Group Theory

Robert Gilmore
Physics Department, Drexel University, Philadelphia, Pennsylvania 19104, USA, revised at CORIA
(Dated: October 8, 2013)

Printed from:Wiley-Mathematical/GroupChapter/group.tex on October 8, 2013

I. INTRODUCTION

Symmetry has sung its siren song to Physicists since the beginning of time, or at least since before there were Physicists. Today the ideas of symmetry are incorporated into a subject with the less imaginative and suggestive name of Group Theory. This Chapter introduces many of the ideas of group theory that are important in the natural sciences.

Natural philosophers in the past have come up with many imaginative arguments for estimating physical quantities. They have often used out-of-the-box methods that were proprietary to pull rabbits out of hats. When these ideas were made available to a wider audience they were often improved upon in unexpected and previously unimaginable ways. A number of these methods are precursors of group theory. These are Dimensional Analysis, Scaling Theory, and Dynamical Similarity. We review these three methods in Sec. II.

In Sec. III we get down to the business at hand, introducing the definition of group and giving a small set of important examples. These range from finite groups to discrete groups to Lie groups. These also include transformation groups, which played an important if underrecognized rôle in the development of classical physics, in particular the theories of Special and General Relativity. The relation between these theories and group theory is indicated in Sec. IX.

Despite this important rôle in the development of Physics, groups existed at the fringe of the Physics of the early 20^{th} century. It was not until the theory of the linear matrix representations of groups was invented that the theory of groups migrated from the outer fringes to play a more central rôle in Physics. Important points in the theory of representations are introduced in Sec. X. Representations were used in an increasingly imaginative number of ways in Physics throughout the 20^{th} century. Early on they were used to label states in Quantum systems with a symmetry group: for example, the rotation group $SO(3)$. Once states were named, degeneracies could be predicted and computations simplified. Such applications are indicated in Sec. XI. Later, they were used when symmetry was not present, or just the remnant of a broken symmetry was present. When used in this sense, they are often called "dynamical groups." This type of use greatly extended the importance of Group Theory in Physics. As a latest tour de force in the development of Physics, groups play a central rôle in the formulation of Gauge Theories. These theories describe the interactions between fermions and the bosons and lie at the heart of the Standard Model. We provide the simplest example of a gauge theory, based on the simplest compact one parameter Lie group $U(1)$, in Sec. XIII.

For an encore, in Sec. XIV we show how the theory of the special functions of mathematical physics (Legendre and associated Legendre functions, Laguerre and associated Laguerre functions, Gegenbauer, Chebyshev, Hermite, Bessel functions, and others) are subsumed under the theory of representations of some low-dimensional Lie groups. The classical theory of special functions came to fruition in the mid 19^{th} century, long before Lie groups and their representations were even invented.

II. PRECURSORS TO GROUP THEORY

The axioms used to define a group were formulated in the second half of the nineteenth century. Long before then the important ideas underlying these axioms were used to derive classical results (for example, Pythagoras' Theorem: see below) in alternative, simpler, and/or more elegant ways, to obtain new results, or to consolidate different results under a single elegant argument. In this Section we survey some of these imaginative lines of thought. We begin with a simple argument due to Barenblatt that has been used to derive Pythagoras' theorem. We continue with a discussion of the central features of dimensional analysis and illustrate how this tool can be used to estimate the size of a hydrogen atom. We continue in the same vein, using scaling arguments to estimate the sizes of other "atom-like" structures based on the known size of the hydrogen atom. We conclude this section with a brief description of dynamical similarity and how the arguments intrinsic to this line of thinking can be used to estimate one of Kepler's laws and to place four classical mechanics laws (Kepler, Newton, Galileo, Hooke) in a common framework.

We emphasize that group theory is not used explicitly in any of these arguments but its fingerprints are everywhere. These digressions should serve as appetizes to indicate the power of the tool called Group Theory in modern physical theories.

A. Classical Geometry

Barenblatt (Barenblatt, 2003) has given a beautiful derivation of Pythagoras' Theorem that is out-of-the-box and suggests some of the ideas behind Dimensional Analysis. The area of the right triangle $\Delta(a, b, c)$ is $\frac{1}{2}ab$ (Fig.

1). Dimensionally, the area is proportional to square of any of the sides, multiplied by some factor. We make a unique choice of side by choosing the hypotenuse, so that $\Delta(a,b,c) = c^2 \times f(\theta)$, θ is one of the two acute angles, and $f(\theta) \neq 0$ unless $\theta = 0$ or $\pi/2$. Equating the two expressions

$$f(\theta) = \frac{1}{2}\left(\frac{a}{c}\right)\left(\frac{b}{c}\right) = \frac{1}{2}\left(\frac{b}{c}\right)\left(\frac{a}{c}\right) \stackrel{\text{symmetry}}{=} f(\frac{\pi}{2} - \theta) \tag{1}$$

This shows (a) that the same function $f(\theta)$ applies for all similar triangles and (b) $f(\theta) = f(\frac{\pi}{2} - \theta)$. The latter result is due to reflection 'symmetry' of the triangle about the bisector of the right angle: the triangle changes but its area does not. We need (a) alone to prove Pythagoras' Theorem. The proof is in the figure caption.

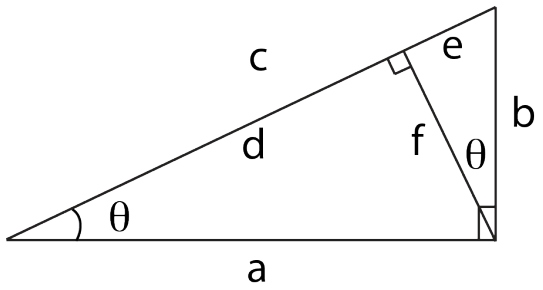

FIG. 1: The area of the large right triangle is the sum of the areas of the two similar smaller right triangles: $\Delta(a,b,c) = \Delta(d,f,a) + \Delta(f,e,b)$, so that $c^2 f(\theta) = a^2 f(\theta) + b^2 f(\theta)$. Since $f(\theta) \neq 0$ for a nondegenerate right triangle, $a^2 + b^2 = c^2$.

B. Dimensional Analysis

How big is a hydrogen atom?

The size of the electron 'orbit' around the proton in the hydrogen atom ought to depend on the electron mass m_e, or more precisely the electron-proton reduced mass $\mu = m_e M_P/(m_e + M_P)$. It should also depend on the value of Planck's constant h or reduced Planck's constant $\hbar = h/2\pi$. Since the interaction between the proton with charge e and the electron with charge $-e$ is electromagnetic, of the form $V(r) = -e^2/r$ (Gaussian units), it should depend on e^2.

