta^{(1)}}{1+\epsilon D^{(0)}\xi} =$$

$$y^{(2)} + \epsilon\left(D^{(1)}\eta^{(1)} - y^{(2)}D^{(0)}\xi\right) \qquad (16.38)$$

As a result,

$$\eta^{(2)}(x,y,y^{(1)},y^{(2)}) = D^{(1)}\eta^{(1)} - y^{(2)}D^{(0)}\xi \qquad (16.39)$$

where

$$D^{(n)} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{dy^{(1)}}{dx}\frac{\partial}{\partial y^{(1)}} + \cdots + y^{(n+1)}\frac{\partial}{\partial y^{(n)}} \tag{16.40}$$

It is explicitly

$$\eta^{(2)} = \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_{yy'})y'' \tag{16.41}$$

The determining equations are

$$F(x, y, y^{(1)}, y^{(2)}) = 0 \qquad X(x, y, y^{(1)}, y^{(2)})F(x, y, y^{(1)}, y^{(2)}) = 0 \tag{16.42}$$

Symmetries are found by following the algorithm described in Sect. 16.2.3 and illustrated in Sect. 16.3.

16.4.5 Reduction of Order

If a higher order equation has a known one-parameter symmetry group, the order of the equation can be reduced by one. We illustrate as usual by example. The general case can easily be inferred from the example.

Suppose a second order equation $F(x, y, y', y'') = 0$ is invariant under the scaling group (16.36). Then the dependent coordinate is $S = \ln y$ and the surface equation can be expressed in terms of three invariant coordinates as $F(R, -, T, U) = 0$. Here as before R depends only on x and y, $T = T(x, y, y')$, and $U = U(x, y, y', y'')$ is another invariant coordinate. How does one construct such an invariant coordinate? It is simple to see that the derivative dT/dR is invariant under the group. Not only is it invariant, but it is of first degree in the second order term $y^{(2)}$, for

$$\frac{dT}{dR} = \frac{dT/dx}{dR/dx} = \frac{T_x + T_y y^{(1)} + T_{y^{(1)}} y^{(2)}}{R_x + R_y y^{(1)}} \tag{16.43}$$

For the scaling group the new invariant coordinate is

$$\frac{dT}{dR} = \frac{2xy' + x^2 y''}{y + xy'} \tag{16.44}$$

and the most general second order equation invariant under this group is

$$G(R, -, T, \frac{dT}{dR}) = 0 \tag{16.45}$$

This is a *first* order equation in the invariant coordinate T. The result is that we have used a one-parameter symmetry group to reduce the order of a second order equation by one. If an additional symmetry can be identified, the equation can be reduced to quadratures a second time (i.e., completely integrated).

The most general second order equation invariant under the group of scaling transformations Eq. (16.36) that is of first degree in y'' is

$$\frac{dT}{dR} = \frac{x^2y'' + 2xy'}{y + xy'} = g(xy, x^2y') = g(R, T) \tag{16.46}$$

This is a first order equation in T. Certain forms of the function g may admit another Lie symmetry. If such a symmetry can be found, the order of the equation can again be reduced by one.

16.4.6 Higher Order Equations

These ideas can be extended to higher order equations. We begin with an nth order equation $F(x, y, \cdots, y^{(n)}) = 0$. As usual, we seek an infinitesimal generator

$$X = \xi\frac{\partial}{\partial x} + \eta^{(0)}\frac{\partial}{\partial y^{(0)}} + \eta^{(1)}\frac{\partial}{\partial y^{(1)}} + \cdots + \eta^{(n)}\frac{\partial}{\partial y^{(n)}} = \xi\frac{\partial}{\partial x} + \sum_{j=0}^{n}\eta^{(j)}\frac{\partial}{\partial y^{(j)}} \tag{16.47}$$

The functions in the prolongation formulas are determined following the procedure demonstrated in Eq. (16.38). They are recursively related:

$$\begin{array}{rcl} \eta^{(0)}(x,y) & = & \eta(x,y) \\ \eta^{(1)}(x,y,y^{(1)}) & = & D^{(0)}\eta^{(0)} - y^{(1)}D^{(0)}\xi \\ \eta^{(2)}(x,y,y^{(1)},y^{(2)}) & = & D^{(1)}\eta^{(1)} - y^{(2)}D^{(0)}\xi \\ \eta^{(3)}(x,y,y^{(1)},y^{(2)},y^{(3)}) & = & D^{(2)}\eta^{(2)} - y^{(3)}D^{(0)}\xi \\ \vdots & \vdots & \vdots \end{array} \tag{16.48}$$

The operator X is used as described in Sect. 16.2 to compute the functions $\xi(x, y)$ and $\eta(x, y)$. There will be as many linearly independent infinitesimal generators as the corank of the set of simultaneous linear equations for the Taylor series coefficients of these functions.

