8
Structure Theory for Lie Algebras

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In this chapter we discuss the structure of Lie algebras. A typical Lie algebra is a semidirect sum of a semisimple Lie algebra and a solvable subalgebra that is invariant. By inspection of the regular representation 'in suitable form,' we are able to determine the maximal nilpotent and solvable invariant subalgebras of the Lie algebra and its semisimple part. We show how to use the Cartan-Killing inner product to determine which subalgebras in the Lie algebra are nilpotent, solvable, semisimple, and compact.

8.1 Regular Representation

A Lie algebra is defined by its commutation relations. The commutation relations are completely encapsulated by the structure constants. These are conveniently summarized in the regular representation

$$[Z, X_i] = R(Z)_i^{\ j} X_j \tag{8.1}$$

Under a change of basis $X_j = A_j^{\ s} Y_s$ this $n \times n$ matrix undergoes a similarity transformation

$$S(Z)_r^{\ s} = (A^{-1})_r^{\ i} R(Z)_i^{\ j} A_j^{\ s} \tag{8.2}$$

It is very useful to find a basis, or construct a similarity transformation, that brings the regular representation of every operator in the Lie algebra *simultaneously* to some standard form. The structure of the Lie algebra can be decided by inspecting this standard form.

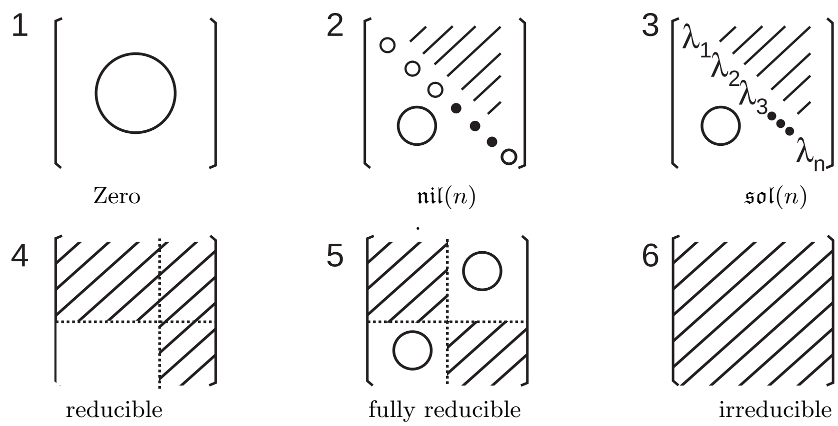

Fig. 8.1. Structure of the regular representation for different types of Lie algebras.

8.2 Some Standard Forms for the Regular Representation

We summarize in Fig. 8.1 the standard forms that the regular representation can assume. We also provide an example for each.

1. Zero. In this case all structure constants vanish and the algebra is commutative.

Example: The Lie algebra $\mathfrak{a}(p, q)$ consists of matrices of the form

$$\begin{array}{cc} \left[\begin{array}{c|c} 0 & A \\ \hline 0 & 0 \end{array}\right] & \begin{array}{c} \uparrow \\ p \\ \downarrow \\ \uparrow \\ q \\ \downarrow \end{array} \end{array} \tag{8.3}$$

This is an $n \times n$ $(n = p + q)$ matrix algebra which is $N = p * q$ dimensional. The independent degrees of freedom are the N independent matrix elements of the $p \times q$ matrix A. The $n \times n$ matrices all commute under matrix multiplication. The group operation is equivalent to addition of the $p \times q$ matrices. The regular representation consists of $N \times N$ matrices, all N of them are zero.

2. $\mathfrak{nil}(n)$ **Strictly Upper Triangular.** In this case the Lie algebra is nilpotent.

Example: We consider the Lie algebra spanned by the photon operators $a, a^\dagger$, and $I = [a, a^\dagger]$ or the isomorphic 3×3 matrix algebra (5.11). The regular representation is a 3×3 matrix

$$\mathfrak{Reg}(la + ra^\dagger + \delta I) = \begin{array}{cc} \left[\begin{array}{ccc} 0 & 0 & l \\ 0 & 0 & -r \\ 0 & 0 & 0 \end{array}\right] & \begin{array}{c} a^\dagger \\ a \\ I \end{array} \end{array} \tag{8.4}$$

3. $\mathfrak{sol}(n)$ **Upper Triangular.** In this case nonzero elements occur on and above the diagonal. The algebra is solvable.