Mass is measured in gm. The dimensions of the charge coupling e^2 are determined by recognizing that e^2/r is a (potential) energy, with dimensions $M^1L^2T^{-2}$. We will use capital letters M (mass), L (length), and T (time) to characterize the three independent dimensional 'directions'. As a result, the charge coupling strength e^2 has dimensions ML^3T^{-2} and is measured in $gm(cm)^3/sec^2$. The quantum of action $\hbar$ has dimensions $[\hbar] = ML^2T^{-1}$. Here and below we use the standard convention that $[*]$ is to be read "the dimensions of * are".

Constant	Dimensions	Value	Units
μ	M	9.10442×10^{-28}	gm
$\hbar$	ML^2T^{-1}	1.05443×10^{-27}	$gm\ cm^2\ sec^{-1}$
e^2	ML^3T^{-2}	2.30655×10^{-19}	$gm\ cm^3\ sec^{-2}$
a_0	L	?	cm

Can we construct something (e.g., Bohr orbit a_B) with the dimensions of length from m, e^2, and $\hbar$? To do this, we introduce three unknown exponents a, b, and c and write

$$\begin{aligned} a_B \simeq m^a\ (e^2)^b\ \hbar^c &= (M)^a\ (ML^3T^{-2})^b\ (ML^2T^{-1})^c \\ &= (M)^{a+b+c}\ L^{0a+3b+2c}\ T^{0a-2b-c} \end{aligned} \tag{2}$$

and set this result equal to the dimensions of whatever we would like to compute, in this case the Bohr orbit a_B (characteristic atomic length), with $[a_B] = L$. This results in a matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

We can invert this matrix to find

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix} \tag{4}$$

This allows us to determine the values of the exponents which provide the appropriate combinations of important physical parameters to construct the characteristic atomic length:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \tag{5}$$

This result tells us that

$$a_0 \sim m^{-1}(e^2)^{-1}(\hbar)^2 = \hbar^2/me^2 \sim 10^{-8}\ cm \tag{6}$$

To construct a characteristic atomic time, we can replace the vector col$[0,1,0]$ in Eq. (5) by the vector col$[0,0,1]$, giving us the result $\tau_0 \sim \hbar^3/m(e^2)^2$. Finally, to get a characteristic energy, we can form the combination $\mathcal{E} \sim ML^2T^{-2} = m(\hbar^2/me^2)^2(\hbar^3/me^4)^{-2} = me^4/\hbar^2$. Another, and more systematic, way to get this result is to substitute the vector col$[1,2,-2]^t$ for $[0,1,0]^t$ in Eq. (5).

Note that our estimate would be somewhat different if we had used h instead of $\hbar = h/2\pi$ in these arguments. We point out that this method is *very* useful for estimating the order of magnitude of physical parameters and in practised hands usually gets the prefactor within a factor of 10. The most critical feature of dimensional analysis is to identify the parameters that are most important in governing the science of the problem, and then to construct a result depending on only those parameters.

C. Scaling

Positronium is a bound state of an electron e with a positron $\bar{e}$, its antiparticle with mass m_e and charge $+e$. How big is positronium?

To address this question we could work very hard and solve the Schrödinger equation for positronium. This is identical to the Schrödinger equation for the hydrogen atom, except for replacing the hydrogen atom reduced mass $m_e M_p/(m_e + M_p) \simeq m_e$ by the positronium reduced mass $m_e m_e/(m_e + m_e) = \frac{1}{2}m_e$. Or we could be lazy and observe that the hydrogen atom radius is inversely proportional to the reduced electron-proton mass, so the positronium radius should be inversely proportional to the reduced electron-positron mass $m_e/2$. Since the reduced electron-proton mass is effectively the electron mass, the positronium atom is approximately twice as large as the hydrogen atom.

In a semiconductor it is possible to excite an electron (charge $-e$) from an almost filled (valence) band into an almost empty (conduction) band. This leaves a 'hole' of charge $+e$ behind in the valence band. The positively charged hole in the valence band interacts with the excited electron in the conduction band through a reduced Coulomb interaction: $V(r) = -e^2/\epsilon r$. The strength of the interaction is reduced by screening effects which are swept into a phenomenological dielectric constant ϵ. In addition, the effective masses m_e^* of the excited electron and the left-behind hole m_h^* are modified from the free-space electron mass values by many-particle effects.

How big is an exciton in Gallium Arsenide (GaAs)? For this semiconductor the phenomenological parameters are $\epsilon = 12.5$, $m_e^* = 0.07m_e$, $m_h^* = 0.4m_e$.

We extend the scaling argument above by computing the reduced mass of the electron hole pair: $\mu_{e-h} = (0.07m_e)(0.4m_e)/(0.07 + 0.4)m_e = 0.06m_e$ and replacing e^2 in the expression Eq. (4) for the Bohr radius a_0 by e^2/ϵ. The effect is to multiply a_0 by $12.5/0.06 = 208$. The ground state radius of the exciton formed in $GaAs$ is about 10^{-6} cm. The ground state binding energy is lower than the hydrogen atom binding energy of 13.6 eV by a factor of $0.06/12.5^2 = 3.8 \times 10^{-4}$ so it is 5.2 meV.

Scaling arguments such as these are closely related to renormalization group arguments as presented in Chapter XXX.

D. Dynamical Similarity

Jupiter is about five times further (5.2AU) from our Sun than the Earth. How many earth years does it take for Jupiter to orbit the Sun?

Landau and Lifshitz provide an elegant solution to this simple question using similarity (scaling) arguments (Landau and Lifshitz, 1960). The equation of motion for the Earth around the Sun is

$$m_E \frac{d^2\mathbf{x_E}}{dt_E^2} = -Gm_E M_S \frac{\hat{\mathbf{x}}_\mathbf{E}}{|\mathbf{x_E}|^2} \tag{7}$$

where $\mathbf{x_E}$ is a vector from the sun to the earth and $\hat{\mathbf{x}}_\mathbf{E}$ the unit vector in this direction. If Jupiter is in a geometrically similar orbit, then $\mathbf{x_J} = \alpha\mathbf{x_E}$, with $\alpha = 5.2$. Similarly, time will evolve along the Jupiter trajectory in a scaled version of its evolution along the Earth's trajectory: $t_J = \beta t_E$. Substituting these scaled expressions into the equation of motion for Jupiter, and cancelling out m_J from both sides, we find

$$\frac{\alpha}{\beta^2}\frac{d^2\mathbf{x_E}}{dt_E^2} = -\frac{1}{\alpha^2}GM_S\frac{\hat{\mathbf{x}}_\mathbf{E}}{|\mathbf{x_E}|^2} \tag{8}$$

This scaled equation for Jupiter's orbit can only be equated to the equation for the Earth's trajectory (the orbits are similar) provided $\alpha^3/\beta^2 = 1$. That is, $\beta = \alpha^{3/2}$, so that the time-scaling factor is $5.2^{3/2} = 12.5$.

