7

EXPonentiation

Contents

Linearization of a Lie group to form a Lie algebra introduces an enormous simplification in the study of Lie groups. The inverse process, reconstructing the Lie group from the Lie algebra, is carried out by the EXPonential map. We return to a more thorough study of the exponential map in this chapter. In particular, we address the three problems raised in Chapter 4: Does the EXPonential operation map the Lie algebra back onto the Lie group? Are Lie groups with isomorphic Lie algebras themselves isomorphic? Are there natural ways to parameterize Lie groups? We close this Chapter with a spectrum of applications of the EXPonential mapping in Physics. Applications include computing the dynamical evolution of quantum systems and their thermal expectation values.

7.1 Preliminaries

In Chapter 4 we saw how the linearization and EXPonentiation operations relate Lie groups and Lie algebras

$$\text{Lie groups} \quad \underset{EXP}{\overset{ln}{\rightleftharpoons}} \quad \text{Lie algebras} \tag{7.1}$$

At that time three questions, and their answers, were briefly raised about the EXPonential mapping. These questions are more thoroughly explored in this chapter.

The three questions, and their answers, are now presented.

Question 1: Does EXP map the Lie algebra onto the entire group?

Answer 1: No—but with some effort and insight, Yes.

Question 2: Are Lie groups with isomorphic Lie algebras isomorphic?

Answer 2: No—but there is a unique Lie group (covering group) and all others with the same Lie algebra are simply related to this unique simply connected Lie group.

Question 3: Are all mappings of the Lie algebra onto the Lie group identical?

Answer: No—but with care they are all analytically related to each other (by Baker–Campbell–Hausdorff formulas).

Each question is now discussed in more detail.

7.2 The Covering Problem

Cartan gave a simple example which showed that it is not always possible to map a Lie algebra onto the entire Lie group through a single mapping of the form $EXP(X)$. We consider the Lie group $SL(2;R)$ with Lie algebra $\mathfrak{sl}(2;R)$:

$$X = \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix} \in \mathfrak{sl}(2;R) \tag{7.2}$$

For this matrix algebra

$$\text{Tr } EXP(X) \geq -2 \tag{7.3}$$

Since $SL(2;R)$ contains group operations of the form

$$\begin{bmatrix} -\lambda & 0 \\ 0 & -1/\lambda \end{bmatrix} \qquad \lambda > 1 \tag{7.4}$$

with trace less than -2, a single exponential cannot map the Lie algebra onto the entire group.

The lower bound (-2) on the trace of the exponential can be seen as follows. Trace is an invariant under similarity transformation, so

$$\text{Tr } e^X = \text{Tr } S\ e^X\ S^{-1} = \text{Tr } e^{SXS^{-1}} \tag{7.5}$$

Now choose S to diagonalize (7.2). Since Tr $X = 0$, the eigenvalues λ can only have the form $\pm\theta$ or $\pm i\theta$ (θ real)

$$\text{Tr } e^{SXS^{-1}} \longrightarrow \begin{array}{ccll} 2\cosh\theta & \geq & 2 & \text{real eigenvalues} \\ 2\cos\theta & \geq & -2 & \text{imaginary eigenvalues} \end{array} \tag{7.6}$$

The problem in attempting to parameterize the Lie group with a single exponential map lies with the compact generators. The compact generators "go around" in circles, while the noncompact generators "go on forever." Furthermore, the compact generators always form a subgroup in the Lie group while the noncompact generators do not.

To make these cryptic statements less mysterious, we compute $EXP(X)$, with X given in (7.2), and find

$$EXP \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} \cosh r + a\sinh r/r & (b+c)\sinh r/r \\ (b-c)\sinh r/r & \cosh r - a\sinh r/r \end{bmatrix} & r^2 = a^2+b^2-c^2 > 0 \\ &= \begin{bmatrix} 1+a & b+c \\ b-c & 1-a \end{bmatrix} & a^2+b^2-c^2 = 0 \\ &= \begin{bmatrix} \cos r + a\sin r/r & (b+c)\sin r/r \\ (b-c)\sin r/r & \cos r - a\sin r/r \end{bmatrix} & -r^2 = a^2+b^2-c^2 < 0 \end{aligned} \tag{7.7}$$

The "light cone" structure of the (a, b, c) coordinate space of the Lie algebra is shown in Fig. 7.1. Points inside this cone map onto 2×2 rotation matrices in the group $SO(2)$. Points outside this cone map onto noncompact group elements. Points on the cone itself map onto some interesting group operations.

Many points inside the cone map onto the same operation in the subgroup $SO(2)$. To see this most easily set $a = b = 0$. Points on the c-axis map onto

$$(0,0,c) \longrightarrow \begin{bmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{bmatrix} \tag{7.8}$$

and therefore points separated by $2\pi n$ along the c-axis map onto the same group operation in $SO(2) \subset SL(2; R)$. The complementary sub-

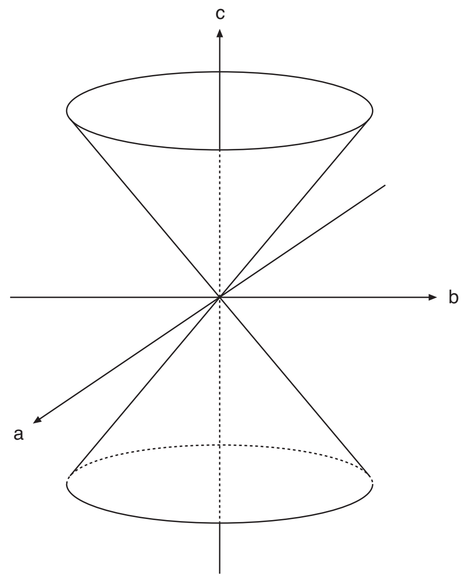

Fig. 7.1. "Light cone" for $SL(2;R)$.

space $(a,b,0)$ maps onto noncompact group operations in $SL(2;R)$

$$(a,b,0) \longrightarrow \begin{bmatrix} \cosh r + (a/r)\sinh r & (b/r)\sinh r \\ (b/r)\sinh r & \cosh r - (a/r)\sinh r \end{bmatrix} \qquad r^2 = a^2 + b^2 \tag{7.9}$$

that are not recurrent. In fact, this 2-parameter set of group operations has the same topology as the subspace $(a,b,0)$ in the Lie algebra. We show this below.

In addition to providing an example that shows that $EXP(X)$ may not map onto the group when the group is noncompact, Cartan provided a theorem that a succession of mappings would always do the job. For simple groups (Chapter 9) the product of two exponential mappings—one of the compact generators, the other of the noncompact generators—will map the algebra onto the group. To separate compact and noncompact generators we use the Cartan-Killing inner product (4.43) computed in the defining matrix representation (7.2)

$$(X,X) = \text{Tr } X^2 = 2(a^2 + b^2 - c^2) \tag{7.10}$$

The metric is positive definite on noncompact generators and negative definite on noncompact generators. This decomposition in the Lie algebra leads to

$$\begin{array}{ccc} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} & + & \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \\ \\ EXP \downarrow & \downarrow & \downarrow EXP \\ \\ \begin{bmatrix} z+y & x \\ x & z-y \end{bmatrix} & \times & \begin{bmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{bmatrix} \end{array} \tag{7.11}$$

For simplicity we have set $z = \cosh r \geq 1$ and $(x, y) = (b, a) \sinh(r)/r$, $r^2 = a^2 + b^2$. We observe that

$$z^2 - x^2 - y^2 = 1 \tag{7.12}$$

which is just the upper sheet of the two-sheeted hyperboloid H^2_{2+}, shown in Fig. 7.2(a). This sheet is topologically equivalent to the space R^2, the plane that it covers. For the compact generator only a small range of parameter values $-\pi \leq c \leq +\pi$ is required to map the subalgebra onto the subgroup $SO(2)$.

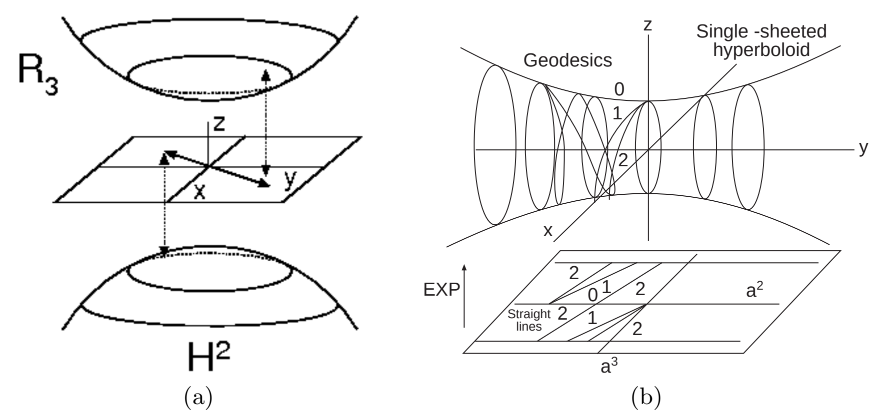

Fig. 7.2. Two-sheeted and single sheeted hyperboloids. Both are quotients (coset spaces) of $SL(2; R)$ by one of its two inequivalent types of subgroups, $SO(2)$ and $SO(1, 1)$.

The connection of $SL(2; R)$ with geometry may be unexpected, but it is not unique to $SL(2; R)$. Moreover, other geometric structures are obtained by exponentiating different subspaces of the algebra $\mathfrak{sl}(2; R)$.

For example

$$\begin{array}{ccccc} \begin{bmatrix} a & c \\ -c & -a \end{bmatrix} & & + & & \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \\ \\ EXP \downarrow & & \downarrow & & \downarrow EXP \\ \\ \begin{bmatrix} z+y & x \\ -x & z-y \end{bmatrix} & & \times & & \begin{bmatrix} \cosh b & \sinh b \\ \sinh b & \cosh b \end{bmatrix} \end{array} \tag{7.13}$$

In this expression for the coset representatives (recall the definition of cosets, or quotients of a group by a subgroup, given in Chapter 1) the three real parameters (x, y, z) obey

$$z^2 + x^2 - y^2 = 1 \tag{7.14}$$

This equation describes the surface of the single-sheeted hyperboloid H_1^2, shown in Fig. 7.2(b). Many other algebraic surfaces can be obtained from Lie algebras in this way.