Example: The algebra spanned by the photon number operator $\hat{n} = a^\dagger a$, creation and annihilation operators $a^\dagger$ and a, and their commutator $I = [a, a^\dagger]$ is isomorphic to the algebra described by the 3×3 matrices (5.9). The regular representation is a 4×4 matrix

$$\mathfrak{Reg}(\eta\hat{n} + la + ra^\dagger + \delta I) = \begin{array}{cc} \left[\begin{array}{cccc} 0 & -r & l & 0 \\ 0 & \eta & 0 & l \\ 0 & 0 & -\eta & -r \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{c} \hat{n} \\ a^\dagger \\ a \\ I \end{array} \end{array} \tag{8.5}$$

4. $\mathfrak{ut}(p, q)$ In this case the regular representation is reducible and the Lie algebra is nonsemisimple.

Example: We consider the algebra consisting of the six photon operators $\hat{n} = \frac{1}{2}\left\{a, a^\dagger\right\} = a^\dagger a + \frac{1}{2}, a^{\dagger\ 2}, a^2, a^\dagger, a, I = [a, a^\dagger]$. This is isomorphic to the algebra of six 4×4 matrices presented in (5.7). The algebra of 4×4 matrices (the "defining" representation) and the regular representation of this algebra are given below

$$\eta(\hat{n}+\tfrac{1}{2}) + Ra^{\dagger\,2} + La^2 + ra^\dagger + la + \delta I$$

$$\mathfrak{Def} = \begin{bmatrix} 0 & l & r & -2\delta \\ 0 & \eta & 2R & -r \\ 0 & -2L & -\eta & l \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathfrak{Reg} = \left[\begin{array}{ccc|ccc} 0 & -2R & 2L & -r & l & 0 \\ 4L & 2\eta & 0 & 2l & 0 & 0 \\ -4R & 0 & -2\eta & 0 & -2r & 0 \\ \hline & & & \eta & 2L & l \\ & & & -2R & -\eta & -r \\ & & & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \hat{n}+\frac{1}{2} \\ a^{\dagger 2} \\ a^2 \\ a^\dagger \\ a \\ I \end{array} \tag{8.6}$$

The subspace spanned by the three operators $a^\dagger, a, I$ is invariant, as is shown by the structure of the regular representation.

5. Block Diagonal. In this case the regular representation is fully reducible and the Lie algebra is semisimple.

Example: The Lie algebra $\mathfrak{so}(4)$ has six generators $X_{ij} = -X_{ji}$, $1 \le i, j \le 4$ and commutation relations

$$[X_{ij}, X_{rs}] = X_{is}\delta_{jr} + X_{jr}\delta_{is} - X_{ir}\delta_{js} - X_{js}\delta_{ir} \tag{8.7}$$

The following two linear combinations of generators

$$\begin{array}{rclrcl} Y_i & = & \frac{1}{2}(X_{i4} + \frac{1}{2}\epsilon_{irs}X_{rs}) & X_{i4} & = & Y_i + Z_i \\ Z_i & = & \frac{1}{2}(X_{i4} - \frac{1}{2}\epsilon_{irs}X_{rs}) & X_{ij} & = & \epsilon_{ijk}(Y_k - Z_k) \end{array} \tag{8.8}$$

obey the commutation relations

$$\begin{array}{rcl} [Y_i, Y_j] & = & -\epsilon_{ijk}Y_k \\ [Z_i, Z_j] & = & +\epsilon_{ijk}Z_k \\ [Y_i, Z_j] & = & 0 \end{array} \tag{8.9}$$