We have derived Kepler's Third Law without even solving the equations of motion! Landau and Lifshitz point out that you can do even better than that. You don't even need to know the equations of motion to construct scaling relations when motion is described by a potential $V(\mathbf{x})$ which is homogeneous of degree k. This means that $V(\alpha\mathbf{x}) = \alpha^k V(\mathbf{x})$. When the equations of motion are derivable from a Variational Principle $\delta I = 0$, where

$$I = \int \left(m\left(\frac{d\mathbf{x}}{dt}\right)^2 - V(\mathbf{x})\right)\, dt \tag{9}$$

then the scaling relations $\mathbf{x} \to \mathbf{x}' = \alpha\mathbf{x}$, $t \to t' = \beta t$ lead to a modified action

$$I' = \frac{\alpha^2}{\beta}\int \left(m\left(\frac{d\mathbf{x}}{dt}\right)^2 - \alpha^{k-2}\beta^2 V(\mathbf{x})\right)\, dt \tag{10}$$

The Action I' is proportional to the original Action I, and therefore leads to the same equations of motion, only when $\alpha^{k-2}\beta^2 = 1$. That is, the time elapsed, T, is proportional to the distance traveled, D, according to $T \simeq D^{(1-k/2)}$. Four cases are of interest.

k=−1 (Coulomb/Gravitational Potential) The period of a planetary orbit scales like the 3/2 power of the distance from the Sun (Kepler's Third Law).

k= 0 (No forces) The distance traveled is proportional to the time elapsed (essentially Newton's First Law). To recover Newton's first law completely it is only necessary to carry out the variation in Eq. (10), which leads to $\frac{d}{dt}\left(\frac{d\mathbf{x}}{dt}\right) = 0$.

k=+1 (Free fall in a homogeneous gravitational field) The potential $V(z) = mgz$ describes free fall in a homogeneous graviational field. Galileo is reputed to have dropped rocks off the Leaning Tower of Pisa to determine

TABLE I: Four important results in the historical development of science are consequences of scaling arguments.

k	Scaling	Law
-1	$T^2 \simeq D^3$	Kepler #3
0	$D \simeq T$	Newton #1
$+1$	$\Delta z \simeq \Delta t^2$	Galileo : Rolling Stones
$+2$	$T \simeq D^0$	Hooke

that the distance fallen was proportional to the square of the time elapsed. The story is apocryphal: in fact, he rolled stones down an inclined plane to arrive at the result $\Delta z \simeq \Delta t^2$.

k=+2 (Harmonic oscillator potential) The period is independent of displacement: $\beta = 1$ independent of α. Hooke's law, $F = -kx$, $V(x) = \frac{1}{2}kx^2$ leads to oscillatory motion whose frequency is independent of the amplitude of motion. This was particularly useful for constructing robust clocks.

These four historical results in the development of early science are summarized in Table I and Fig. 2.

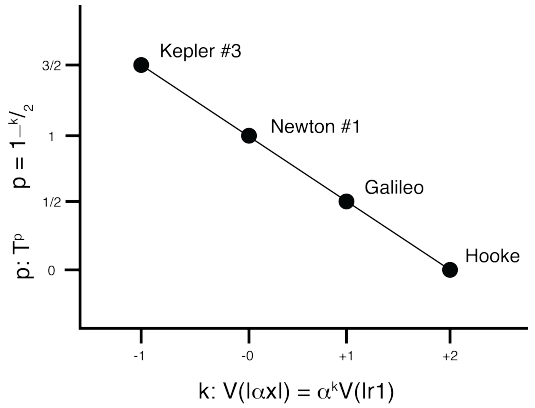

FIG. 2: Four substantial advances in the development of early Physics are summarized. Each is a consequence of using a homogeneous potential with a different degree k in a variational description of the dynamics. Scaling relates the size scale of the trajectory α to the time scale $\beta = \alpha^p$, $p = 1 - \frac{1}{2}k$ of the motion.

III. GROUPS: DEFINITIONS

In this Section we finally get to the point of defining what a group is by stating the group axioms (see: Wigner, 1959; Hamermesh, 1962; Tinkham, 1964; Gilmore, 2008; Sternberg, 1994). These are illustrated in the following sections with a number of examples: finite groups, including the two-element group, the group of transformations that leaves the equilateral triangle invariant, the permutation group, point groups, and discrete groups with a countable infinite number of group operations, such as space groups. Then we introduce groups of transformations in space as matrix groups. Lie groups are introduced and examples of matrix Lie groups are presented. Lie groups are linearized to form their Lie algebras, and groups are recovered from their algebras by reversing the linearization procedure using the exponential mapping. Many of the important properties of Lie algebras are introduced, including isomorphisms among different representations of a Lie algebra. A powerful disentangling theorem is presented and illustrated in a very simple case that plays a prominent rôle in the field of Quantum Optics. We will use this result in the Section on Special Functions.

A. Group Axioms

A group G consists of:

- a set of group elements: $g_0, g_1, g_2, g_3, \cdots \in G$
- a group operation, $\circ$, called group multiplication

that satisfy the following four axioms:

Closure: $g_i \in G,\ g_j \in G \Rightarrow g_i \circ g_j \in G$

Associativity: $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$

Identity: $g_0 \circ g_i = g_i = g_i \circ g_0$

Unique Inverse: $g_k \circ g_l = g_0 = g_l \circ g_k$

Group multiplication $\circ$ has two inputs and one output. The two inputs must be members of the set. The first axiom (**Closure**) requires that the output must also be a member of the set.

The composition rule $\circ$ does not allow us to multiply three input arguments. Rather, two can be combined to one, and that output can be combined with the third. This can be done in two different ways that preserves the order (i, j, k). The second axiom (**Associativity**) requires that these two different ways give the same final output.