If one or more generators can be constructed, a dependent coordinate S can be computed by solving Eq. (16.15). The remaining invariant coordinates are obtained from the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta^{(0)}} = \frac{dy^{(1)}}{\eta^{(1)}} = \cdots = \frac{dy^{(n)}}{\eta^{(n)}} \tag{16.49}$$

In fact, only the first two invariant coordinates $R(x,y)$ and $T(x,y,y^{(1)})$ need be computed. The remaining invariant coordinates are $dT^{(j)}/dR^{(j)}$, $j=0$ (for T) and $j=1,2,\cdots,n-1$. Each of these latter is of first degree in $y^{(j+1)}$. As a result, the existence of a Lie symmetry can be used to reduce an nth order equation to an $(n-1)$st order equation.

16.4.7 Partial Differential Equations: Laplace's Equation

Lie's metehods can be extended to partial differential equations. We illustrate a small part of the theory by treating Laplace's equation in this subsection and the heat equation in the following.

In n dimensions, Laplace's equation with a source term is

$$\nabla^2 u(x^1, x^2, \cdots, x^n) = \delta(x) \tag{16.50}$$

This equation is clearly invariant under rotations, so that the infinitesimal generators of rotations are Lie symmetries. The equation is also invariant under scaling transformations $x^i \to \lambda x^i$, $u \to \alpha u$. Under the scaling transformation $\delta(x) \to \delta(\lambda x) = \lambda^{-n}\delta(x)$, so that

$$\nabla^2 u = \delta(x) \longrightarrow \frac{\alpha}{\lambda^2}\nabla^2 u = \lambda^{-n}\delta(x) \tag{16.51}$$

The equation is invariant provided $\alpha = \lambda^{2-n}$. The infinitesimal generators of symmetries for this equation therefore consist of generators of rotations and scale transformations [14]:

$$\begin{aligned} X_{ij} &= x^i\partial_j - x^j\partial_i \\ Z &= x^i\partial_i + (2-n)u\frac{\partial}{\partial u} \end{aligned} \tag{16.52}$$

A new independent coordinate $R = R(x,u)$ satisfies $XR = 0$, where X is any linear combination of the generators in Eq. (16.52). A solution is $R \sim u|x|^{n-2}$. As a result, $u \sim |x|^{2-n} = k|x|^{2-n}$. The constant of proportionality can be computed using the divergence theorem. Both sides of Eq. (16.50) are integrated over the interior of a unit sphere in R^n. The volume integral on the right is $+1$. The volume integral on the left is transformed into a surface integral using the divergence theorem:

$$\int_V k\nabla^2|x|^{2-n}dV = \int_{S=\partial V} k(2-n)\frac{\hat{\mathbf{n}}\cdot\, d\mathbf{S}}{|x|^{n-1}} = (2-n)kV(S^n) = 1 \tag{16.53}$$

Here $V(S^n) = 2\pi^{n/2}/\Gamma(\frac{n}{2})$ is the surface area of a unit sphere in R^n.

As a result, the solution of Laplace's equation in R^n $(n \neq 2)$ with unit source term at the origin is

$$u(x) = \frac{k}{|x|^{n-2}} \qquad k = \frac{-1}{(n-2)V(S^n)} \tag{16.54}$$

16.4.8 Partial Differential Equations: Heat Equation

The heat equation on R^n for $u(x,t)$ with source term

$$u_t - \nabla^2 u = \delta(x,t) \tag{16.55}$$

is treated similarly [55]. It is invariant under rotations, so the operators X_{ij} are Lie symmetries. Under the scaling transformation $u \rightarrow \alpha u$, $t \rightarrow \beta t$, and $x^i \rightarrow \lambda x^i$ the equation transforms as follows:

$$u_t - \nabla^2 u = \delta(x,t) \longrightarrow \frac{\alpha}{\beta}u_t - \frac{\alpha}{\lambda^2}\nabla^2 u = \frac{1}{\lambda^n \beta}\delta(x,t) \tag{16.56}$$