We point out that the EXPonential function maps the sum of two subspaces in the algebra into the product of the associated group operations [cf. (7.11) and (7.13)]. We can regard one of the subspaces as the difference between the full space (Lie algebra) and the other subspace (subalgebra). The EXPonential maps the difference of spaces into the quotient of group operations. For example

$$\begin{array}{ccccc} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} & = & \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix} & - & \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix} \\ \\ EXP \downarrow & & EXP \downarrow & \downarrow EXP & \downarrow EXP \\ \\ \begin{bmatrix} z+y & x \\ x & z-y \end{bmatrix} & = & SL(2;R) & / & SO(2) \end{array} \tag{7.15}$$

The "quotient" means that all elements in $SL(2;R)$ that differ only by multiplication by a 2×2 rotation matrix on the right are identified with each other. It is convenient to choose one such group operation to represent this entire set. This group operation [on the left in (7.15)] is called a **coset representative**. The entire one-dimensional set parameterized by c, $0 \leq c < 2\pi$, is the coset. In the theory of Lie groups, cosets and coset representatives are usually interesting spaces.

From this discussion we conclude that the group $SL(2;R)$ can be viewed in various different ways involving coset decompositions. In the parameterization (7.11) obtained from the coset decomposition $SL(2;R)/SO(2)$, the manifold parameterizing the group is the direct product of the upper sheet of the two-sheeted hyperboloid with a circle. Since the upper sheet of a two-sheeted hyperboloid is topologically (but not geometrically!) equivalent to R^2, the manifold that parameterizes $SL(2;R)$ is the direct product $R^2 \times S^1$. A different parameterization (7.13) based on the coset decomposition $[SL(2;R)/SO(1,1)] \times SO(1,1)$ $(SO(1,1) \simeq R^1)$ shows that the manifold underlying $SL(2;R)$ is the direct product of the single-sheeted hyperboloid (equivalent to $R^1 \times S^1$) with R^1. This product is once again $R^2 \times S^1$.

Since matrix Lie groups are defined by algebraic constraints, so are their subgroups and quotient spaces. This means that the underlying manifold for each matrix Lie group is an algebraic manifold. For example, for subgroups of $GL(n;R)$ the underlying manifold is a subset of R^N, $N = n^2$, that is defined by algebraic constraints. This manifold can be expressed as products of algebraic submanifolds, each parameterizing a subgroup or coset.

We conclude this discussion of the covering problem by stating a theorem due to Cartan. It is always possible to map a Lie algebra onto its Lie group with a product of exponential mappings. In fact, if the algebra can be written in the form

$$\begin{array}{cccccc} \text{algebra} & = & \text{noncompact generators} & + & \text{compact generators} \\ EXP\downarrow & & EXP\downarrow & \downarrow & \downarrow EXP \\ \text{group} & = & \text{coset representatives} & \times & \text{compact subgroup} \end{array} \tag{7.16}$$

then the product of two exponential maps, one of the noncompact generators, the other of the compact generators (which form a subalgebra), maps onto the entire Lie group. The algebraic manifold parameterizing the EXPonential of the noncompact generators is R^m, for suitable m (= number of noncompact generators). The manifold that parameterizes the EXPonential of the compact generators is compact.

7.3 The Isomorphism Problem and the Covering Group

Isomorphic Lie groups have isomorphic Lie algebras, but two Lie groups with isomorphic Lie algebras need not be isomorphic. To illustrate this point, we treat the groups $SO(2,1)$ and $SU(1,1)$ with Lie algebras

$$\mathfrak{so}(2,1) = \begin{bmatrix} 0 & a_3 & a_2 \\ -a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{bmatrix} \qquad \mathfrak{su}(1,1) = \frac{i}{2}\begin{bmatrix} b_3 & ib_1+b_2 \\ ib_1-b_2 & -b_3 \end{bmatrix} \tag{7.17}$$

The Lie algebras are isomorphic but the Lie groups are not. The group $SO(2,1)$ is covered by the map

$$\begin{array}{ccc} \begin{bmatrix} 0 & 0 & a_2 \\ 0 & 0 & a_1 \\ a_2 & a_1 & 0 \end{bmatrix} & + & \begin{bmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ EXP\downarrow & \downarrow & \downarrow EXP \\ [SO(2,1)/SO(2)] & \times & \begin{bmatrix} \cos a_3 & \sin a_3 & 0 \\ -\sin a_3 & \cos a_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{7.18}$$

The group $SU(1,1)$ is similarly covered by

$$\begin{array}{ccc} \frac{i}{2}\begin{bmatrix} 0 & ib_1+b_2 \\ ib_1-b_2 & 0 \end{bmatrix} & + & \frac{i}{2}\begin{bmatrix} b_3 & 0 \\ 0 & -b_3 \end{bmatrix} \\ EXP\downarrow & \downarrow & \downarrow EXP \\ [SU(1,1)/U(1)] & \times & \begin{bmatrix} e^{+ib_3/2} & 0 \\ 0 & e^{-ib_3/2} \end{bmatrix} \end{array} \tag{7.19}$$

The cosets $SO(2,1)/SO(2)$ and $SU(1,1)/U(1)$ are both isomorphic to R^2 and have a 1:1 correspondence. The subgroups $SO(2)$ and $U(1)$ have a 2:1 correspondence. This can be seen by increasing b_3 by 2π and noticing that the 2×2 unitary matrix in (7.19) goes to its negative: $U(b_3+2\pi) = -U(b_3)$. However, increasing a_3 by 2π does not change the 3×3 rotation matrix in (7.18). The 2:1 correspondence can be seen in a better and simpler way. One can ask: How far along a straight line through the origin does one have to go to return to the identity? For the subgroup $U(1) \subset SU(1,1)$ the result is 4π; for the subgroup $SO(2) \subset SO(2,1)$ the result is 2π. Therefore, $SU(1,1)$ is 'twice as large' as $SO(2,1)$. More formally, there is a $2 \to 1$ homomorphism of $SU(1,1)$ onto $SO(2,1)$.

Once again there is a result due to Cartan that is useful for comparing

Lie groups that have isomorphic Lie algebras. Since the noncompact parts of the Lie algebras map to elements of the group with the topology of a Euclidean space, a comparison of the largest compact subgroups of the two groups is sufficient to determine if the groups are isomorphic.

The most familiar example of nonisomorphic groups with isomorphic Lie algebras is the pair $SO(3)$ and $SU(2)$ with algebras

$$\mathfrak{so}(3) = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \qquad \mathfrak{su}(2) = \frac{i}{2}\begin{bmatrix} b_3 & b_1 - ib_2 \\ b_1 + ib_2 & -b_3 \end{bmatrix} \tag{7.20}$$

It can be checked that all points in the interior of a sphere of radius $\sqrt{a_1^2 + a_2^2 + a_3^2} \leq \pi$) map onto $SO(3)$ provided antipodal points at $|\mathbf{a}| = \pi$ are identified

$$\pi(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \sim -\pi(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

with θ = latitude, ϕ = longitude on a sphere. For $SU(2)$ all points within a sphere of radius 2π [$\sqrt{b_1^2 + b_2^2 + b_3^2} < 2\pi$] are mapped onto distinct elements of $SU(2)$ and all points at a radius of 2π are mapped onto $-I_2$. There is an easier way to verify the $2 \rightarrow 1$ nature of the map $SU(2)$ to $SO(3)$. All straight lines through the origin of the Lie algebra are equivalent (since the algebra has rank 1, cf. Chapter 8). Therefore, we can compare how a convenient line (z-axis) maps onto the two groups. This has already been done for the comparison of $SU(1,1)$ with $SO(2,1)$.

Another convenient parameterization of $SO(3)$ and $SU(2)$ can be used to show the 2:1 map. This is analogous to (7.18)

$$\begin{array}{ccccc} \mathfrak{so}(3) & = & \begin{bmatrix} 0 & 0 & -a_2 \\ 0 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} & + & \begin{bmatrix} 0 & a_3 & 0 \\ -a_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & EXP\downarrow & \downarrow EXP & & \downarrow EXP \\ & & \begin{bmatrix} \star & \star & -x \\ \star & \star & y \\ x & -y & z \end{bmatrix} & \times & \begin{bmatrix} \cos a_3 & \sin a_3 & 0 \\ -\sin a_3 & \cos a_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{7.21}$$

A similar parameterization for $SU(2)$ gives

$$\begin{array}{ccccc}
\mathfrak{su}(2) & = & \frac{i}{2}\left[\begin{array}{cc} 0 & b_1 - ib_2 \\ b_1 + ib_2 & 0 \end{array}\right] & + & \frac{i}{2}\left[\begin{array}{cc} b_3 & 0 \\ 0 & -b_3 \end{array}\right] \\
 & & EXP \downarrow \qquad\qquad \downarrow EXP & & \downarrow EXP \\
 & & \left[\begin{array}{cc} z' & i(x'-iy') \\ i(x'+iy') & z' \end{array}\right] & \times & \left[\begin{array}{cc} e^{ib_3/2} & 0 \\ 0 & e^{-ib_3/2} \end{array}\right]
\end{array} \tag{7.22}$$

The coset representatives $SO(3)/SO(2)$, parameterized by the real numbers (x, y, z) subject to $x^2+y^2+z^2 = 1$, and $SU(2)/U(1)$, parameterized by the real numbers (x', y', z') subject to $x'^2 + y'^2 + z'^2 = 1$, are in 1:1 correspondence with points in the same geometric space—a sphere in this case. As a result, the 2:1 nature of the mapping $SU(2) \rightarrow SO(3)$ can be seen from the 2:1 nature of the rotations around the “3” axis.

Yet another result of Cartan establishes a unique connection between Lie groups and Lie algebras. There is a unique Lie algebra for every Lie group. For each Lie algebra there may be many inequivalent Lie groups. But there is a unique Lie group, $\overline{G}$, called the **universal covering group**. This group is simply connected: every loop starting and ending at the identity can be continuously deformed to the identity. Moveover, every other Lie group with this Lie algebra is either identical to this simply connected Lie group, or else has the form of a quotient $\overline{G}/D$, where D is a discrete invariant subgroup of $\overline{G}$ whose elements commute with $\overline{G}$: $gd_i = d_i g$ for $d_i \in D$ and $g \in G$. If $\overline{G}$ is compact it is useful to determine the largest such subgroup, D_{MAX}, of $\overline{G}$. Then all compact Lie groups with the same Lie algebra as $\overline{G}$ are obtained by “dividing” $\overline{G}$ by all possible subgroups of D_{MAX}, as shown in Fig. 7.3.

For simple matrix Lie groups G, computation of the discrete invariant subgroup D is a simple matter. The only discrete group operations d_i that commute with all $g \in G$ are multiples of the identity, by Schur’s Lemma

$$g \in G, \qquad d_i \in D, \qquad G \text{ simple}, \qquad gd_i = d_i g \Rightarrow d_i = \lambda I_n \tag{7.23}$$

Two Lie groups with isomorphic Lie algebras are **locally isomorphic**. If G_1 and G_2 have the same Lie algebra, $G_1 = \overline{G}/D_1$ and G_1 is locally isomorphic with $\overline{G}$. By the same argument G_2 is locally isomorphic with $\overline{G}$, and therefore also with G_1. If $\overline{G}$ is compact, G_1 and G_2 are also locally isomorphic with $\overline{G}/D_{\mathrm{MAX}}$, which is a universal image Lie group.