The third axiom (**Identity**) requires that a special group element exists. This, combined with any other group element, gives back exactly that group element.

The fourth axiom (**Unique Inverse**) guarantees that for each group element g_l, there is another uniquely defined group element g_k, with the property that the product of the two is the unique identity element $g_k \circ g_l = g_0$. It is a simple matter to prove that $g_l \circ g_k = g_0$.

Remark 1 — Indexes: The notation (subscripts $i, j, k, \cdots$) may suggest that the indices are integers. This is not generally true: for continuous groups the indices are points in some subspace of a Euclidean space or more complicated manifold.

Remark 2 — Commutativity: In general the output of the group multiplication depends on the order of the inputs: $g_i \circ g_j \neq g_j \circ g_i$. If the result is independent of the order the group is said to be **commutative**.

It is not entirely obvious that the **Unique Inverse** axiom is needed. It is included among the axioms because many of our uses involve relating measurements made by two observers. For example, if Allyson on the Earth can predict something about the length of a year on Jupiter, then Bob on Jupiter should just as well be able to predict the length of Allyson's year on Earth. Basically, this axiom is an implementation of Galileo's Principle of Relativity.

B. Isomorphisms and Homomorphisms

It is often possible to compare two different groups. When it is possible it is very useful. Suppose we have two groups G with group operations $g_0, g_1, g_2, \cdots$ and group composition law $g_i \circ g_j = g_k$ and H with group operations $h_0, h_1, h_2, \cdots$ and group composition law $h_i \diamond h_j = k_k$. A mapping f from G to H is a **homomorphism** if it preserves the group operation:

$$f(g_i \circ g_j) = f(g_i) \diamond f(g_j) \tag{11}$$

In this expression $f(g_*) \in H$, so the two group elements $f(g_i)$ and $f(g_j)$ can only be combined using the combinatorial operation $\diamond$. If Eq. (11) is true for all pairs of group elements in G the mapping f is a homomorphism.

If G has four elements I, C_4, C_4^2, C_4^3 and H has two Id, C_2, the mapping $f(I) = f(C_4^2) = Id$, $f(C_4) = f(C_4^3) = C_2$, the mapping f is a homomorphism. If the mapping f is a homomorphism *and* is also 1:1, it is called an **isomorphism**. Under these conditions the inverse mapping also is an ismorphism:

$$f^{-1}(h_p \diamond h_q) = f^{-1}(h_p) \circ f^{-1}(h_q) \tag{12}$$

As an example, an isomorphism exists between the four group elements I, C_4, C_4^2, C_4^3 and the 2×2 matrices with $f(C_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

IV. EXAMPLES OF DISCRETE GROUPS

A. Finite Groups

1. The Two-Element Group Z_2

The simplest nontrivial group has one additional element beyond the identity e: $G = \{e, g\}$ with $g \circ g = e$. This group can act in our three-dimensional space R^3 in several different ways:

Reflection: $(x, y, z) \stackrel{g=\sigma_Z}{\rightarrow} (+x, +y, -z)$

Rotation: $(x, y, z) \stackrel{g=R_Z(\pi)}{\rightarrow} (-x, -y, +z)$

Inversion: $(x, y, z) \stackrel{g=\mathcal{P}}{\rightarrow} (-x, -y, -z)$

These three different actions of the order-two group on R^3 describe: reflections in the x-y plane, σ_Z; rotations around the Z axis through π radians, $R_Z(\pi)$; and inversion in the origin, the parity operation, $\mathcal{P}$. They can be distinguished by their *matrix representations*, which are

$$\begin{matrix} \sigma_Z \\ \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{matrix} \quad \begin{matrix} R_Z(\pi) \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix} \end{matrix} \quad \begin{matrix} \mathcal{P} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{matrix} \tag{13}$$

2. Group of Equilateral Triangle C_{3v}

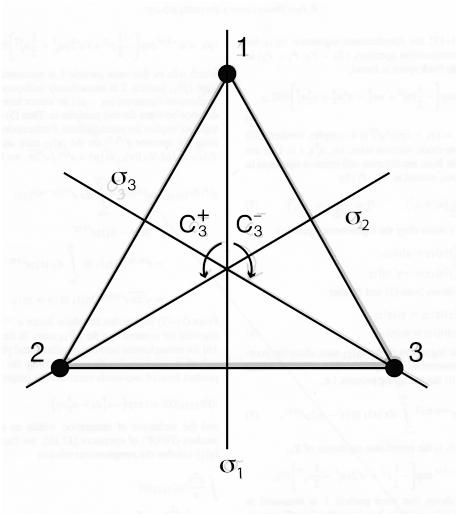

FIG. 3: The group of the equilateral triangle consists of: (a) the identify group operation e; (b) two rotations $C_3^{\pm}$ by $\pm\frac{2\pi}{3}$ about the centroid of the triangle; and (c) three reflections σ_i in straight lines between the centroid and each of the vertices i.

The six operations that map the equilateral triangle to itself constitute the group C_{3v} (c.f. Fig. 3). There are three distinct types of operations:

Identity: e This operation does nothing: it maps each vertex into itself.

Rotations: Two rotations $C_3^{\pm}$ about the center of the triangle through $\pm 2\pi/3$ radians.

Reflections: There are three reflections σ_i, each in a straight line through the center of the triangle and the vertex i ($i = 1, 2, 3$, c.f. Fig. 3).

These operations can be defined by their action on the vertices of the triangle. For example

$$C_3^+ \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{14}$$

The first description (( )) says that the rotation C_3^+ maps vertex 1 to vertex 2, $2 \to 3$ and $3 \to 1$. The second description ([ ]) can be understood as follows:

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{15}$$

The group multiplication law can be represented through a 6×6 matrix (Cayley multiplication table) that describes the output of $g_i \circ g_j$, with g_i listed by column and g_j by row:

g_j / g_i	e	C_3^+	C_3^-	σ_1	σ_2	σ_3
e	e	C_3^+	C_3^-	σ_1	σ_2	σ_3
C_3^+	C_3^+	C_3^-	e	σ_2	σ_3	σ_1
C_3^-	C_3^-	e	C_3^+	σ_3	σ_1	σ_2
σ_1	σ_1	σ_3	σ_2	e	C_3^-	C_3^+
σ_2	σ_2	σ_1	σ_3	C_3^+	e	C_3^-
σ_3	σ_3	σ_2	σ_1	C_3^-	C_3^+	e

(16)

This table makes clear that the group is not commutative: $C_3^- = \sigma_1 \circ \sigma_2 \neq \sigma_2 \circ \sigma_1 = C_3^+$.