The 4×4 defining matrix representation and the 6×6 regular representation have the structure

$$X = \sum y_i Y_i + \sum z_j Z_j$$

$$\mathfrak{Def}(X) \rightarrow \begin{bmatrix} 0 & +(y_3 - z_3) & -(y_2 - z_2) & +(y_1 + z_1) \\ -(y_3 - z_3) & 0 & +(y_1 - z_1) & +(y_2 + z_2) \\ +(y_2 - z_2) & -(y_1 - z_1) & 0 & +(y_3 + z_3) \\ -(y_1 + z_1) & -(y_2 + z_2) & -(y_3 + z_3) & 0 \end{bmatrix}$$

$$\mathfrak{Reg}(X) \rightarrow \begin{array}{c} \left[\begin{array}{ccc|ccc} 0 & -y_3 & +y_2 & & & \\ +y_3 & 0 & -y_1 & & & \\ -y_2 & +y_1 & 0 & & & \\ \hline & & & 0 & +z_3 & -z_2 \\ & & & -z_3 & 0 & +z_1 \\ & & & +z_2 & -z_1 & 0 \end{array}\right] \end{array} \begin{array}{c} Y_1 \\ Y_2 \\ Y_3 \\ Z_1 \\ Z_2 \\ Z_3 \end{array} \tag{8.10}$$

Since the regular representation has a block diagonal structure, the algebra is semisimple. It is not at all obvious that the Lie algebra $\mathfrak{so}(4)$ is semisimple and can be written as the direct sum of two simple algebras. This is not true for the other orthogonal Lie algebras, $\mathfrak{so}(n)$, $n > 4$. We will have to wait until Chapter 10 to be able to see that $\mathfrak{so}(4)$ is semisimple, not simple.

6. Irreducible. In this case the regular representation is irreducible and the Lie algebra is simple.

Example: The Lie algebras $\mathfrak{su}(n)$ $(n \geq 2)$, $\mathfrak{so}(n)$ $(n > 4)$, and $\mathfrak{sp}(n)$ $(n \geq 1)$ are all simple. To be concrete, the Lie algebra for $SU(2)$ has defining and regular representations

$$\mathfrak{Def} = \frac{i}{2} \begin{bmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{bmatrix} \quad \mathfrak{Reg} = \begin{bmatrix} 0 & -a_3 & +a_2 \\ +a_3 & 0 & -a_1 \\ -a_2 & +a_1 & 0 \end{bmatrix} \begin{array}{c} X_1 \\ X_2 \\ X_3 \end{array} \tag{8.11}$$

8.3 What These Forms Mean

Reducing the regular representation to one of the standard forms described in the previous section means that the structure constants, and therefore the commutation relations, have also been reduced to some standard form. We discuss in this section what each of the standard forms implies about the commutation relations and structure of the Lie algebra.

1. Commutative Case. If all the structure constants are zero, then

$$[X_i, X_j] = 0 \tag{8.12}$$

for each element in the Lie algebra.

2. & 3. Nilpotent and Solvable. In these cases

$$\begin{array}{rcllll} [Z, X_i] & = & R(Z)_i^{\ j} X_j & & & \\ R(Z)_i^{\ j} & = & 0 & \text{unless} & j > i & \text{nilpotent} \\ & & & & j \geq i & \text{solvable} \end{array} \tag{8.13}$$

This means that $[Z, X_i]$ can be expressed as a linear combination of basis vectors X_j with $j \geq i$. This in turn means that the basis vectors $X_i, X_{i+1}, \cdots, X_n$ span a subalgebra for each value of $i = 1, 2, \cdots, n$. Since this subalgebra is mapped onto itself by every element Z in the original algebra, each subalgebra is an invariant subalgebra. The result is shown schematically in Fig. 8.2 and is summarized by

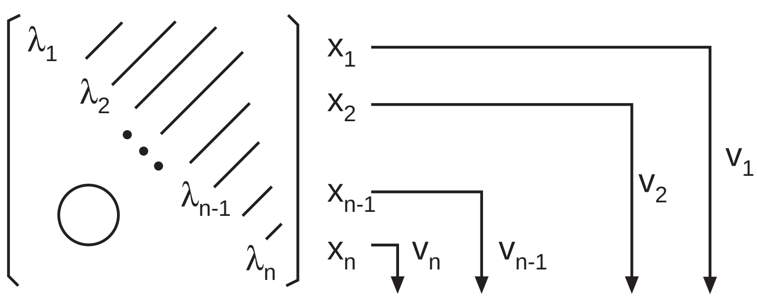

Fig. 8.2. Structure of nilpotent and solvable algebras.