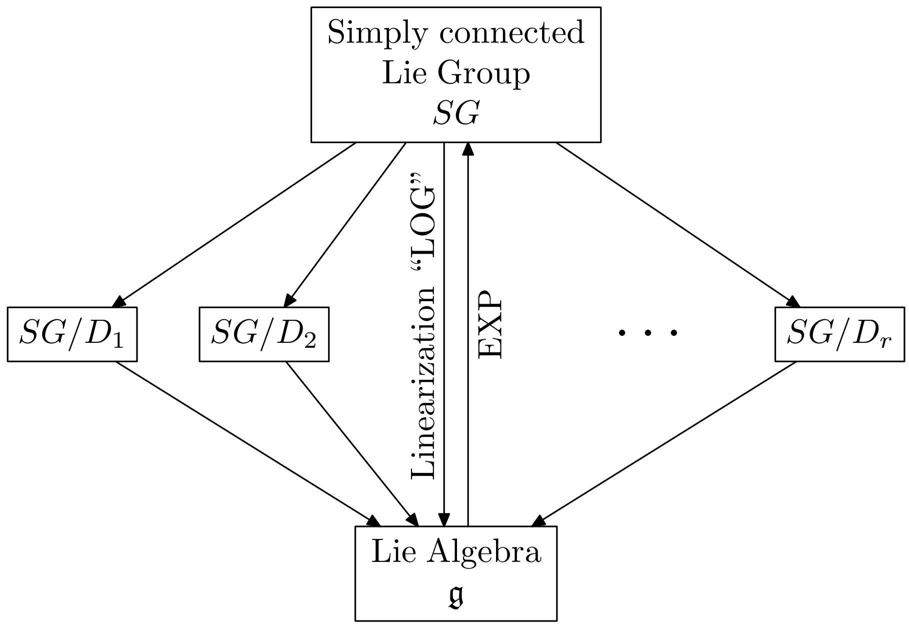

Fig. 7.3. Cartan's covering theorem. There is a unique correspondence between Lie algebras $\mathfrak{g}$ and simply connected Lie groups $\overline{G}$. Every other Lie group with this Lie algebra is a quotient of the universal covering group by one of the discrete invariant subgroups D_i of $\overline{G}$.

$$G_1 = \overline{G}/D_1 \rightarrow \overline{G}/D_{\mathrm{MAX}} \leftarrow \overline{G}/D_2 = G_2$$

Example: The maximal discrete invariant subgroup of $SU(2)$ consists of matrices λI_2 that obey $\lambda^*\lambda = 1$ and $\det(\lambda I_2) = +1$, so that $\lambda = \pm 1$. D is the two-element subgroup $D = \{I_2, -I_2\}$. For the locally isomorphic Lie group $SO(3)$, $D = \lambda I_3$ with $\lambda = +1$. As a result $SU(2)/\{I_2, -I_2\} = SO(3)/I_3 = SO(3)$. For each group operation in $SO(3)$ there are two matrices in $SU(2)$ that differ in sign.

Remark: The maximal compact subgroups $SO(2)$ of $SO(2,1)$ and $U(1)$ of $SU(1,1)$ are not simply connected. Their simply connected covering group is R^1, the group of translations of the line. The covering group $\overline{SO(2,1)} = \overline{SU(1,1)}$ has no compact subgroup at all. Its underlying group manifold is $\overline{SO(2,1)/SO(2)} \times \overline{SO(2)} = \overline{SU(1,1)/U(1)} \times \overline{U(1)} = [SO(2,1)/SO(2)] \times \overline{SO(2)} = SU(1,1)/U(1) \times \overline{U(1)} = R^2 \times R^1$. It is the only group we will encounter in this book that is not a matrix group.

The covering group $\overline{SO(2,1)} = \overline{SU(1,1)}$ has many discrete invariant subgroups but does not have a maximal discrete invariant subgroup.

7.4 The Parameterization Problem and BCH Formulas

A Lie algebra can be mapped onto a Lie group in many different ways. More generally, points in the underlying topological space can be identified with group operations in an unlimited number of ways. These different parameterizations of a Lie group can be related to each other by analytic transformations in a way that can often be used to simplify computations. Reparameterization formulas involving products of exponentials of operators are called **Baker-Campbell-Hausdorff** (BCH) formulas for historical reasons. Once again we illustrate by example rather than present a general theory.

As a first example we consider the affine group of transformations of the line, and two different parameterizations of this group. One maps a point (x, y) in the right half-plane R^2_+ into the group operator

$$(x,y) \rightarrow \left[\begin{array}{cc} x & y \\ 0 & 1 \end{array}\right] \qquad x > 0 \tag{7.24}$$

The second maps a point (w, z) in R^2 into the group under the EXPonential map

$$(w,z) = EXP \left[\begin{array}{cc} w & z \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} e^w & (e^w-1)z/w \\ 0 & 1 \end{array}\right] \tag{7.25}$$

We ask: Is there some mapping of the half-plane R^2_+ $(x > 0, y)$ into R^2 (w, z) that makes these two group operations, and the group multiplication laws derived from them, equivalent? The transformation between these two parameterizations is obtained by identifying matrix elements:

$$(x,y) \rightarrow \left[\begin{array}{cc} x & y \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} e^w & (e^w-1)z/w \\ 0 & 1 \end{array}\right] \leftarrow (w,z) \tag{7.26}$$

The mapping ("diffeomorphism") between the half-plane R^2_+ and the plane R^2, or the coordinates (x, y) and (w, z) is

$$\begin{array}{rcl} x & = & e^w \\ y & = & (e^w-1)z/w = z\left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \end{array} \tag{7.27}$$

and the inverse transformation is

$$\begin{aligned} w &= \ln x \\ z &= y\ln(x)/(x-1) \qquad z=0 \quad \text{for} \quad x=1 \end{aligned} \tag{7.28}$$

These transformations are analytic for $x > 0$.

As a second example we treat the algebra of upper triangular 3×3 matrices

$$\begin{bmatrix} 0 & l & \delta \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} = lX_l + rX_r + \delta X_\delta \tag{7.29}$$

The commutation relations of these three generators are

$$[X_l, X_r] = X_\delta, \qquad [X_l, X_\delta] = [X_r, X_\delta] = 0 \tag{7.30}$$

The single-mode photon operators $a, a^\dagger, I$ obey isomorphic commutation relations

$$\left[a, a^\dagger\right] = I, \qquad [a, I] = \left[a^\dagger, I\right] = 0 \tag{7.31}$$

The two Lie algebras are isomorphic under

$$\begin{aligned} X_l &\rightarrow a \\ X_r &\rightarrow a^\dagger \\ X_\delta &\rightarrow I \end{aligned} \tag{7.32}$$

For many quantum computations it is convenient to relate several different parameterizations of the Lie group. For example, the following "disentangling" results are useful

$$\begin{array}{ccc} & e^{ra^\dagger + la + \delta I} & \\ \| & & \| \\ e^{r'a^\dagger} e^{\delta' I} e^{l'a} & = & e^{l''a} e^{\delta'' I} e^{r''a^\dagger} \end{array} \tag{7.33}$$

This reparameterization computation can be carried out using 3×3

matrices

$$
\begin{array}{ccc}
EXP\begin{bmatrix} 0 & l & \delta \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} & = & \begin{bmatrix} 1 & l & \delta+\frac{1}{2}lr \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \\
\| & & \| \\
e^{r'a^\dagger}e^{\delta' I}e^{l'a} \rightarrow \begin{bmatrix} 1 & l' & \delta' \\ 0 & 1 & r' \\ 0 & 0 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & l'' & \delta''+l''r'' \\ 0 & 1 & r'' \\ 0 & 0 & 1 \end{bmatrix} \leftarrow e^{l''a}e^{\delta'' I}e^{r''a^\dagger}
\end{array} \tag{7.34}
$$

We see immediately that $l = l' = l''$, $r = r' = r''$, $\delta' = \delta + \frac{1}{2}lr = \delta'' + l''r''$, and obtain the Heisenberg identity (for $\delta = 0$)

$$
e^{ra^\dagger}e^{+\frac{1}{2}lrI}e^{la} = e^{ra^\dagger+la} = e^{la}e^{-\frac{1}{2}lrI}e^{ra^\dagger} \tag{7.35}
$$

As a third example we treat the four-parameter Lie group of solvable 3×3 matrices with Lie algebra

$$
\begin{bmatrix} 0 & l & \delta \\ 0 & \eta & r \\ 0 & 0 & 0 \end{bmatrix} = \eta X_\eta + lX_l + rX_r + \delta X_\delta \tag{7.36}
$$

This Lie algebra is isomorphic with the Lie algebra spanned by the four single-mode photon operators $\hat{n} = a^\dagger a, a, a^\dagger, I$ under the identification

$$
\begin{array}{ccc}
X_\eta & \rightarrow & \hat{n} \\
X_l & \rightarrow & a \\
X_r & \rightarrow & a^\dagger \\
X_\delta & \rightarrow & I
\end{array} \tag{7.37}
$$

If for some reason $EXP(\eta a^\dagger a + ra^\dagger + la)$ needed to be rewritten in the more conveniently ordered form $EXP(r'a^\dagger)EXP(\eta' a^\dagger a + \delta' I)EXP(l'a)$, then the reparameterization computation could be carried out in the 3×3

matrix representation

$$EXP(\eta a^\dagger a + ra^\dagger + la) = EXP(r'a^\dagger)EXP(\eta' a^\dagger a + \delta' I)EXP(l'a)$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\begin{bmatrix} 1 & (e^\eta - 1)l/\eta & (e^\eta - 1 - \eta)lr/\eta^2 \\ 0 & e^\eta & (e^\eta - 1)r/\eta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & l' & \delta' \\ 0 & e^{\eta'} & r' \\ 0 & 0 & 1 \end{bmatrix} \tag{7.38}$$

By inspection, we obtain

$$\begin{array}{rclcrcl} \eta' & = & \eta & \qquad & l' & = & (e^\eta - 1)l/\eta \\ \delta' & = & (e^\eta - 1 - \eta)lr/\eta^2 & & r' & = & (e^\eta - 1)r/\eta \end{array} \tag{7.39}$$

If it is necessary to compute the expectation value of $EXP(\eta a^\dagger a + ra^\dagger + la)$ in the ground state of the harmonic oscillator, then

$$\langle 0|e^{\eta a^\dagger a + ra^\dagger + la}|0\rangle = \langle 0|e^{r'a^\dagger} e^{\eta' a^\dagger a + \delta' I} e^{l'a}|0\rangle \tag{7.40}$$

Since $e^{l'a}|0\rangle = |0\rangle$, $\langle 0|e^{r'a^\dagger} = \langle 0|$ and $e^{\eta' a^\dagger a}|0\rangle = |0\rangle$, the expectation value is

$$\langle 0 \left| e^{\eta a^\dagger a + ra^\dagger + la} \right| 0\rangle = e^{\delta'} = EXP\left(\frac{(e^\eta - 1 - \eta)lr}{\eta^2}\right) \tag{7.41}$$

This result is not easy to derive by other techniques.