The partition of the six elements in this group into three subsets of geometrically equivalent transformations is typical of any group. These subsets are called **classes**. Classes are defined by the condition

$$\textbf{Class}: \ \{h_1, h_2, \cdots\} \quad g \circ h_i \circ g^{-1} = h_j \quad \text{all } g \in G \tag{17}$$

All elements in the same class have essentially the same properties. They are **equivalent** under a group transformation. The three classes for the finite group C_{3v} are: $\{e\}, \{C_3^+, C_3^-\}, \{\sigma_1, \sigma_2, \sigma_3\}$.

It is clear from the group multiplication table (Eq. (16)) that C_{3v} has a number of (proper) **subgroups**: three subgroups of order two $\{e, \sigma_1\}, \{e, \sigma_2\}, \{e, \sigma_3\}$ and one of order three $\{e, C_3^+, C_3^-\}$. For technical reasons the single eelment $\{e\}$ and the entire group C_{3v} are also considered to be subgroups of C_{3v} (they are not proper subgroups). Whenever a group G has a subgroup H, it is always possible to write each group operation in G as the product of an operation in H with "something else": $g_i = h_j C_k$. For example, if H is the subgroup of order three we can choose the two elements C_1, C_2 ($2 = 6/3$) as $\{e, \sigma_1\}$. Then from Eq. (16)

$$\begin{aligned} \{e, C_3^+, C_3^-\} \circ e &= \{e, C_3^+, C_3^-\} \\ \{e, C_3^+, C_3^-\} \circ \sigma_1 &= \{\sigma_1 \sigma_2, \sigma_3\} \end{aligned} \tag{18}$$

Since in some sense G is a product of group operations in the subgroup H with group elements in C ($G = H \circ C$), we can formally write C as the "quotient" of G by the subgroup H: $C = H \backslash G$. If we composed in the reversed order: $G = CH$, then we could write $C = G/H$.

The set C is called a **coset**. It is not unique, but for finite groups its **order** (number of elements in the set) is unique: the quotient of the order G by the order of H. A coset may or may not be a group, depending whether the subgroup H is invariant in G ($gHg^{-1} \subset H$ for all $g \in G$) or not.

Remark': When G and H are Lie groups of dimensions d_G and d_H, G/H is a manifold of dimension d_G/d_H, and under a broad set of conditions this manifold has a geometric structure imparted by a Riemannian metric derived from the geometric properties of the two groups G and H (Gilmore, 2008).

3. Cyclic Groups C_n

The cyclic group consists of all rotations of the circle into itself through the angle $2\pi/n$ radians, and integer multiples of this angle. There are n such operations. The rotation through $2\pi k/n$ radians is obtained by applying the "smallest" rotation (also called C_n or C_n^1) k times. This smallest rotation is called a **generator** of the group. The group is commutative. There are therefore as many classes as group elements. The group operations can be put in 1 : 1 correspondence with the complex numbers and also with real 2×2 matrices:

$$\left[e^{i2\pi k/n}\right] \overset{1\times1}{\leftarrow} C_n^k \overset{2\times2}{\rightarrow} \begin{bmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ -\sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{bmatrix} \tag{19}$$

with $k = 0, 1, 2 \cdots, n-1$ or $k = 1, 2, \cdots n$. Every element in the group can be obtained by multiplying C_n^1 by itself. In the same way the 1×1 complex matrix with $k = 1$ is the generator for the 1×1 matrix representation of this group and the 2×2 real matrix with $k = 1$ is the generator for the 2×2 matrix representation of the group.

4. Permutation Groups S_n

Permutation groups act to interchange things. For example, if we have n numbers $1, 2, 3, \cdots n$, each permutation group operation will act to scramble the order of the

integers differently. Two useful ways to describe elements in the permutation group are shown in Eq. (14) for the permutation group on three vertices of an equilateral triangle. In the first case, the extension of this notation for individual group elements consists of a matrix with two rows, the top showing the ordering before the operation is applied, the bottom showing the ordering after the group operation has been applied. In the second case shown in Eq. (14) the extension consists of $n \times n$ matrices with exactly one $+1$ in each row and each column. The order of S_n is $n!$. Permutation groups are noncommutative for $n > 2$. $S_3 = C_{3v}$.

The permutation group plays a fundamental role in both mathematics and physics. In mathematics it is used to label the irreducible tensor representations of all Lie groups of interest. In physics it is required to distinguish among different states that many identical particles (either bosons or fermions) can assume.

5. Generators and Relations

If G is a discrete group, with either a finite or a countable number of group elements, it is useful to introduce a small set of **generators** $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ to describe the group. Every element in the group can be represented as a product of these generators and/or their inverses in some order.

For example, if there is only one generator $\{\sigma\}$ and every group element can be written in the form $g_n = \sigma^n$, $n = \cdots, -2, -1, 0, 1, 2, \cdots$ then G has a countable number of group elements. It is called a **free group** with one generator. If there are two generators $\{\sigma_1, \sigma_2\}$, the two generators commute $\sigma_1\sigma_2 = \sigma_2\sigma_1$, and every group operation can be expressed in the form $g_{m,n} = \sigma_1^m\sigma_2^n$ (m, n integers), the group is the free group with two commuting generators. Free groups with $k > 2$ generators are defined similarly. Free groups with 1, 2, 3 $\cdots$ generators are isomorphic to groups that act on periodic lattices in 1, 2, 3 $\cdots$ dimensions.

Often the generators satisfy **relations**. For example, a single generator σ may satisfy the relation $\sigma^p = I$. Then there are exactly p distinct group operations $I = \sigma^0, \sigma^1, \sigma^2, \cdots, \sigma^{p-1}$. The group with one generator and one relation is the cyclic group C_p. Generators $\{\sigma_1, \sigma_2, \cdots, \sigma_k\}$ and relations $f_l(\{\sigma_1, \sigma_2, \cdots, \sigma_k\}) = I$, $l = 1, 2, \cdots$ have been used to define many different groups. In fact, every discrete group is either defined by a set of generators and relations, or else a subgroup of such a group. The symmetric group S_n is defined by $n-1$ generators $\sigma_i, i = 1, 2, \cdots, n-1$ and the relations $\sigma_i^2 = I$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $j \neq i \pm 1$, and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. The tetrahedral ($T$), octahedral ($O$), and icosahedral ($I$) point groups are defined by two generators and three relations: $\sigma_1^2 = I, \sigma_2^3 = I, (\sigma_1\sigma_2)^p = I$ with $p = 3, 4, 5$, respectively. The quaternion group Q_8 can be defined with two generators and two relations $\sigma_1\sigma_2\sigma_1 = \sigma_2, \sigma_2\sigma_1\sigma_2 = \sigma_1$ or in terms of three generators and four relations $\sigma_1^4 = \sigma_2^4 = \sigma_3^4 = I, \sigma_1\sigma_2\sigma_3 = I$. In the latter case the three generators can be chosen as 2×2 matrices that are the three Pauli spin matrices, multiplied by $i = \sqrt{-1}$.