$$
\begin{array}{ccl}
V_1 & \text{spanned by} & X_n, X_{n-1}, X_{n-2}, \cdots, X_2, X_1 \\
\cup & & \\
V_2 & \text{spanned by} & X_n, X_{n-1}, X_{n-2}, \cdots, X_2 \\
\cup & & \\
\vdots & \vdots & \quad\vdots \\
\cup & & \\
V_{n-2} & \text{spanned by} & X_n, X_{n-1}, X_{n-2} \\
\cup & & \\
V_{n-1} & \text{spanned by} & X_n, X_{n-1} \\
\cup & & \\
V_n & \text{spanned by} & X_n
\end{array}
\tag{8.14}
$$

Each V_i is an invariant subalgebra in V_j, $i > j$. The original algebra is V_1.

4. Reducible or Nonsemisimple. The block diagonal form for the regular representation requires the commutation relations

$$
\left[\begin{array}{c|c} \star & \star \\ \hline 0 & \star \end{array}\right]
\begin{array}{c} \uparrow \\ V_1 \\ \downarrow \\ \hline \uparrow \\ V_2 \\ \downarrow \end{array}
\qquad \Rightarrow \qquad [\text{Any}, V_2] \subset V_2 \tag{8.15}
$$

Since in particular $[V_2, V_2] \subseteq V_2$, V_2 is a subalgebra in the original algebra. Further, since the commutator of anything in the original algebra with V_2 is in V_2, V_2 is an *invariant* subalgebra. The complementary subspace V_1 is not generally even a subalgebra of the original algebra.

5. Fully Reducible or Semisimple. In this case the block diagonal form for the regular representation requires the commutation relations

$$
\left[\begin{array}{c|c} \star & 0 \\ \hline 0 & \star \end{array}\right]
\begin{array}{c} \uparrow \\ V_1 \\ \downarrow \\ \hline \uparrow \\ V_2 \\ \downarrow \end{array}
\qquad \Rightarrow \qquad
\begin{array}{rcl}
[V_1, V_1] & \subseteq & V_1 \\
[V_2, V_2] & \subseteq & V_2 \\
[V_1, V_2] & = & 0
\end{array}
\tag{8.16}
$$

Both V_1 and V_2 are invariant subalgebras. Moreover, every element in V_1 commutes with every operator in V_2. Therefore the two subalgebras V_1 and V_2 can be studied separately and independently.

6. Irreducible or Simple. In this case every generator X can be written as the commutator of some pair of operators Y and Z in the Lie algebra:

$$X = [Y, Z] \tag{8.17}$$

It is this ability of an algebra to reproduce itself under commutation that distinguishes simple and semisimple Lie algebras from solvable and nilpotent algebras. Nonsemisimple algebras are composed of a semisimple subalgebra and a solvable invariant subalgebra.

8.4 How to Make This Decomposition

There is a systematic procedure for decomposing a Lie algebra into its semisimple component and its maximal solvable invariant subalgebra. This is a simple two-step procedure. In the first step we identify the subspace of the Lie algebra on which the Cartan-Killing inner product is identically zero. If there is no such subspace the algorithm stops here and the algebra is semisimple. If there is a nontrivial subspace, it forms the maximal nilpotent invariant subalgebra of the algebra. This subspace is 'removed' from the algebra, and the commutation relations and Cartan-Killing inner product for the remaining operators are computed. The algorithm stops here, regardless of the outcome. If there is a nontrivial subspace on which the new Cartan-Killing inner product is identically zero, the elements in this subspace, together with the previously identified nilpotent invariant subalgebra, span a solvable algebra. This is the maximal solvable invariant subalgebra in the original Lie algebra. The complementary subspace on which the new Cartan-Killing inner product is nonsingular is the semisimple part of the original Lie algebra.