As a final example we treat the Lie algebra $\mathfrak{su}(2)$. First, we show how to compute the matrix element of an arbitrary rotation between 'ground state' wave functions ($|j, -j\rangle$)

$$\langle \begin{array}{c} j \\ -j \end{array} |e^{i\theta\cdot\mathbf{J}}| \begin{array}{c} j \\ -j \end{array} \rangle \tag{7.42}$$

This expectation would be easy to compute if the exponential were written in a "normally ordered form"

$$(7.42) = \langle \begin{array}{c} j \\ -j \end{array} |e^{i\theta'_+ J_+} e^{i\theta'_z J_z} e^{i\theta'_- J_-}| \begin{array}{c} j \\ -j \end{array} \rangle \tag{7.43}$$

Since

$$e^{i\theta'_- J_-}| \begin{array}{c} j \\ -j \end{array} \rangle = (I + i\theta'_- J_- + \cdots)| \begin{array}{c} j \\ -j \end{array} \rangle = | \begin{array}{c} j \\ -j \end{array} \rangle \tag{7.44}$$

with a similar result for J_+ acting on the left, we find

$$(7.42) = \langle \begin{array}{c} j \\ -j \end{array} |e^{i\theta'_z J_z}| \begin{array}{c} j \\ -j \end{array} \rangle = e^{-ij\theta'_z} \tag{7.45}$$

The only problem that remains is to compute θ'_z as a function of θ. To do this we carry out the operator disentangling calculations in the faithful 2×2 matrix representation $\mathbf{J} \rightarrow \frac{1}{2}\sigma$, where σ are the Pauli spin matrices (5.14).

$$e^{i\theta\cdot\mathbf{J}} \rightarrow EXP\frac{i}{2}\left[\begin{array}{cc} \theta_z & \theta_x - i\theta_y \\ \theta_x + i\theta_y & -\theta_z \end{array}\right] =$$

$$\left[\begin{array}{cc} \cos(\theta/2) + i(\theta_z/\theta)\sin(\theta/2) & i[(\theta_x - i\theta_y)/\theta]\sin(\theta/2) \\ i[(\theta_x + i\theta_y)/\theta]\sin(\theta/2) & \cos(\theta/2) - i(\theta_z/\theta)\sin(\theta/2) \end{array}\right] \tag{7.46}$$

In a similar way we find

$$\begin{array}{ccc} EXP(i\theta'_+ J_+) & EXP(i\theta'_z J_z) & EXP(i\theta'_- J_-) \\ \downarrow & \downarrow & \downarrow \\ \left[\begin{array}{cc} 1 & i\theta'_+ \\ 0 & 1 \end{array}\right] & \left[\begin{array}{cc} e^{i\theta'_z/2} & 0 \\ 0 & e^{-i\theta'_z/2} \end{array}\right] & \left[\begin{array}{cc} 1 & 0 \\ i\theta'_- & 1 \end{array}\right] \end{array} =$$

$$\left[\begin{array}{cc} e^{i\theta'_z/2} - \theta'_+\theta'_- e^{-i\theta'_z/2} & i\theta'_+ e^{-i\theta'_z/2} \\ i\theta'_- e^{-i\theta'_z/2} & e^{-i\theta'_z/2} \end{array}\right] \tag{7.47}$$

where $\theta_\pm = \theta_1 \pm i\theta_2$. Comparison of the two matrices gives immediately

$$e^{-i\theta'_z/2} = \cos(\theta/2) - i(\theta_z/\theta)\sin(\theta/2) \tag{7.48}$$

As a result, we find

$$\langle \begin{array}{c} j \\ -j \end{array} |e^{i\theta\cdot\mathbf{J}}| \begin{array}{c} j \\ -j \end{array} \rangle = e^{-ij\theta'_z} = (e^{-i\theta'_z/2})^{2j} = [\cos(\theta/2) - i(\theta_z/\theta)\sin(\theta/2)]^{2j} \tag{7.49}$$

This result is useful in the field of quantum optics but is not easy to compute by other means.

To illustrate the use of Baker–Campbell–Hausdorff formulas in another situation we compute the matrix elements

$$\langle \begin{array}{c} j \\ j \end{array} |J_+^k J_-^k| \begin{array}{c} j \\ j \end{array} \rangle \tag{7.50}$$

To do this we construct a generating function

$$\langle \begin{array}{c} j \\ j \end{array} |e^{\alpha J_+} e^{\beta J_-}| \begin{array}{c} j \\ j \end{array} \rangle = \sum_{rs} \frac{\alpha^r \beta^s}{r!s!} \langle \begin{array}{c} j \\ j \end{array} |J_+^r J_-^s| \begin{array}{c} j \\ j \end{array} \rangle \tag{7.51}$$

The operator product $e^{\alpha J_+} e^{\beta J_-}$ is written in normally ordered form

$EXP(\beta' J_-)\ EXP(n' J_z) EXP(\alpha' J_+)$ and the parameters α', β', n' computed. We find

$$\langle \begin{matrix} j \\ j \end{matrix} | e^{\beta' J_-} e^{n' J_z} e^{\alpha' J_+} | \begin{matrix} j \\ j \end{matrix} \rangle = e^{jn'} = (1+\alpha\beta)^{2j} \tag{7.52}$$

By expanding $(1+\alpha\beta)^{2j}$ and invoking analyticity, we find

$$\langle \begin{matrix} j \\ j \end{matrix} | J_+^r J_-^s | \begin{matrix} j \\ j \end{matrix} \rangle = \frac{(2j)! r!}{(2j-r)!} \delta_{rs} \tag{7.53}$$

Other matrix elements of products of angular momentum operators can similarly be constructed from appropriate generating functions.

The general computational procedure should now be clear. Given a Lie algebra of operators and the associated group operations that are exponentials of the elements in the Lie algebra, it is possible to carry out all calculations in either the algebra or the group using a faithful matrix representation of the operator algebra. In general, the smaller the size of the matrices, the easier the computation.

For example, if operators $\mathcal{A}, \mathcal{B}$ belong to two complementary subspaces in some operator Lie algebra $\mathfrak{g}$ then the operator product $e^{\mathcal{A}} e^{\mathcal{B}}$ can be reparameterized as $e^{\mathcal{B}'} e^{\mathcal{A}'}$ ($\mathcal{A}', \mathcal{B}'$ different operators in the same subspaces as $\mathcal{A}, \mathcal{B}$) by:

(i) Finding a faithful matrix representation of the operator algebra;

(ii) Identifying the operators $\mathcal{A}, \mathcal{B}$ with matrices A, B;

(iii) Carrying out the matrix calculations $e^A e^B$ and $e^{B'} e^{A'}$;

(iv) Determining the matrices A', B' by comparing matrix elements; and

(v) Using the isomorphism $A' \leftrightarrow \mathcal{A}'\ B' \leftrightarrow \mathcal{B}'$.

This procedure will produce a local analytic reparameterization $(\mathcal{A}, \mathcal{B}) \leftrightarrow (\mathcal{A}', \mathcal{B}')$. If the matrix group used to construct this reparameterization is simply connected (the covering group) the analytic reparameterization will be global. Otherwise, some care must be taken to compare the maximal discrete invariant subgroups of the operator group and the matrix group. When the operators $\mathcal{A}, \mathcal{B}, ...$ are related to matrices $A, B, \ldots$ by a matrix - operator mapping (c.f., Chapter 6) $\mathcal{A} \leftrightarrow A$, the disentangling formulas can be constructed using the matrices $A, B, \ldots$.

7.5 EXPonentials and Physics

By the greatest good fortune — or perhaps by the deepest possible connections between mathematics and physics — the exponential function also plays a most fundamental role in physics. In fact, it plays two roles: one in dynamics and another in equilibrium statics (thermo"dynamics"). More fundamental yet, these two roles are related by analytic continuation ("Wick rotation"). We describe both roles in this section, in terms of two examples: one related to fermions, the other related to bosons.

7.5.1 Dynamics

The dynamics of quantum systems is governed by the time-dependent Schrödinger equation:

$$H|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle \tag{7.54}$$

The state of the system at time $t+\delta t$ is related to the state at time t by

$$|\psi(t+\delta t)\rangle = (I - \frac{i}{\hbar}H\delta t)|\psi(t)\rangle = e^{-\frac{i}{\hbar}H\delta t}|\psi(t)\rangle \tag{7.55}$$

The exponential is unitary since the hamiltonian operator H is hermitian. The state $|\psi(t_f)\rangle$ at some final time t_f is related to the state at initial time t_i by $|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle$. The finite time unitary operator is built up from small displacements

$$U(t_f,t_i) = U(t_f,t_f-\delta t)\ \dots\ U(t_i+2\delta t, t_i+\delta t)U(t_i+\delta t, t_i) =$$

$$\prod\ U(t_i+(n+1)\delta t, t_i+n\delta t) = \text{``}\int_{t_i}^{t_f}\text{''}\ U(\tau)d\tau = T\ \int_{t_i}^{t_f} e^{-\frac{i}{\hbar}H(t)}\ dt \tag{7.56}$$

Care must be taken with the formal integration in this equation, as in general $H(t')$ does not commute with $H(t)$, $t' \neq t$. It is for this reason that the symbol "T" preceeds the integral: this signifies a time-ordered product. If the Hamiltonian is not explicitly time-dependent then the integral in Eq. (7.56) reduces to an everyday Riemann integral.

Expression of the time dependence in terms of a unitary evolution operator is useful for two very different reasons:

(i) The evolution is decoupled from the initial state.
(ii) In special cases it is very simple to construct this unitary evolution operator when it would be much more difficult to construct the evolution of a specific state.

The second case becomes important when the Hamiltonian is a linear superposition of operators that exist in a Lie algebra. In that case the unitary operator is a group operation, and it may be possible to find some shortcuts for its computation. We give two examples.