The study of discrete groups defined by generators and relations has a long and very rich history.

B. Infinite Discrete Groups

1. Translation Groups: 1 Dimension

Imagine a series of points at locations na along the straight line, where a is a physical parameter with dimensions of length ($[a] = L$) and n is an integer. The group that leaves this set invariant consists of rigid displacements through integer multiples of the fundamental length. The operation T_{ka} displaces the point at na to position $(n + k)a$. This group has a single generator T_a, and $T_{ka} = T_a \circ T_a \circ \cdots \circ T_a = T_a^k$. It is convenient to represent these group operations by 2×2 matrices

$$T_{ka} \to \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix} \tag{20}$$

In this representation group composition is equivalent to matrix multiplication. The group is commutative. The generator for the group and this matrix representation is obtained by setting $k = 1$. There is also an entire set of 1×1 complex matrix representations indexed by a real parameter p with generator $T_a \to \left[e^{ipa}\right]$. The representations with $p' = p + 2\pi/a$ are equivalent, so all the inequivalent complex representations can be parameterized by real values of p in the range $0 \leq p < 2\pi/a$ or, more symmetrically $-\pi/a \leq p \leq \pi/a$, with the endpoints identified. The real parameter p is in the dual space to the lattice, called the first Brillouin zone.

2. Translation Groups: 2 Dimensions

Now imagine a series of lattice points in a plane at positions $\mathbf{x} = i_1\mathbf{f}_1 + i_2\mathbf{f}_2$. Here i_1, i_2 are integers and the vectors $\mathbf{f}_1, \mathbf{f}_2$ are not colinear but otherwise arbitrary. Then the set of rigid displacements (j_1, j_2) move lattice points $\mathbf{x}$ to new locations as per

$$T_{j_1\mathbf{f}_1 + j_2\mathbf{f}_2}\,(i_1\mathbf{f}_1 + i_2\mathbf{f}_2) = (i_1 + j_1)\mathbf{f}_1 + (i_2 + j_2)\mathbf{f}_2 \tag{21}$$

Generalizing Eq. (20), there is a simple $1:1$ (or **faithful**) matrix representation for this group of rigid translations:

$$T_{j_1\mathbf{f}_1 + j_2\mathbf{f}_2} \to \begin{bmatrix} 1 & 0 & j_1|\mathbf{f}_1| \\ 0 & 1 & j_2|\mathbf{f}_2| \\ 0 & 0 & 1 \end{bmatrix} \tag{22}$$

Extension to groups of rigid displacements of lattices in higher dimensions is straightforward.

3. Spacegroups

When $|\mathbf{f}_1| = |\mathbf{f}_2|$ and the two vectors are orthogonal, rotations through $k\pi/2$ ($k = 1, 2, 3$) radians about any lattice point map the lattice into itself. So also do reflections in lines perpendicular to $|\mathbf{f}_1|$ and $|\mathbf{f}_2|$ as well as lines perpendicular to $\pm|\mathbf{f}_1| \pm |\mathbf{f}_2|$. This set of group operations contains displacements, rotations, and reflections. It is an example of a two-dimensional *space group*. There are many other space groups in two dimensions and very many more in three dimensions. These groups were first used to enumerate the types of regular lattices that Nature allows in two and three dimensions (Tinkham, 1964). After the development of Quantum Mechanics they were used in another way (depending on the theory of representations): to give names to wavefunctions that describe electrons (and also phonons) in these crystal lattices.

V. EXAMPLES OF MATRIX GROUPS

A. Translation Groups

The group of rigid translations of points in R^3 through distances a_1 in the x-direction, a_2 in the y-direction, and a_3 in the z-direction can be described by simple block 4×4 ($4 = 3 + 1$) matrices:

$$T_{a_1,a_2,a_3} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 1 & a_3 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{23}$$

If the $\mathbf{a}$ belong to a lattice the group is discrete. If they are continuous the group is continuous and has dimension three.

B. Heisenberg Group H_3

The Heisenberg group H_3 plays a fundamental role in quantum mechanics. As it appears in the quantum theory it is described by "infinite-dimensional" matrices. However, the group itself is three dimensional. In fact, it has a simple faithful description in terms of 3×3 matrices depending on three parameters:

$$h(a,b,c) = \left[\begin{array}{c|cc} 1 & a & c \\ \hline 0 & 1 & b \\ 0 & 0 & 1 \end{array}\right] \tag{24}$$

The matrix representation is faithful because any matrix of the form (24) uniquely defines the abstract group operation $h(a, b, c)$. The group is not commutative. The group multiplication law can be easily seen via matrix multiplication:

$$h_1 h_2 = h_3 = h(a_3, b_3, c_3) = \left[\begin{array}{c|cc} 1 & a_1 + a_2 & c_2 + a_1 b_2 \\ \hline 0 & 1 & b_1 + b_2 \\ 0 & 0 & 1 \end{array}\right] \tag{25}$$

The group composition law given in Eq. (25) defines the Heisenberg group. The result $c_3 = a_1 b_2 + c_2$ leads to remarkable noncommutativity properties among canonically conjugate variables in the quantum theory: $[p, x] = \hbar/i$.

C. Rotation Group $SO(3)$

The set of rigid rotations of R^3 forms a group. It is conveniently represented by a faithful 3×3 matrix. The 3×3 matrix describing rotations about an axis of unit length $\hat{\mathbf{n}}$ through an angle θ, $0 \le \theta \le \pi$ is

$$(\hat{\mathbf{n}}, \theta) \rightarrow I_3 \cos\theta + \hat{\mathbf{n}} \cdot \mathbf{L} \sin\theta + \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} \times \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{bmatrix} (1 - \cos\theta) \tag{26}$$

Here $\mathbf{L}$ is a set of three 3×3 angular momentum matrices

$$L_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad L_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad L_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

The matrix multiplying $(1 - \cos\theta)$ in Eq. (26) is a 3×3 matrix: it is the product of a 3×1 with a 1×3 matrix. We will show later how this marvelous expression has been derived.