Invariance under the scaling transformations places the following two constraints on the three scaling variables (since there is only one equation): $\alpha\lambda^n = 1$ and $\beta/\lambda^2 = 1$. From these relations it is possible to construct $n+1$ additional Lie symmetries, so that the entire set is

$$\begin{aligned} X_{ij} &= x^i \partial_j - x^j \partial_i \\ Y_i &= 2t\frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u} \\ Z &= 2t\frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - nu\frac{\partial}{\partial u} \end{aligned} \tag{16.57}$$

An invariant coordinate depending on the x^i, t and u is $R = ut^{n/2}e^{|x|^2/4t}$, from which we obtain as before

$$u = kt^{-n/2}e^{-|x|^2/4t} \qquad k = \left(\frac{1}{2\sqrt{\pi}}\right)^n \tag{16.58}$$

16.4.9 Closing Remarks

Galois resolved the problem of determining if an algebraic equation could be solved by radicals, and if so how, between 1829 and 1832. His manuscripts were lost, rejected, or filed for posterity. His accomplishments were unrecognized at his death in 1832. They were rescued from oblivion, the black hole of French indifference to its greatest mathematician, by Cauchy in 1843.

Lie's discoveries began in 1874. He realized that the hodgepodge of seemingly different techniques for solving differential equations that existed at that time (and still does) were almost all special manifestations of one single principle — the invariance of solutions of ordinary differential equations under a continuous group. Lie was luckier than Galois when it came to recognition during his lifetime.

There are several problems in the implementation of Lie's algorithms that have either been lightly addressed or passed over in our discussion. These are:

(i) Under what conditions is it possible to solve the determining equations for the surface? That is, when is it possible — or impossible — to solve the linear partial differential equations for $\xi(x, y)$ and $\eta(x, y)$?
(ii) Under what conditions is it possible to solve the determining equations for the canonical variables?
(iii) Under what conditions is it possible to solve the canonical surface equation $F(R, -, T) = 0$ for T as a function of R? When it is possible, what is the algorithm for accomplishing this?
(iv) Under what conditions is it possible to integrate a function of a single variable: $\int f(R,\text{-}, T(R))dR$?

The final question has been resolved for algebraic functions by Risch in 1969 [58]. He exploited the tools of Galois theory in a heavy way to provide an algorithm for determining when an algebraic function can be integrated in closed form, and determining the integral when the answer to the first question is positive. We summarize the dates of these accomplishments here

1830	Galois	Solve algebraic equations.
1874	Lie	Solve differential equations.
1969	Risch	Integrate in closed form.
?	—	Solve determining equations for ξ, η.
?	—	Solve determining equations for R, S, T.
?	—	Solve $F(R, -, T) = 0$ for R.

It is clear that additional algorithms are possible and desirable.

16.5 Conclusion

Lie set out to extend Galois' treatment of algebraic equations to the field of ordinary differential equations. Galois observed that an algebraic equation has a symmetry group: a set of operations that maps solutions into solutions. If the symmetry group has certain properties, these properties can be used to generate an algorithm for solving the equation.

It was Lie's genius to see that the "trivial" additive constant that occurs in the solution of a differential equation that has been reduced to quadratures is in fact a group operation. The symmetry group in this simplest case is simply the one parameter group of translations. Armed with this observation, he developed algorithmic methods to attack ordinary differential equations by searching for their symmetry groups. Lie in fact studied local groups of transformations. The even more beautiful study of global Lie groups was a later development.

In Sect. 16.2 we presented Lie's algorithm for solving first order ordinary differential equations in a number of simple steps. These involve:

(i) Introduce a set of point transformations in the x-y plane. These are defined by the functions $\xi(x,y)$ and $\eta(x,y)$.

(ii) Construct the first prolongation $\zeta(x,y,p) = \eta^{(1)}(x,y,y^{(1)})$ from the functions defining the local change of variables.

(iii) Introduce the operator $X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial p}$. This describes a Taylor series expansion of the surface equation $F(x,y,p) = 0$ that defines the first order ODE.

(iv) Solve the determining equation $XF = 0$ when $F = 0$ for the functions $\xi(x,y)$ and $\eta(x,y)$.

(v) Solve the determining equations $XR = 0$, $XS = 1$, $XT = 0$ for the canonical coordinates. These are the coordinates in which the surface is a "cylinder:" The surface equation is independent of the new dependent variable: $F \to F(R,-,T) = 0$.

(vi) Construct the constraint equation $dS/dR = f(R,-,T)$ in this new coordinate system.

(vii) Solve the surface equation for T as a function of R: $T = T(R)$.

(viii) Solve the constraint equation for S: $S = \int f(R,-,T(R)) + c$.