Example 1. A Hamiltonian acts in a $2j+1$ dimensional space through a set of three operators $J_z, J_\pm$ that obey angular momentum commutation relations. We wish to determine the evolution of some particular state $|j, m_j\rangle$. The Hamiltonian is

$$H = \epsilon(t)J_z + \alpha(t)J_+ + \alpha^*(t)J_- \stackrel{j\to\frac{1}{2}}{\longrightarrow} \left[\begin{array}{cc} \frac{1}{2}\epsilon(t) & \alpha(t) \\ \alpha^*(t) & -\frac{1}{2}\epsilon(t) \end{array}\right] \qquad (7.57)$$

The unitary operator acting in the $2j+1$ dimensional space is a unitary representation of some operation in the group $SU(2)$. It is simpler to determine how $g(t) \in SU(2)$ evolves, and then construct its unitary representation, than it is to determine the time evolution of the $(2j+1) \times (2j+1)$ unitary matrix. Specifically, the equation of motion *in the group* is

$$\frac{d}{dt}\left[\begin{array}{cc} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{array}\right] = -\frac{i}{\hbar}\left[\begin{array}{cc} \frac{1}{2}\epsilon(t) & \alpha(t) \\ \alpha^*(t) & -\frac{1}{2}\epsilon(t) \end{array}\right]\left[\begin{array}{cc} a(t) & b(t) \\ -b^*(t) & a^*(t) \end{array}\right] \qquad (7.58)$$

After some algebraic manipulation this matrix equation reduces to two equations for the complex coefficients $a(t)$ and $b(t)$ or three equations for the real coefficients of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$. These are first order equations and can be solved by standard integration methods (e.g., RK4). The initial conditions are $a(t_i) = 1, b(t_i) = 0$. The final 2×2 unitary matrix is determined by $a(t_f), b(t_f)$. This is a group operation in $SU(2)$ that can subsequently be mapped into the $(2j+1) \times (2j+1)$ unitary irreducible representation of this group. At this point the problem is solved, independent of the initial state $|\psi(t_i)\rangle$.

Example 2. As a second example we treat a Hamiltonain that is a linear combination of the boson number, creation, and annihilation operators (and their commutator):

$$H = \omega(t)a^\dagger a + \alpha(t)a^\dagger + \alpha^*(t)a + \delta(t)I \rightarrow \left[\begin{array}{ccc} 0 & \alpha^*(t) & \delta(t) \\ 0 & \omega(t) & \alpha(t) \\ 0 & 0 & 0 \end{array}\right] \qquad (7.59)$$

The boson operators act as a hermitian superposition in an infinite-dimensional space with basis vectors $|n\rangle$, $n = 0, 1, 2, \ldots$. The matrix on the right is a faithful finite-dimensional *nonhermitian* representation of these operators. The most general unitary operator that can be constructed from these operators is $U = EXP(i[n(t)a^\dagger a + r(t)a^\dagger + r^*(t)a + d(t)I])$. This exponential is easy to compute in the faithful 3×3 nonunitary representation. The matrix equation of motion analogous to Eq. (7.58) is explicitly

$$\frac{d}{dt}\left[\begin{array}{ccc} 1 & r^*\frac{(e^{in}-1)}{(in)} & r^*r\left(\frac{(e^{in}-1-in)}{(in)^2}\right)+id \\ 0 & e^{in} & r\frac{(e^{in}-1)}{(in)} \\ 0 & 0 & 1 \end{array}\right] =$$

$$-\frac{i}{\hbar}\left[\begin{array}{ccc} 0 & \alpha^*(t) & \delta(t) \\ 0 & \omega(t) & \alpha(t) \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{ccc} 1 & r^*\frac{(e^{in}-1)}{(in)} & r^*r\left(\frac{(e^{in}-1-in)}{(in)^2}\right)+id \\ 0 & e^{in} & r\frac{(e^{in}-1)}{(in)} \\ 0 & 0 & 1 \end{array}\right] \tag{7.60}$$

This matrix equation leads to an ugly but manageable set of coupled nonlinear equations in four real variables (n, r, r^*, d) that can be integrated by standard methods. In the case that $d\omega(t)/dt = 0$ the equations simplify considerably, and can almost be solved by inspection.

7.5.2 Equilibrium Thermodynamics

In classical and quantum physics expectation values are expressed in terms of a density operator ρ

$$\langle \mathcal{O} \rangle = \mathrm{tr}\ \rho\mathcal{O} \tag{7.61}$$

In thermodynamic equilibrium the density operator is expressed in terms of the Hamiltonian describing the system as $\rho = e^{-\beta H}/Z$, where the normalization constant, or partition function, is $Z = \mathrm{tr}\ e^{-\beta H}$ and $\beta = 1/k_BT$, k_B is the Boltzmann constant and T is the absolute temperature. When H is an element in a finite dimensional Lie algebra, many simplifications in the computation of thermal expectation values occur. Again, we give two examples.

Example 1. We choose a Hamiltonian constructed from angular momentum operators

$$H = \epsilon J_z + \alpha J_+ + \alpha^* J_- \stackrel{j\rightarrow\frac{1}{2}}{\longrightarrow} \left[\begin{array}{cc} \frac{1}{2}\epsilon(t) & \alpha(t) \\ \alpha^*(t) & -\frac{1}{2}\epsilon(t) \end{array}\right] \tag{7.62}$$

We would like to be able to compute thermal expectation values of various moments of the angular momentum operators. The simplest way to go about this is to compute *generating functions* for these expectation values. To do this we compute $\langle e^{\Lambda} \rangle$, where $\Lambda = \lambda \cdot \mathbf{J}$. All symmetric moments can be constructed by taking derivatives of this generating function. We first compute this generating function in the smallest faithful matrix representation:

$$e^{-\beta H} e^{\Lambda} \rightarrow \left(I_2 \cosh(\beta |H|) - \beta \left[\begin{array}{cc} \epsilon/2 & \alpha \\ \alpha^* & -\epsilon/2 \end{array} \right] \frac{\sinh(\beta |H|)}{\beta |H|} \right) \times$$

$$\left(I_2 \cosh(|\Lambda|) + \left[\begin{array}{cc} \lambda_3/2 & \lambda \\ \lambda^* & -\lambda_3/2 \end{array} \right] \right) \frac{\sinh(|\Lambda|)}{|\Lambda|} \tag{7.63}$$

The trace of this expression is

$$\text{tr } e^{-\beta H} e^{\Lambda} \rightarrow$$

$$2\cosh(\beta |H|)\cosh(|\Lambda|) - 2\frac{H \cdot \Lambda}{\sqrt{H \cdot H}\sqrt{\Lambda \cdot \Lambda}} \sinh(\beta |H|)\sinh(|\Lambda|) \tag{7.64}$$

In these expressions $H \cdot \Lambda = (H, \Lambda) = \frac{1}{2}$ tr $H\Lambda$, and similarly for $|H| = \sqrt{(H,H)}$ and $|\Lambda| = \sqrt{(\Lambda, \Lambda)}$.

The trace of this 2×2 matrix can be written in another useful way after a similarity transform that diagonalizes it:

$$\text{tr } e^{-\beta H} e^{\lambda \cdot \mathbf{J}} = \text{tr } \left[\begin{array}{cc} e^{+\mu(H,\Lambda)/2} & 0 \\ 0 & e^{-\mu(H,\Lambda)/2} \end{array} \right] = 2\cosh(\mu(H,\Lambda)/2) \tag{7.65}$$

If N two-level atoms are acting incoherently, the trace over the 2^N states of all N atoms is the Nth power of the trace expressed in (7.65). On the other hand, if all N atoms are acting coherently, there are $2J+1$ states, where $N = 2J$. The trace over these states is [1]

$$\chi(H, \Lambda, J) = \frac{\sinh(J + \frac{1}{2})\mu(H, \Lambda)}{\sinh(\frac{1}{2})\mu(H, \Lambda)} \tag{7.66}$$

where $\mu(H, \Lambda, T)$ is determined from Eq. (7.65). The thermodynamic generating function is

$$\langle e^{\Lambda} \rangle = \frac{\chi(H, \Lambda, J)}{\chi(H, 0, J)} \tag{7.67}$$

To construct explicit expectation values (e.g., $\langle J_- \rangle$) it is sufficient to differentiate the generating function (e.g., $\frac{\partial}{\partial \lambda^*} \langle e^{\Lambda} \rangle / \langle e^0 \rangle$) and evaluate the result at $\Lambda = 0$. It is even more convenient to differentiate the logarithm and evaluate at $\Lambda = 0$: $\frac{\partial}{\partial \lambda^*} \log(\langle e^{\Lambda} \rangle)|_{\Lambda=0}$.

Example 2. As a second example we treat a harmonic oscillator described by a time-independent Hamiltonian of the form (7.68) in thermodynamic equilibrium at temperature T

$$H = \hbar\omega a^\dagger a + \alpha a^\dagger + \alpha^* a + \delta I \rightarrow \begin{bmatrix} 0 & \alpha^* & \delta \\ 0 & \hbar\omega & \alpha \\ 0 & 0 & 0 \end{bmatrix} \tag{7.68}$$

The density operator is $\rho = e^{-\beta(\hbar\omega a^\dagger a + \alpha a^\dagger + \alpha^* a + \delta I)}/Z$. The generating function for operator expectation values is $\chi(H, \Lambda, T) = \text{tr } e^{-\beta H} e^{\lambda_n a^\dagger a + \lambda a^\dagger + \lambda^* a + dI}/Z = \langle e^\Lambda \rangle$. The trace is taken in the infinite dimensional Hilbert space with Fock basis $|0\rangle, |1\rangle, |2\rangle, \ldots$. It would be insane to attempt to compute this expectation value without exploiting opportunities allowed by choice of a smaller, more convenient faithful matrix representation M of the group. The calculation proceeds according to the following steps:

(i) Write each of the operators H, Λ in the 3×3 matrix representation M (cf., 7.59);

(ii) Compute the exponential of each. For example

$$e^{-\beta M(H)} = EXP{-\beta} \begin{bmatrix} 0 & \alpha^* & \delta \\ 0 & \hbar\omega & \alpha \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha^* \frac{e^{-\beta\hbar\omega}-1}{\hbar\omega} & \frac{e^{-\beta\hbar\omega}-1+\beta\hbar\omega}{(\hbar\omega)^2}\alpha^*\alpha - \beta\delta \\ 0 & e^{-\beta\hbar\omega} & \alpha\frac{e^{-\beta\hbar\omega}-1}{\hbar\omega} \\ 0 & 0 & 1 \end{bmatrix} \tag{7.69}$$