There is a 1:1 correspondence between points in the interior of a ball of radius π and rotations through an angle in the range $0 \le \theta < \pi$. Two points on the surface $(\hat{\mathbf{n}}, \pi)$ and $(-\hat{\mathbf{n}}, \pi)$ describe the same rotation. The parameter space describing this group is not a simply connected submanifold of R^3: it is a doubly connected manifold. The relation between continuous groups and their underlying parameter space involves some fascinating topology.

D. Lorentz Group $SO(3,1)$

The Lorentz group is the group of linear transformations that leave invariant the square of the distance between two nearby points in spacetime: (cdt, dx, dy, dz) and (cdt', dx', dy', dz'). The distance can be written in matrix form:

$$(cd\tau)^2 = (cdt)^2 - (dx^2 + dy^2 + dz^2) =$$

$$\begin{bmatrix} cdt & dx & dy & dz \end{bmatrix} \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix} \quad (28)$$

If the infinitesimals in the primed coordinate system are related to those in the unprimed coordinate system by a linear transformation — $dx^{'\mu} = M^\mu_\nu dx^\nu$ — then the matrices M must satisfy the constraint (^{t} means matrix transpose and the summation convention has been used: doubled indices are summed over.)

$$M^t I_{1,3} M = I_{1,3} \quad (29)$$

where $I_{1,3}$ is the diagonal matrix $diag(+1,-1,-1,-1)$. The matrices M belong to the orthogonal group $O(1,3)$. This is a six-parameter group. Clearly the rotations (three dimensions worth) form a subgroup, represented by matrices of the form $\left[\begin{array}{c|c} \pm 1 & 0 \\ \hline 0 & \pm R(\hat{\mathbf{n}},\theta) \end{array}\right]$, where $R(\hat{\mathbf{n}},\theta)$ is given in Eq.(26). This group has four disconnected components, each connected to a 4×4 matrix of the form $\begin{bmatrix} 1 & 0 \\ 0 & I_3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -I_3 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -I_3 \end{bmatrix}$. We choose the component connected to the identity I_4. This is the special Lorentz group $SO(1,3)$. A general matrix in this group can be written in the form

$$SO(1,3) = B(\boldsymbol{\beta}) R(\boldsymbol{\theta}) \quad (30)$$

where the matrices $B(\boldsymbol{\beta})$ describe **boost** transformations and $R(\boldsymbol{\theta}) = R(\hat{\mathbf{n}},\theta)$. A boost transformation maps a coordinate system at rest to a coordinate moving with velocity $\mathbf{v} = c\boldsymbol{\beta}$ and with axes parallel to those in the stationary coordinate system. We will describe these transformations in more detail below.

Since every group operation in $SO(1,3)$ can be expressed as the product of a rotation operation with a boost, we can formally write boost transformations as elements in a coset: $B(\boldsymbol{\beta}) = SO(1,3)/SO(3)$.

A general boost transformation can be written in the form

$$B(\boldsymbol{\beta}) = \left[\begin{array}{c|c} \gamma & \gamma\boldsymbol{\beta} \\ \hline \gamma\boldsymbol{\beta} & I_3 + (\gamma-1)\frac{\beta_i\beta_j}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} \end{array}\right] \quad (31)$$

For example, a boost in the x-direction with $\mathbf{v}/c = (\beta,0,0)$ has the following effect on coordinates ($\beta = |\boldsymbol{\beta}|$):

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}' = \left[\begin{array}{c|ccc} \gamma & \gamma\beta & 0 & 0 \\ \hline \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma(ct+\beta x) \\ \gamma(x+\beta ct) \\ y \\ z \end{bmatrix} \quad (32)$$

Here $\gamma^2 - (\beta\gamma)^2 = 1$ so $\gamma = 1/\sqrt{1-\beta^2}$. In the nonrelativistic limit $x' = \gamma(x+\beta ct) \to x + vt$, so β has an interpretation of $\boldsymbol{\beta} = \mathbf{v}/c$.

The product of two boosts in the same direction is obtained by matrix multiplication. This can be carried out on a 2×2 submatrix of that given in Eq. (32):

$$B(\beta_1)B(\beta_2) = \begin{bmatrix} \gamma_1 & \beta_1\gamma_1 \\ \beta_1\gamma_1 & \gamma_1 \end{bmatrix} \begin{bmatrix} \gamma_2 & \beta_2\gamma_2 \\ \beta_2\gamma_2 & \gamma_2 \end{bmatrix} = \begin{bmatrix} \gamma_{\rm tot} & \beta_{\rm tot}\gamma_{\rm tot} \\ \beta_{\rm tot}\gamma_{\rm tot} & \gamma_{\rm tot} \end{bmatrix} \quad (33)$$

Simple matrix multiplication shows $\beta_{\rm tot} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$, which is the relativistic velocity addition formula for parallel velocity transformations.

When the boosts are not parallel, their product is a transformation in $SO(1,3)$ that can be written as the product of a boost with a rotation:

$$B(\boldsymbol{\beta}_1)B(\boldsymbol{\beta}_2) = B(\boldsymbol{\beta}_{\rm tot})R(\boldsymbol{\theta}) \quad (34)$$

Multiplying two boost matrices of the form given in Eq. (31) leads to a simple expression for $\gamma_{\rm tot}$ and a more complicated expression for $\boldsymbol{\beta}_{\rm tot}$

$$\begin{aligned} \gamma_{\rm tot} &= \gamma_1\gamma_2(1+\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2) \\ \gamma_{\rm tot}\boldsymbol{\beta}_{\rm tot} &= \left[\gamma_1\gamma_2 + (\gamma_1-1)\gamma_2\frac{\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2}{\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_1}\right]\boldsymbol{\beta}_1 + \gamma_2\boldsymbol{\beta}_2 \end{aligned} \quad (35)$$

This shows what is intuitively obvious: the boost direction is in the plane of the two boosts. Less obvious is the rotation required by noncollinear boosts. It is around an axis parallel to the crossproduct of the two boosts. When the two boosts are perpendicular the result is

$$\hat{\mathbf{n}}\ \sin(\theta) = -\boldsymbol{\beta}_1\times\boldsymbol{\beta}_2\ \cdot\ \frac{\gamma_1\gamma_2}{1+\gamma_1\gamma_2} \quad (36)$$