In small, easy-to-digest steps, this two-step algorithm takes the following form:

a. From the structure constants of the original Lie algebra $\mathfrak{g}$ form the Cartan-Killing inner product.

b. Determine the subspace on which this inner product is positive-definite, negative-definite, and zero:

$$\mathfrak{g} = (V_- + V_+) + V_0 \tag{8.18}$$

c. If V_0 is zero, stop. If not, V_0 is the maximal nilpotent invariant subalgebra in $\mathfrak{g}$.

d. Form the difference $\mathfrak{g}' = \mathfrak{g} - V_0$. This is a Lie algebra (under the 'mod' operation: set to zero any part of the commutator that is in V_0). Compute the structure constants and Cartan-Killing inner product for $\mathfrak{g}'$.

e. Effect another decomposition:

$$\mathfrak{g}' = (V'_- + V'_+) + V'_0 \tag{8.19}$$

f. The original Lie algebra has the following structure

$$\mathfrak{g} = \underbrace{\underbrace{\underbrace{V'_-}_{\text{compact subalgebra}} + \underbrace{V'_+}_{\text{noncompact generators}}}_{\text{semisimple}} + \underbrace{\underbrace{V'_0}_{\text{abelian}} + \underbrace{V_0}_{\text{nilpotent}}}_{\text{max. solvable invariant subalgebra}}}_{\text{nonsemisimple Lie algebra}} \tag{8.20}$$

This algorithm cannot distinguish semisimple Lie algebras from simple Lie algebras. We will develop tools in Chapter 10 that will make this distinction possible simply by inspection of the algebra's (canonical) commutation relations.

8.5 An Example

To illustrate this procedure, we compute the structure of the six-dimensional Lie algebra of two photon operators. The regular representation is given in (8.6). The inner product of a vector with itself is

$$(X, X) = -40RL + 10\eta^2 \tag{8.21}$$

The inner product is identically zero on the subspace V_0 spanned by $a^\dagger, a$ and I. The three remaining operators have regular representation

$$\eta(a^\dagger a + \frac{1}{2}) + Ra^{\dagger 2} + La^2 \longrightarrow \begin{array}{cc} \begin{bmatrix} 0 & -2R & 2L \\ 4L & 2\eta & 0 \\ -4R & 0 & -2\eta \end{bmatrix} & \begin{array}{c} \hat{n} + \frac{1}{2} \\ a^{\dagger 2} \\ a^2 \end{array} \end{array} \tag{8.22}$$

with inner product

$$(X, X)' = -32RL + 8\eta^2 \tag{8.23}$$

In this case $V'_0 = 0$ and the two photon algebra has the decomposition

$$\mathfrak{g} = \underbrace{(\hat{n} + \frac{1}{2}, a^{\dagger 2}, a^2)}_{\text{su(1,1)}} + \underbrace{(a^\dagger, a, I)}_{\text{nilpotent invariant subalgebra}} \tag{8.24}$$

The Cartan-Killing inner product can be diagonalized by choosing two linear combinations of the operators $a^{\dagger 2}$ and a^2. Then $a^{\dagger 2} + a^2$ is a compact generator, since the Cartan-Killing form is negative-definite on it. The other two operators, $a^\dagger a + \frac{1}{2}$ and $a^{\dagger 2} - a^2$, are noncompact.

8.6 Conclusion

An arbitrary Lie algebra is a semidirect sum of a semisimple Lie algebra and a solvable invariant subalgebra. The structure of a Lie algebra can be determined by inspecting its regular representation, once this has been brought to suitable form by a similarity transformation. To facilitate constructing this transformation, we have shown how to use the Cartan-Killing inner product to determine the linear vector subspaces in the Lie algebra that are maximal nilpotent invariant subalgebras, the maximal solvable invariant subalgebra, the semisimple subalgebra, and its maximal compact subalgebra.

8.7 Problems

1. Compute the decomposition (8.20) for

i. The photon algebra $\hat{n}, a^\dagger, a, I$ (8.5).
ii. The algebra $\mathfrak{so}(3,1)$.
iii. The algebra for the Poincaré group (3.26).
iv. The algebra for the Galilei group (3.27).

2. Compute the decomposition (8.20) for Lie algebras spanned by various combinations of the boson creation and annihilation operators (**i. - vii.** below). These satisfy $[b_i, b_j^\dagger] = I\delta_{ij}$, $1 \le i, j \le n$. Commutators involving bilinear (trilinear, ...) products are computed in the usual way.

i. $b_i, b_j^\dagger, I$.
ii. $b_i^\dagger b_j$.
iii. $b_i^\dagger b_j, b_i, b_j^\dagger, I$.
iv. $b_i^\dagger b_j^\dagger, b_i^\dagger b_j + \frac{1}{2}\delta_{ij}, b_i b_j$.
v. $b_i^\dagger b_j^\dagger, b_i^\dagger b_j + \frac{1}{2}\delta_{ij}, b_i b_j, b_i, b_j^\dagger, I$.
vi. $b, b^\dagger b, b^\dagger b^\dagger b$.
vii. $b^\dagger, b^\dagger b, b^\dagger b b$.