(iii) Multiply the group operations together:

$$e^{-\beta M(H)} e^{M(\Lambda)} = \begin{bmatrix} 1 & Z_l & * \\ 0 & * & Z_r \\ 0 & 0 & 1 \end{bmatrix}$$

(iv) Find a similarity transformation, S, that zeroes out Z_l and Z_r:

$$M(S) \begin{bmatrix} 1 & Z_l & * \\ 0 & * & Z_r \\ 0 & 0 & 1 \end{bmatrix} M(S^{-1}) = \begin{bmatrix} 1 & 0 & B \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(v) Map this group operation to the infinite dimensional matrix representation acting on the Fock space

$$\begin{bmatrix} 1 & 0 & B \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow e^{Aa^\dagger a + BI}$$

(vi) Take the trace. Assuming $A < 0$ the sum converges to

$$\mathrm{tr}\ e^{Aa^\dagger a + BI} = \frac{e^B}{1 - e^A}$$

(vii) Take the logarithm to find

$$\log(\chi(H, \Lambda, T)) = B - A - \log(e^{-A} - 1)$$

(viii) These steps can be implemented easily using symbol manipulation codes. The result is

$$\begin{aligned}
-A &= \beta\hbar\omega - \lambda_n \\
B &= \frac{e^{-\beta\hbar\omega} - 1 + \beta\hbar\omega}{(\hbar\omega)^2}\alpha^*\alpha - \beta\delta + d + \frac{e^{\lambda_n} - 1 - \lambda_n}{\lambda_n^2}\lambda^*\lambda \\
&+ \frac{e^{-\beta\hbar\omega} - 1}{\hbar\omega}\frac{e^{\lambda_n} - 1}{\lambda_n}(\alpha^*\lambda + \alpha\lambda^*) \,/\, \left(1 - e^{-(\beta\hbar\omega - \lambda_n)}\right) \\
&+ \left[e^{-\beta\hbar\omega}\left(\frac{e^{\lambda_n} - 1}{\lambda_n}\right)^2 \lambda^*\lambda + e^{\lambda_n}\left(\frac{e^{-\beta\hbar\omega} - 1}{\hbar\omega}\right)^2 \alpha^*\alpha\right] / \left[1 - e^{-(\beta\hbar\omega - \lambda_n)}\right]
\end{aligned} \tag{7.70}$$

The generating function for only the creation and annihilation operators ($\lambda_n = d = 0$) is considerably simpler.

7.6 Conclusion

The EXPonential mapping from a Lie algebra to a Lie group is generally not onto. It is not in general possible to recover the entire Lie group by taking a single exponential of the Lie algebra. However, a sequence of exponential mappings from various linear vector subspaces in the Lie algebra can be found that covers the Lie group. This sequence of exponential mappings can be used to determine the structure of the underlying manifold of the Lie group. It also provides a useful parameterization for the Lie group.

Associated with every Lie algebra $\mathfrak{g}$ is a unique Lie group $\overline{G}$ that is simply connected. Every matrix group with this Lie algebra is locally isomorphic to this covering group. Every Lie group G with Lie algebra

$\mathfrak{g}$ has the structure $\overline{G}/D$, where D is a discrete invariant subgroup of $\overline{G}$. If $D = Id$, G is isomorphic to $\overline{G}$, otherwise it is a homomorphic image of $\overline{G}$. For simple matrix groups, D consists of multiples of the identity matrix, λI_n, and is simple to compute. If G_1 and G_2 have isomorphic Lie algebras they are locally isomorphic with the universal covering group and with each other.

Many different parameterizations of a Lie group are possible. The most useful ones typically involve a sequence of exponential mappings of linear vector subspaces of the Lie algebra into the Lie group. These are 'linear' in the sense that the coordinates parameterizing elements in the Lie group are components of a vector in a linear vector space (the Lie algebra). Different parameterizations are related by analytic reparameterization formulas — called Baker–Campbell–Hausdorff formulas for historical reasons. These BCH formulas can be constructed by finding a faithful matrix representation of the Lie algebra, then carrying out the reparameterization computation using products of exponentials of these matrices.

Exponentials play a fundamental role in physics as well as mathematics. We have explored two of the most useful applications of the exponential function in physics. These describe dynamics and statics. The dynamical evolution of a quantum system is governed by a unitary transformation that can be written as a time-ordered exponential. If the Hamiltonian is a linear superposition of basis vectors in a finite dimensional Lie algebra many useful computational methods are available for its simple computation. We have provided two illustrations of the methods that are available. If the physical system is in thermodynamic equilibrium, the density operator is also the exponential of the Hamiltonian. The two (dynamics and statics) are related by a "Wick rotation": $it/\hbar \leftrightarrow 1/k_BT$. We have used the same two physical systems as vehicles to illustrate how the exponential mapping, and suitable stepping back and forth through large and small unitary or nonunitary but faithful representations, has been used to simplify computation of partition functions and generating functions for symmetrized operator expectation values.

7.7 Problems

1. Construct the analytic group mapping $\phi((x_1, y_1), (x_2, y_2))$ for the parameterization (7.24) of the affine group. Construct the mapping $\phi((w_1, z_1), (w_2, z_2))$ for the parameterization (7.25) of this group.

2. Show that a straight line through the origin of the parameter space (a, b, c) that is inside the light cone $a^2 + b^2 - c^2 < 0$ Eq.(7.7) maps onto the subgroup $SO(2) \subset SL(2; R)$. Show that if $a = b = 0$, the basic 'repetition period' in the c-direction, c_T, in the subgroup is 2π but if $a^2 + b^2 > 0$ ($\sqrt{a^2 + b^2} = \beta \times c$, $|\beta| < 1$), the basic repetition period in the c-direction is increased to $2\pi\gamma$, where $\gamma = 1/\sqrt{1 - \beta^2}$ and $\beta^2 = (a^2 + b^2)/c^2$. Compare this renormalization of periodicity with "time dilation."

3. Compute the maximal discrete invariant subgroup D_{MAX} of $SU(3)$ and show that it is $\left\{I_3, \lambda I_3, \lambda^2 I_3\right\}$, where $\lambda = e^{2\pi i/3}$. Next, show that $SU(3)/D_{MAX}$ is isomorphic to the group of real 8×8 matrices $EXP[\mathfrak{Reg}(\mathfrak{su}(3))]$ ("eight-fold way").

4. Compute the maximal discrete invariant subgroup for the special unitary groups $SU(n)$ and show that it is the cyclic group of order n generated by ϵI_n, $\epsilon = e^{2\pi i/n}$. What real matrix group is $SU(n)/D_{MAX}$ equivalent to?

5. Show that the covering group $\overline{SU(1,1)}$ does not have a maximum discrete invariant subgroup.

6. It is convenient to introduce the creation and annihilation operators $a^\dagger, a$ to study the one dimensional quantum oscillator. These two operators are defined by

$$a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right) \qquad a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$$

Computation of the matrix elements of the moments of x in the harmonic oscillator basis, $\langle n'|x^k|n\rangle$, can be simplified using disentangling theorems. This problem indicates how.

a. The function $e^{\lambda x}$ is a generating function for matrix elements of x^k. Show that

$$\langle n'|x^k|n\rangle = \frac{d^k}{d\lambda^k}\langle n'|e^{\lambda x}|n\rangle_{\lambda=0}$$

b. Use the 3×3 matrix representation for the photon creation and annihilation operators and their commutator $[a, a^\dagger] = I$ to show

$$e^{\lambda x} = e^{\lambda(a^\dagger + a)/\sqrt{2}} = exp\left(\frac{\lambda}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & \lambda/\sqrt{2} & \lambda^2/4 \\ 0 & 1 & \lambda/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

c. Construct a disentangling theorem that expresses this group operator in the form $e^{ra^\dagger}e^{\delta I}e^{la}$ by constructing the matrix product of these three operators:

$$e^{ra^\dagger}e^{\delta I}e^{la} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & l & \delta \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}$$

d. By comparing the matrices in **b** and **c**, conclude

$$e^{\lambda(a^\dagger + a)/\sqrt{2}} = e^{\lambda a^\dagger/\sqrt{2}}e^{\lambda^2/4}e^{\lambda a/\sqrt{2}}$$

e. Use the disentangling theorem in **d** to compute $\langle n'|x^4|n\rangle$. In particular, show

$$\langle n'|x^4|n\rangle = \frac{d^4}{d\lambda^4}\sum_{p,q,r}\frac{\lambda^{p+2q+r}}{p!q!r!}2^{-(p/2+2q+r/2)}\langle n'|(a^\dagger)^p a^r|n\rangle_{\lambda=0}$$

$$\to \sum_{p+2q+r=4}\frac{4!}{p!q!r!}\,\frac{\langle n'|(a^\dagger)^p(a)^r|n\rangle}{2^{(p/2+q+r/2)}}$$

The point of this exercise is that the computation of the matrix elements is simplified because the operators are in *normally ordered* form (all annihilation operators first, on the right and all creation operators last, on the left). As a result, the calculation reduces to summing a descending series with no more than three nonzero terms.

7. In order to describe the scattering of X-rays from an atom moving in a harmonic potential it is necessary to compute a structure factor $\langle e^{ikx}\rangle$. The expectation value is thermal: $P_n \simeq e^{-n\beta\hbar\omega}$. This expectation value can be written in algebraic form as

$$\langle e^{ikx}\rangle = \frac{\text{Tr }e^{ikx}\ e^{-\beta\mathcal{H}}}{\text{Tr }e^{-\beta\mathcal{H}}} \tag{7.71}$$

We concentrate on the numerator, as the denominator is obtained in the limit $k \to 0$.

a. Show

$$\mathrm{Tr}\ e^{ikx}\ e^{-\beta\mathcal{H}} = \sum_{n=0}^{\infty}\langle n|e^{ikx}|n\rangle\ e^{-n\beta\hbar\omega} = \sum_{n=0}^{\infty}\langle n|e^{ikx}\ e^{-n\beta\hbar\omega}|n\rangle \tag{7.72}$$

b. The trace is invariant under similarity transform (the operator is bounded). Show that

$$\mathrm{Tr}\ e^{ikx}\ e^{-\beta\hbar\omega a^\dagger a} = \mathrm{Tr}\ e^{-\beta\hbar\omega a^\dagger a}e^{\delta} = e^{\delta}\ \mathrm{Tr}\ e^{-\beta\hbar\omega a^\dagger a} \tag{7.73}$$

As a result $\langle e^{ikx}\rangle = e^{\delta}$.

c. Compute δ using 3×3 nonunitary matrix multiplications to carry out multiplications *in the group* rather than in an $\infty\times\infty$ unitary representation of the group.

$$M(S)\ M(e^{ikx})\ M(e^{-\beta\hbar\omega\hat{n}})\ M(S^{-1}) = M(e^{-\beta\hbar\omega'\hat{n}})\ M(e^{\delta}) \tag{7.74}$$

$$\begin{array}{cccc} S & e^{ikx} & e^{-\beta\hbar\omega a^\dagger a} & S^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1 & \alpha & \alpha\beta/2 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & ik/\sqrt{2} & -k^2/4 \\ 0 & 1 & ik/\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-\beta\hbar\omega} & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -\alpha & \alpha\beta/2 \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{7.75}$$

Carry out the multiplication of 3×3 matrices in this nonunitary representation M. Show that $\omega' = \omega$. Determine α, β, and compute γ. Show

$$\langle e^{ikx}\rangle = e^{\delta} \qquad \delta = -\frac{1}{2}k^2\coth(\frac{1}{2}\beta\hbar\omega) \tag{7.76}$$

8. A finite set of operators X_i closes under commutation: $[X_i, X_j] = \sum_{k=1}^{N} C_{ij}{}^{k}\ X_k$. These operators span a finite dimensional Lie algebra $\mathfrak{g}$ of Lie group G. Assume that this set of operators has two representations R and S with the properties:

- R is hermitian: $\big(R(a^iX_i)\big)^\dagger = \big(a^iR(X_i)\big)^\dagger = (a^i)^*R^\dagger(X_i)$.
- S is faithful: $S(a^iX_i) = 0 \Rightarrow a^i = 0$.