(ix) Backsubstitute the original coordinates for the new coordinates: $x = x(R,S)$ and $y = y(R,S)$ to obtain the solution of the original equation.

The steps in this algorithm have been illustrated by working out a simple example in Sect. 16.3.

These methods extend in any number of ways. We have indicated a number of useful directions by example in Sect. 16.4.

16.6 Problems

1. Show that invariance under a one-parameter group of transformations can also be expressed in the form

$$\frac{d^n}{d\epsilon^n}F[\bar{x}(\epsilon),\bar{y}(\epsilon),\bar{p}(\epsilon)]|_{\epsilon=0}=0, \qquad n=0,1,2,\cdots \tag{16.59}$$

Show that the first two terms $n = 0, 1$ are exactly the determining equations 16.12.

2. Construct the invariance group for each of the transformations presented in Table 16.1.

3. **Mechanical Similarity:** The classical newtonian equation of motion for a particle of mass m in the presence of a potential $V(\mathbf{x})$ is

$$m\frac{d^2\mathbf{x}}{dt^2} = -\nabla V(\mathbf{x})$$

Assume that under a scaling transformation, the mass scales with a factor α (i.e., $m \rightarrow \alpha m$), $\mathbf{x} \rightarrow \beta\mathbf{x}$, $t \rightarrow \gamma t$. Assume also that the potential is homogeneous of degree k: $V(\beta\mathbf{x}) \rightarrow \beta^k V(\mathbf{x})$ [49]. Under this scaling transformation show that the equation of motion transforms to

$$\alpha^1\beta^1\gamma^{-2}m\frac{d^2\mathbf{x}}{dt^2} = -\beta^{k-1}\nabla V(\mathbf{x})$$

a. Show that the scaled equation is identical to the original provided $\alpha^1\beta^{2-k}\ \gamma^{-2} = 1$.

b. Set $\alpha = 1$. Show that trajectories are invariant under the scaling transformation with $\gamma^2 = \beta^{2-k}$. Show that in the cases $k = -1, k = 0, k, = +1, k = +2$ the following scaling results hold:

k	Potential Type	transformation
-1	Coulomb	$\gamma^2 = \beta^3$
0	No Force	$\gamma^2 = \beta^2$
$+1$	Local Gravitational Potential	$\gamma^2 = \beta^1$
$+2$	Harmonic Oscillator	$\gamma^2 = \beta^0$

The first line is a statement of Kepler's Third Law: for closed planetary orbits, the square of the period (γ^2) is proportional to the cube of the semiaxis (β^3). If R' and T' are the semiaxis and period of planet P' and R and T are the semiaxis and period of planet P, and the two planets

P and P' have geometrically similar orbits, $\beta^3 \rightarrow (R'/R)^3 = (T'/T)^2 \leftarrow \gamma^2$. The second line is a statement of the integral of Newton's second law in the absence of forces in an inertial frame: the distance traveled (β) is proportional to the time elapsed (γ). The third line is a statement that in a local gravitational potential of the form $V = mgz$, the distance fallen increases like the square of the time elapsed. The fourth line is a statement of Hooke's Law: in harmonic motion the period (γ) is independent of the size of the orbit.

c. Fix $\gamma = 1$ and construct a table relating the mass and orbital scale under the four forces described in the table above.

d. Fix $\beta = 1$ and show that the period scales like $\sqrt{M}$ for all homogeneous potentials. Reconcile this result with the well-known result that the period of a planet is independent of its mass in lowest order.

e. If the motion is bounded for all times, show

$$2\langle T\rangle = \langle \mathbf{x}\cdot\nabla V(\mathbf{x})\rangle = \langle kV(\mathbf{x})\rangle$$

where T is the kinetic energy. This is the Virial Theorem for homogenoeous potentials.

f. Show that the kinetic energy scales like $\alpha\beta^2\gamma^{-2} = \beta^k$ (use **a.**). Since the potential energy scales the same way, the total energy has this scaling property.

4. Assume that the dynamics of a system are derivable from an action principle: for example, the Euler-Lagrange equations are derived from the variation of an action: $\delta \int \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}})d\mathbf{x} = 0$. Show that if a scaling transsformation leaves the Lagrangian invariant up to an overall scaling factor, the trajectories will scale under this transformation.