3. Fermion creation and annihilation operators obey anticommutation relations $\left\{f_i, f_j^\dagger\right\} = \delta_{ij}$, but their bilinear combinations close under the same commutation relations as do boson operators. Compute the structure of these fermion algebras:

i. $f_i^\dagger f_j$.
ii. $f_i^\dagger f_j^\dagger, f_i^\dagger f_j + \frac{1}{2}\delta_{ij}, f_i f_j$.

4. Compute the decomposition (8.20) for Lie algebras spanned by various combinations of the position (x_i) and momentum (∂_j) operators for n independent directions:

i. x_i, ∂_j, I.
ii. $x_i \partial_j$.
iii. $x_i \partial_j, x_i, \partial_j, I$.
iv. $x_i x_j, x_i \partial_j + \frac{1}{2} I \delta_{ij}, \partial_i \partial_j$.
v. $x_i x_j, x_i \partial_j, \partial_i \partial_j, x_i, \partial_j, I$.
vi. $\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}$.
vii. $x, x\frac{d}{dx}, x\frac{d^2}{dx^2}$.

5. What is the relation between the Cartan-Killing inner product computed using the defining matrix representation of a matrix Lie algebra and using the regular matrix representation of the Lie algebra?

6. The Lorentz, Poincaré, and Galilei groups in $2+1$ dimensions (x, y and t) have Lie algebras with matrix structures:

$$\underset{\text{Lorentz}}{\left[\begin{array}{cc|c} 0 & \theta & v_1 \\ -\theta & 0 & v_2 \\ \hline v_1 & v_2 & 0 \end{array}\right]} \quad \underset{\text{Poincare}}{\left[\begin{array}{cc|cc} 0 & \theta & v_1 & t_1 \\ -\theta & 0 & v_2 & t_2 \\ \hline v_1 & v_2 & 0 & t_3 \\ 0 & 0 & 0 & 0 \end{array}\right]} \quad \underset{\text{Galilei}}{\left[\begin{array}{cc|cc} 0 & \theta & v_1 & t_1 \\ -\theta & 0 & v_2 & t_2 \\ \hline 0 & 0 & 0 & t_3 \\ 0 & 0 & 0 & 0 \end{array}\right]} \tag{8.25}$$

a. Compute the matrix infinitesimal generators for each.
b. Construct their commutation relations.
c. Decompose each Lie algebra into the standard form (8.20).
d. For each Lie algebra, express the generators in terms of the operators x_i, ∂_j.

e. For each Lie algebra, express the generators in terms of the boson operators $b_i^\dagger, b_j$, $1 \leq i,j \leq 3$.

7. In a semisimple Lie algebra the Cartan-Killing metric $g_{ij} = C_{ir}{}^s C_{js}{}^r$ is nonsingular and therefore the contravariant metric g^{ij} is well defined. Show that the bilinear operator $\mathcal{C}^2 = g^{ij} X_i X_j$ satisfies $[\mathcal{C}^2, X_k] = 0$. If there is one quadratic Casimir operator, it must therefore be proportional to $\mathcal{C}^2$.

8. Show that $C_{ijk} = C_{ij}{}^r g_{rk}$ is a third order antisymmetric tensor: $C_{ijk} = C_{jki} = C_{kij} = -C_{kji} = -C_{jik} = -C_{ikj}$. (Hint: use the Jacobi identity.)

9. Determine the structure of the Lie algebra defined by the following operators (c.f., Eq. (16.57):

$$\begin{aligned} X_{ij} &= x^i \partial_j - x^j \partial_i \\ Y_i &= 2t\frac{\partial}{\partial x^i} - x^i u \frac{\partial}{\partial u} \\ Z &= 2t\frac{\partial}{\partial t} + x^i \frac{\partial}{\partial x^i} - nu\frac{\partial}{\partial u} \end{aligned} \tag{8.26}$$