We require S to be finite dimensional so that simple matrix computations are possible. We require R to be hermitian to make an immediate connection with Quantum Mechanics.

a. It happens frequently that $\mathcal{H} = R(a^i X_i)$ describes the physics of some quantum mechanical system. Show that if $H_1, H_2, \cdots, H_r \in \mathfrak{g}$ span a maximal commutative subspace, so that $[H_i, H_j] = 0$, $1 \leq i, j \leq r$, then the hermitian operators $R(H_i)$ are mutually commutative and can all be made diagonal simultaneously in this representation: $[R(H_i)]_{\alpha\beta} = r_\alpha(i)\delta_{\alpha\beta}$.

b. Show that $[S(H_i), S(H_j)] = 0$, but show by example that the r matrices $S(H_i)$ cannot always be simultaneously diagonal.

c. Show the time evolution of the quantum system is given by the unitary operator $U(t) = R(e^{-\frac{i}{\hbar}\mathcal{H}t}) = e^{-\frac{i}{\hbar}R(\mathcal{H})t}$.

d. Show that the density operator for thermal expectation values is $\rho(T) = e^{-\beta\mathcal{H}}/Z = R(e^{-\beta\mathcal{H}})/Z = e^{-\beta R(\mathcal{H})}/Z$. What is Z?

e. Show that the unitary time evolution operator $U(t)$ and the hermitian density operator $\rho(T)$ are related by a Wick rotation $it/\hbar \leftrightarrow \beta = 1/k_B T$.

f. A generating function for thermal expectation values has the form

$$\langle e^{x^i X_i} \rangle = \frac{\text{Tr } e^{R(x^i X_i)} e^{-\beta\mathcal{H}}}{\text{Tr } e^{-\beta\mathcal{H}}} \rightarrow \frac{\text{Tr } R(e^{x^i X_i}) \; R(e^{-\beta a^i X_i})}{\text{Tr } R(e^{-\beta a^i X_i})} \quad (7.77)$$

g. The operator product in the numerator is in the group $G = e^{\mathfrak{g}}$ or its complex extension. If this operator product can be transformed to "diagonal" form (i.e., expressed in terms of the operators H_i) the trace can easily be constructed. Show that for x^i sufficiently small it is always possible to construct a similarity transformation $S = e^{y^k X_k}$ with the property

$$S e^{x^j X_j} e^{-\beta a^i X_i} S^{-1} = e^{-\beta d^i(x,a) H_i} \quad (7.78)$$

h. The thermal expectation value then reduces to

$$\langle e^{x^i X_i} \rangle = \frac{\text{Tr } R(e^{-\beta d^i(x,a) H_i})}{\text{Tr } R(e^{-\beta d^i(0,a) H_i})} \quad (7.79)$$

Since the H_i are diagonal in the representation R, the sums are straightforward.

i. Relate the steps in the algorithm described in this problem to the steps followed in the previous problem for computing the result derived in Eq. (7.76). In particular, identify the operators X_i, the "diagonal" operators H_i, the hermitian representation R (it is invisible), the faithful representation S (it is given explicitly), the generating function $e^{x^i X_i}$, and the Wick rotation.

8. Coherent states were first discussed by Schrödinger in 1926. For many purposes it is useful to apply a unitary transformation to the harmonic oscillator ground state. The unitary transformation has the form $U(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}$, where $a^\dagger$ and a are the usual photon creation and annihilation operators. This unitary operator, acting on the ground state, is relatively simple to compute if it can be disentangled as follows

$$U(\alpha)|0\rangle = e^{(\alpha a^\dagger - \alpha^* a)}|0\rangle = e^{\beta a^\dagger} e^{\delta I} e^{\beta' a}|0\rangle \tag{7.80}$$

This disentangling theorem can be worked out easily in the 3×3 *nonunitary* representation. (It is the group multiplication property that we are after; unitarity is an additional structure that is applied to the *representation* of the group.)

a. Show that the left hand side of Eq. (7.80) simplifies to

$$EXP \begin{bmatrix} 1 & -\alpha^* & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\alpha^* & -\alpha^*\alpha/2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \tag{7.81}$$

b. Show that the right hand side of Eq. (7.80) becomes

$$\begin{bmatrix} 1 & \beta' & \delta \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \tag{7.82}$$

c. Use this result to compute

$$e^{(\alpha a^\dagger - \alpha^* a)}|0\rangle = e^{\alpha a^\dagger} e^{-\alpha^* \alpha I/2} e^{-\alpha^* a}|0\rangle = \sum \frac{(\alpha a^\dagger)^n}{n!}|0\rangle e^{-\alpha^*\alpha/2} \tag{7.83}$$

d. Use a further property of the creation operators (this is a representation-dependent property, so the calculation has now moved back into the infinite-dimensional Hilbert space and out of the nonunitary 3×3 matrix representation), $a^\dagger|n\rangle = |n+1\rangle\sqrt{n+1}$ to conclude

$$|\alpha\rangle = U(\alpha)|0\rangle = e^{-\alpha^*\alpha/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{7.84}$$

e. Compute the inner product $\langle\beta|\alpha\rangle$ and show $\langle\alpha|\alpha\rangle = 1$
f. Show $a|\alpha\rangle = \alpha|\alpha\rangle$.
g. Show $\langle\alpha|x|\alpha\rangle = (\alpha^* + \alpha)/\sqrt{2}$.

9. An $SU(2)$ coherent state (also called atomic coherent state) is constructed by the action of an arbitrary $SU(2)$ group operation on the ground state, or lowest lying state, in a $2j+1$ dimensional invariant space [1,31]:

$$\left|\begin{matrix} j \\ \theta \end{matrix}\right\rangle = SU(2)\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle \tag{7.85}$$

a. Show that rotations by ϕ around the z-axis serve only to multiply the fiducial state by a phase angle: $e^{i\phi J_z}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-ij\phi}$. This simply "renormalizes" the fiducial state, and is generally not important.

b. Rotations about an axis in the x-y plane produce a two-parameter family of coherent states parameterized by coset representatives in $SU(2)/U(1)$:

$$\left|\begin{matrix} j \\ \theta \end{matrix}\right\rangle = e^{i(\theta_x J_x + \theta_y J_y)}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle, \quad i(\theta_x J_x + \theta_y J_y) = \frac{i}{2}\begin{bmatrix} 0 & \theta_x - i\theta_y \\ \theta_x + i\theta_y & 0 \end{bmatrix} \tag{7.86}$$

c. Rewrite $e^{i(\theta_x J_x + \theta_y J_y)}$ in the form $e^{i\alpha_+ J_+}e^{i\alpha_z J_z}e^{i\alpha_- J_-}$ and compute the analytic relation between the angles θ and the parameters α.

d. Show $e^{i\alpha_- J_-}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle$.

e. Show $e^{i\alpha_z J_z}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle = \left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-ij\alpha_z}$.

f. Compute finally

$$U(\alpha)|0\rangle = e^{i\alpha_+ J_+}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-ij\alpha_z} = \sum_{m=-j}^{m=+j}\frac{(i\alpha_+ J_+)^{j+m}}{(j+m)!}\left|\begin{matrix} j \\ -j \end{matrix}\right\rangle e^{-ij\alpha_z} \tag{7.87}$$

g. Show that $J_- | \begin{smallmatrix} j \\ \theta_x\theta_y \end{smallmatrix} \rangle$ cannot be proportional to $| \begin{smallmatrix} j \\ \theta_x\theta_y \end{smallmatrix} \rangle$ because the state $| \begin{smallmatrix} j \\ +j \end{smallmatrix} \rangle$ is not occupied. This is different from the harmonic oscillator (photon operator) case. The difference arises because $SU(2)$ is compact with finite-dimensional unitary irreducible representations and the harmonic oscillator group H_4 is not compact with only an infinite-dimensional unitary irreducible representation of interest.

h. Compute the inner product and show

$$\langle \begin{smallmatrix} j \\ \theta'_x\theta'_y \end{smallmatrix} \Big| \begin{smallmatrix} j \\ \theta_x\theta_y \end{smallmatrix} \rangle = \left[\cos(\frac{\theta'}{2})\cos(\frac{\theta}{2}) + e^{i(\phi'-\phi)}\sin(\frac{\theta'}{2})\sin(\frac{\theta}{2})\right]^{2j} \tag{7.88}$$

where $e^{-i\phi} = (\theta_x - i\theta_y)/\theta$, and similarly for θ' (c.f., Eq. (7.46)).

10. A number of important quantum eigenvalue equations can be expressed in algebraic format. A toy example is

$$(EJ_3 + pJ_1 - Z)\,|u\rangle = 0$$

Here E is an energy eigenvalue, p is some sort of coupling strength, Z could (and sometimes does) represent a charge, and $|u\rangle$ is an eigenfunction. In this toy example, the operators J_3 and J_1 are assumed to belong to the Lie algebra $\mathfrak{su}(2)$ and the equation applies to half-integer spin spaces ($(2j+1)$ is even).

a. Show that a unitary transformation U transforms this equation to the diagonal form $(E'J_3 - Z)|v\rangle = 0$, where $E' = \sqrt{E^2 + p^2}$ and $|v\rangle = U\,|u\rangle$.

b. Show that $E = \pm\sqrt{(Z/m)^2 - p^2}$.

c. Compare this spectrum with the unperturbed spectrum ($p \to 0$).

d. Under what conditions on j, p, Z are these solutions valid?

e. Construct the unitary transformation that diagonalizes the eigenvalue equation, and show that $|u\rangle = e^{i\theta J_2} | \begin{smallmatrix} j \\ m \end{smallmatrix} \rangle$. Compute θ for each E.