5. The heat equation in one dimension is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Show that the following six differential operators v_i are infinitesimal generators of the invariance group of this equation. Show that $e^{\epsilon v_i} f(x,t)$ has the action shown for each of the six generators [55]:

v_i	Infinitesial	$e^{\epsilon v_i} f(x,t) =$
v_1	∂_x	$f(x-\epsilon, t)$
v_2	∂_t	$f(x, t-\epsilon)$
v_3	$u\partial_u$	$e^{\epsilon} f(x,t)$
v_4	$x\partial_x + 2t\partial_t$	$f(e^{-\epsilon}x, e^{-2\epsilon}t)$
v_5	$2t\partial_x - xu\partial_u$	$e^{-\epsilon x+\epsilon^2 t} f(x-2\epsilon t, t)$
v_6	$4xt\partial_x + 4t^2\partial_t - (x^2+2t)u\partial_u$	$\lambda e^{-\epsilon\lambda^2 x^2} f(\lambda^2 x, \lambda^2 t)$
		where $\lambda^2 = 1/(1+4\epsilon t)$

6. The two dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2}$$

Show that the following vector fields map solutions into solutions:

Displacements	P_i	$\partial_x, \partial_y, \partial_t$
Rotations	L_z	$x\partial_y - y\partial_x$
Boosts	B_i	$x\partial_t + t\partial_x, y\partial_t + t\partial_y$
Dilations	D_i	$x\partial_x + y\partial_y + t\partial_t, u\partial_u$

Inversions $$\begin{bmatrix} i_x \\ i_y \\ i_t \end{bmatrix} = \begin{bmatrix} x^2-y^2+t^2 & 2xy & 2xt & -xu \\ 2yx & -x^2+y^2+t^2 & 2yt & -yu \\ 2tx & 2ty & x^2+y^2+t^2 & -tu \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_t \\ \partial_u \end{bmatrix}$$

Show that $D_2 = u\partial_u$ commutes with all remaining generators. Construct the commutation relations of the remaining ten generators, and show they satisfy the commutation relations of the conformal group in 2+1 dimensions. Show that this group is $SO(2+1, 1+1) = SO(3,2)$.

7. Construct the invariance group for the wave equation in 3+1 dimensions. This is the Maxwell equation without sources in space time. There are 16 infinitesimal generators. Show that 15 satisfy the commutation relations for the conformal group $SO(3+1, 1+1) = SO(4,2)$ [9]. The extra generator commutes with all the rest, and is $u\partial_u$.

8. The heat equation in one dimension is $u_{xx} - u_t = 0$. The infinitesimal generator of symmetries for this equation is $X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \cdots = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \cdots$. Show that [66]

$$\begin{aligned}\xi^1 &= a_1 + a_2x + a_3t + a_4xt \\ \xi^2 &= 2a_2t + a_4t^2 + a_5 \\ \eta &= -\tfrac{1}{2}a_3xu - a_4(\tfrac{1}{2}t + \tfrac{1}{4}x^2)u + a_6u + h(x,t)\end{aligned}$$

Here $h(x,t)$ is any function that satisfies the homogeneous heat equation. Construct the infinitesimal generators corresponding to the arbitrary real coordinates a_i and compute their commutation relations. What is the structure of this Lie algebra?

9. Show that the scalar operator

$$S = t^2\frac{\partial}{\partial t} + t\mathbf{x}\cdot\nabla - \frac{1}{4}(\mathbf{x}\cdot\mathbf{x} + 2nt)\; u\frac{\partial}{\partial u}$$

is also a Lie symmetry of Eq. (16.55) with source term.

10. **Noether's Theorem for Physicists:** Many dynamical problems can be expressed in an action principle format:

$$I = \int_{t_1}^{t_2} L(t,x,\dot{x})dt, \qquad \delta I = 0$$

Specifically, the action I is stationary on a physically allowed trajectory. The first variation leads to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

Under a one parameter family of change of variables: $t \to t' = T(t,x,\epsilon) = t + \epsilon\xi(t,x)$, $x_i \to x_i' = X_i(t,x,\epsilon) = x_i + \epsilon\eta_i(t,x)$ the action integral transforms to

$$I = \int_{t_1'}^{t_2'} L(t',x',\dot{x'})dt' = \int_{t_1}^{t_2} L(t',x',\dot{x'})\frac{dt'}{dt}dt$$

where $dt'/dt = \partial T/\partial t + (\partial T/\partial x_i)dx_i/dt$. Show that if you differentiate the action integral with respect to ϵ, then set $\epsilon = 0$ the result is

$$\int_{t_1}^{t_2}\left(\xi\frac{\partial L}{\partial t} + \eta_i\frac{\partial L}{\partial x_i} + \eta_i^{(1)}\frac{\partial L}{\partial \dot{x}_i} + \frac{d\xi}{dt}L\right)dt = 0$$