11. Compute the matrix elements of the rotation matrices in the $2j+1$ unitary irreducible representations of $SU(2)$ and show

$$EXP(i\beta J_y)_{mn} = D^j_{mn}(\beta) = P^j_{mn}(z) = \frac{(-)^{j-m}}{2^j(j-n)!}\left[\frac{(j-n)!(j+m)!}{(j+n)!(j-m)!}\right]^{1/2} \times$$

$$(1+z)^{-(m+n)/2}(1-z)^{-(m-n)/2}\left(\frac{d}{dz}\right)^{j-m}\left[(1-z)^{j-n}(1+z)^{j+n}\right]$$

where $z = \cos(\beta)$. The Wigner matrix elements D^j_{mn} are related to the Jacobi polynomials when $j = l$, where l is an integer.

12. Use the decompositions (7.21) for $SO(3)$ and (7.22) for $SU(2)$ to show:

a. Geodesics through $I_2 \in SU(2)$ focus at $-I_2$ and geodesics through $I_3 \in SO(3)$ focus at I_3. Conclude that $SU(2)$ is a 2-fold covering group of $SO(3)$.

b. Geodesics through the "north pole" of $SU(2)/U(1)$ ($z = 1, x = y = 0$) focus at its "south pole" ($z = -1, x = y = 0$) and geodesics through the north pole of $SO(3)/SO(2)$ ($z = 1, x = y = 0$) focus at its south pole ($z = -1, x = y = 0$).

c. Conclude that $SU(2)/U(1) = S^2 = SO(3)/SO(2)$ and the $2 \rightarrow 1$ nature of the covering $SU(2) \downarrow SO(3)$ is contained in the subgroup of rotations about the z-axis $U(1) \downarrow SO(2)$.

$$\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \xrightarrow{2\rightarrow 1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

13. Show that the discrete invariant subgroups of $SU(n)$ are all commutative groups of order r, with group elements $e^{2\pi ik/r}I_n$, with n/r integer. Compute the foci in $SU(n)$. How are the foci related to the group operations of the form $e^{2\pi ik/n}I_n$?

14. Show that the matrix $\begin{bmatrix} -\lambda & 0 \\ 0 & -1/\lambda \end{bmatrix}$ in $SL(2;R)$ cannot be reached by exponentiating any element in the Lie algebra if $\lambda > 1$. Show that it can be reached by following a "broken geodesic" $e^A e^B$. Find matrices A and B that do this (Hint: don't work too hard).

15. A simple model has been introduced to describe the interaction of light with matter. In this model (Dicke model) N atoms interact with a single mode of the electromagnetic field. Each atom is modeled as a

2-level system, with energy separation ϵ. A single photon has energy $\hbar\omega$. The hamiltonian is chosen as

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\epsilon}{2}\sigma_z^{(i)} + \hbar\omega a^\dagger a + \frac{\lambda}{\sqrt{N}} \sum_{i=1}^{N} \sigma_+^{(i)} a + \sigma_-^{(i)} a^\dagger$$

The operator $\sigma_z^{(i)}$ describes the two states of atom i and the operator $a^\dagger a$ describes the number of photons in the field mode. The operator $\sigma_+^{(j)}$ ($\sigma_\pm^{(j)} = \frac{1}{2}(\sigma_x^{(j)} \pm i\sigma_y^{(j)})$) describes transitions of the jth atom from the ground to its excited state. This atomic transition is accompanied by the absorption (annihilation) of a single photon. The operator $\sigma_-^{(j)} a^\dagger$, describes deexcitation of an atom with emission (creation, $a^\dagger$) of a photon. The strength of interaction of the atom with the electromagnetic field (the dipole moment) is parameterized by λ.

a. Assume the atoms are independent and show

$$[\sigma_z^{(i)}, \sigma_\pm^{(j)}] = \pm\sigma_\pm^{(i)}\delta_{ij} \qquad [\sigma_+^{(i)}, \sigma_-^{(j)}] = \sigma_z^{(i)}\delta_{ij}$$

b. If all the atoms behave cooperatively it is possible to replace $\sum_{i=1}^{N} \frac{1}{2}\sigma_z^{(i)} \rightarrow J_z$, $\sum_{i=1}^{N} \sigma_\pm^{(i)} \rightarrow J_\pm$. Show that the operators $J_z, J_\pm$ satisfy the usual $\mathfrak{su}(2)$ commutation relations.

c. Assume the atoms "behave classically." This means that the quantum mechanical operators $J_z, J_\pm$ can be replace by their c-number expectation values: $J_z \rightarrow \langle J_z(t)\rangle$, $J_+ \rightarrow \langle J_+(t)\rangle$, $J_- \rightarrow \langle J_-(t)\rangle = \langle J_+(t)\rangle^*$. Show that this semiclassical hamiltonian

$$\mathcal{H}_{\text{field}} = \epsilon\langle J_z(t)\rangle + \hbar\omega a^\dagger a + \frac{\lambda}{\sqrt{N}}(\langle J_+(t)\rangle a + \langle J_-(t)\rangle a^\dagger)$$

maps the ground state of the field (the state with no photons) into a coherent state of the electromagnetic field: $|\alpha(t)\rangle = U(\alpha(t))|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$. Use the disentangling theorems to compute the relation between the coherent state parameter $\alpha(t)$ and the classical driving fields $\langle J_z(t)\rangle$ and $\langle J_+(t)\rangle = \langle J_-(t)\rangle^*$.

d. Show that if the initial state of the field is not the ground state, but rather a coherent state $|\beta\rangle$, the state obtained by the action of the classical current is still a coherent state. How are the parameters β, describing the initial condition and α, describing the unitary evolution of the field, related?

e. Suppose now that the atoms are considered quantum mechanically but the field is considered classically. Show that this amounts to the substitutions $a^\dagger \rightarrow \langle a(t)\rangle^*$, $a \rightarrow \langle a(t)\rangle$, and $a^\dagger a \rightarrow \langle a(t)\rangle^* \langle a(t)\rangle$.

f. Show that the resulting semiclassical Hamiltonian is

$$\mathcal{H}_{\text{atoms}} = \epsilon J_z + \frac{\lambda}{\sqrt{N}}(J_+ \langle a(t)\rangle + J_- \langle a(t)\rangle^*)$$

Show that under this semiclassical hamiltonian, if the atoms are in their collective ground state ($m = -\frac{1}{2}$ for each atom, or $M = -J, J = N/2$ for the ensemble of N atoms) the ground state will evolve into a coherent state of the group $SU(2)$ parameterized by a point in the coset $SU(2)/U(1)$.

g. Show that, under the action of this semiclassical Hamiltonian a coherent state will evolve into a coherent state: $|\theta(t)\rangle = e^{i\theta(t)\cdot\mathbf{J}}|J, -J\rangle$, where $J = N/2$. How are the angles $\theta(t)$ related to the classical field variables $\langle a(t)\rangle$ and $\langle a(t)\rangle^*$?

h. Conclude that there is a duality between the atoms and the field in this model: a classical current will produce a coherent state of the electromagnetic field; a classical electromagnetic field will produce a coherent atomic state.

i. The semiclassical Hamiltonian for the field can be used to construct time-dependent field expectation values $\langle a\rangle$ and $\langle a\rangle^*$. Conversely, the semiclassical Hamiltonian for the atoms can be used to construct time-dependent atomic expectation values $\langle J_+\rangle = \langle J_-\rangle^*$. Construct a self-consistent model by requiring that both sets of time-dependent quantities are equal.

16. The thermodynamic properties of the Dicke model can be studied in a similar fashion. Assume N identical atoms interacting with a single field mode are in thermodynamic equilibrium at temperature T ($\beta = 1/k_B T$).

a. Assume $\langle \sigma_+^{(i)}\rangle_T$ has some fixed unknown value, and similarly for the other atomic thermal expectation values. Use these values in the semiclassical approximation for the field hamiltonian to compute the density operator. Compute the thermal expectation values for the operators $a^\dagger, a, a^\dagger a$.

b. Dualize. Assume the field operators have fixed but unknown expectation values. Use these values in the semiclassical approximation for the atomic hamiltonian to compute the density operator. Compute the thermal expectation values for the operators $\sigma_z, \sigma_+, \sigma_-$.

c. Impose self-consistency. Require that if a set of field thermal expectation values produces specific atomic expectation values, these atomic expectation values produce the same set of field expectation values. This leads to a nonlinear set of self-consistency equations. These self-consistent equations may have more than one solution.

d. To lift the self-consistent solution degeneracy, construct the thermal expectation value for $\mathcal{H}$. Choose the minimum energy solution. Under what conditions on $\epsilon, \hbar\omega, \lambda, N$ is there a nontrivial solution (e.g., $\langle J_+ \rangle_T \neq 0$)?

e. Show that a thermodynamic phase transition occurs as $\lambda^2/\epsilon\hbar\omega$ increases through $+1$. Is this a first- or second-order phase transition?

17. The two complex parameters $a(t), b(t)$ in the evolution equation (7.58) can be expressed in terms of their real and imaginary parts. These obey $a_r^2 + a_i^2 + b_r^2 + b_i^2 = 1$ (unitarity condition). This condition simply reflects that the state of the system is given by a unit quaternion. As numerical integration proceeds, imprecisions may cause these parameters to depart slightly from the unitarity condition. Devise a self-correcting integration procedure to correct for this type of error. After N small integration steps, compute the length of the vector (a_r, a_i, b_r, b_i) and scale this length back to $+1$.

18. The thermodynamic generating functions for $SU(2)$ and H_4 given by expressions (7.67) and (7.70) simplify considerably if the "diagonal operator" is not included. Simplify (7.67) by taking the limit $\lambda_3 \to 0$. Simplify (7.70) by taking the limit $\lambda_n \to 0$ and setting $d = \delta = 0$.

19. For many reasons it is less desirable to compute thermal expectation values for *symmetric* operator products such as $\langle J_+J_- + J_-J_+ \rangle$ or $\langle aa^\dagger + a^\dagger a \rangle$ than it is to construct generating functions for *ordered* products of operators such as $\langle J_+J_- \rangle$ or $\langle a^\dagger a \rangle$. Show how to use disentangling theorems to transform the generating functions for symmetric operator products in (7.67) and (7.70), or their simplified forms constructed in the previous problem, into generating functions for ordered products of operators.