Show that by standard arguments the integrand must itself be zero. Show that along an allowed trajectory the vanishing of the integrand can be expressed in the form

$$\frac{d}{dt}\left[\xi L + (\eta_i - \xi \dot{x}_i) L_{\dot{x}_i}\right] = 0$$

The expression within the square brackets is a constant of the motion. Apply this theorem to a Lagrangian that is invariant under space displacements, time displacements, and rotations around a space axis to construct the following conserved quantities:

Symmetry	Conserved Quantity
Space Displacements	Momentum
Time Displacements	Energy
Space-Time Displacements	Four-Momentum
Rotations	Angular Momentum

11. **Noether's Theorem, More General:** We present a more general form of Noether's theorem than is presented above. This form is very powerful and sufficient for most physical applications. It is not the most general form of Noether's theorem. Suppose the dynamics of a system is derivable from an action integral of the form $L[u] = \int \mathcal{L}(x,u)dx$, $x \in R^p$, $u \in R^q$, and suppose the infinitesimal generators that leave the dynamics invariant has the from

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

Show that the components P_i defined by

$$P^i = \xi^i \mathcal{L} + \sum_{\alpha=1}^{q} \phi^\alpha(x,u)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - \sum_{\alpha=1}^{q}\sum_{j=1}^{p} \xi^j u_j^\alpha \frac{\partial \mathcal{L}}{\partial u_i^\alpha}$$

satisfy a conservation law of the form

$$\nabla P = \text{div } P = \frac{\partial P^i}{\partial x^i} = 0$$

12. **Representation Theory:** G is a compact Lie group with invariant measure $d\rho(g)$ and volume $Vol(G) = \int d\rho(g)$, $\Gamma^\lambda_{\mu\nu}(g)$ are the irreducible representations of G constructed by reduction of tensor products (Wigner-Stone theorem), and $\phi(g), \psi(g)$ are functions defined on the group manifold. The orthogonality and completeness relations are

$$\int \frac{\dim \lambda}{Vol(G)}\, \Gamma^{\lambda' *}_{\mu'\nu'}(g)\Gamma^{\lambda}_{\mu\nu}(g)d\rho(g) = \delta^{\lambda'\lambda}\delta_{\mu'\mu}\delta_{\nu'\nu}$$

$$\sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{\dim\lambda}{Vol(G)}\Gamma^{\lambda *}_{\mu\nu}(g')\Gamma^{\lambda}_{\mu\nu}(g)=\delta(g',g)$$

Introduce Dirac notation for these matrix elements:

$$\langle g|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle=\sqrt{\frac{\dim\lambda}{Vol(G)}}\Gamma^{\lambda}_{\mu\nu}(g)\qquad\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}|g\rangle=\sqrt{\frac{\dim\lambda}{Vol(G)}}\Gamma^{\lambda *}_{\mu\nu}(g)$$

a. Write the orthogonality and completeness relations in Dirac notation and show:

$$\int d\rho(g)\langle\begin{matrix}\lambda'\\ \mu'\nu'\end{matrix}|g\rangle\langle g|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle=\langle\begin{matrix}\lambda'\\ \mu'\nu'\end{matrix}|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle$$

$$\sum_{\lambda}\sum_{\mu}\sum_{\nu}\langle g'|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}|g\rangle=\langle g'|g\rangle$$

b. Show that the orthogonality and completeness relations can be expressed in the form of "resolutions of the identity" in appropriate spaces:

$$|g\rangle\langle g| \;=\; \int|g\rangle d\rho(g)\langle g| \;=\; I \quad \text{in group space}$$

$$|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}| \;=\; \sum_{\lambda}\sum_{\mu}\sum_{\nu}|\begin{matrix}\lambda\\ \mu\nu\end{matrix}\rangle\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}| \;=\; I \quad \text{in representation space}$$

c. Carry out a Fourier decomposition on the functions $\psi(g)=\langle g|\psi\rangle$ and $\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}|\psi\rangle=\int d\rho(g)\langle\begin{matrix}\lambda\\ \mu\nu\end{matrix}|g\rangle\langle g|\psi\rangle$ (and similarly for $\phi(g)=\langle g|\phi\rangle$) using the Dirac representation. Write down the Parseval equality for the inner product $\int\phi^*(g)\psi(g)d\rho(g)$ expressed in terms of the discrete and continuous basis vectors in this Hilbert space.