When one of the boosts is infinitesimal we find

$$B(\boldsymbol{\beta})B(\delta\boldsymbol{\beta}) = B(\boldsymbol{\beta}+d\boldsymbol{\beta})R(\hat{\mathbf{n}}d\theta) \quad (37)$$

Multiplying out these matrices and comparing the two sides gives:

$$
\begin{aligned}
d\boldsymbol{\beta} &= \gamma^{-1}\delta\boldsymbol{\beta} + \left(\frac{\gamma^{-1}-1}{\gamma\beta^2}\right)(\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta})\boldsymbol{\beta} \\
\hat{\mathbf{n}}d\theta &= \left(\frac{1-\gamma^{-1}}{\beta^2}\right)\delta\boldsymbol{\beta}\times\boldsymbol{\beta}
\end{aligned}
\tag{38}
$$

In the nonrelativistic limit, when $\boldsymbol{\beta}$ is also small, $1-\gamma^{-1}/\beta^2 \to \frac{1}{2}$. This (in)famous factor of 1/2 is known as the "Thomas factor" in atomic physics.

VI. LIE GROUPS

The group elements g in a Lie group are parameterized by points $\mathbf{x}$ in a manifold $\mathcal{M}^n$ of dimension n: $g = g(\mathbf{x})$, $\mathbf{x} \in \mathcal{M}^n$. The product of two group operations $g(\mathbf{x})$ and $g(\mathbf{y})$ is parameterized by a point $\mathbf{z}$ in the manifold: $g(\mathbf{x}) \circ g(\mathbf{y}) = g(\mathbf{z})$, where $\mathbf{z} = \mathbf{z}(\mathbf{x}, \mathbf{y})$. This composition law can be very complicated. It is necessarily nonlinear (c.f., Eq. (25) for H_3) unless the group is commutative. For example, the parameter space for the group $SO(3)$ consists of points in R^3: $\boldsymbol{\theta} = (\hat{\mathbf{n}}, \theta)$. Only a compact subspace consisting of a sphere of radius π is needed to parameterize this Lie group.

Almost all of the Lie groups of use to physicists exist as matrix groups. For this reason it is possible for us to skip over the fundamental details of whether the composition law must be analytic and the elegant details of their definition and derivations. The composition law can be constructed as follows:

1. Construct a useful way to parameterize each group element as a matrix depending on a suitable number of parameters.
2. Perform matrix multiplication of two group elements.
3. Find the group element that corresponds to the product of the two matrices given in Step 2.

We list several types of matrix groups below.

GL(n;R), GL(n;C), GL(n;Q): These groups consist of $n \times n$ invertible matrices, each of whose n^2 matrix elements are real numbers, complex numbers, or quaternions. The group composition law is matrix multiplication. The numbers of real parameters required to specify an element in these groups are: $n^2, 2\times n^2, 4\times n^2$, respectively.

SL(n;R), SL(n;C): The letter "S" here means "special" as opposed to G meaning "general" in the matrix groups described above. These groups are subgroups of $GL(n;R)$ and $GL(n;C)$ containing the subset of matrices with determinant +1. The real dimensions of these groups are $(n^2-1)\times \dim(F)$ where $\dim(F) = (1, 2)$ for $F = (R, C)$.

O(n), U(n), Sp(n): Three important classes of groups are defined by placing quadratic constraints on matrices. The orthogonal group $O(n)$ is the subgroup of $GL(n;R)$ containing only matrices M that satisfy $M^t I_n M = I_n$. Here we use previously introduced notation: I_n is the unit $n \times n$ matrix and ^{t} signifies the transpose of the matrix. This constraint arises in a natural way when requiring that linear transformations in a real n-dimensional linear vector space preserve a positive definite inner product. The unitary group $U(n)$ is the subgroup of $GL(n;C)$ for which the matrices M satisfy $M^\dagger I_n M = I_n$, where † signifies the adjoint, or complex conjugate transpose matrix. The symplectic group $Sp(n)$ is defined similarly for the quaternions. In this case † signifies quaternion conjugate transpose. The real dimensions of these groups are $n(n-1)/2, n^2$, and $n(2n+1)$, respectively.

SO(n), SU(n): as above, "S" stands for *special*. For the group $O(n)$ the determinant of any group element is a real number whose modulus is +1: i.e., ± 1. Placing the special constraint on the group of orthogonal transformations reduces the "number" of elements in the group by one half (in a measure theoretic sense) but does not reduce the dimension of the space required to parameterize the elements in this group. For the group $U(n)$ the determinant of any group element is a complex number whose modulus is +1: i.e., $e^{i\phi}$. Placing the special constraint on $U(n)$ reduces the dimension by one: $\dim SU(n) = n^2-1$. The symplectic group $Sp(n)$ has determinant +1.

O(p,q), U(p,q), Sp(p,q): These groups are defined by replacing I_n in the definitions for $O(n), U(n), Sp(n)$ by the matrix $I_{p,q} = \left[\begin{array}{c|c} +I_p & 0 \\ \hline 0 & -I_q \end{array}\right]$. These groups preserve an *indefinite* nonsingular metric in linear vector spaces of dimension $(p+q)$. The groups $O(n), U(n), Sp(n)$ are *compact* (a useful topological concept) and so are relatively easy to deal with. This means effectively that only a finite volume of parameter space is required to parameterize every element in the group. The groups $O(p,q), U(p,q), Sp(p,q)$ are not compact if both p and q are nonzero. Further, $O(p,q) \simeq O(q,p)$ by a simple similarity transformation, and similarly for the others.

VII. LIE ALGEBRAS

A Lie algebra is a linear vector space on which an additional composition law $[,]$ is defined. If X, Y, Z are elements in a Lie algebra $\mathcal{L}$, then linear combinations are in the Lie algebra: $\alpha X + \beta Y \in \mathcal{L}$ (this is the linear vector space property), the commutator of two operations $[X, Y]$ is in $\mathcal{L}$, and in addition the new composition law satisfies the following axioms:

$$
\begin{aligned}
[\alpha X + \beta Y, Z] &= \alpha\,[X, Z] + \beta\,[Y, Z] \\
[X, Y] + [Y, X] &= 0 \\
[X, [Y, Z]] + [Y, [Z, Z]] + [Z, [X, Y]] &= 0
\end{aligned}
\tag{39}
$$

The first of these properties preserves the linear vector space properties of $\mathcal{L}$. The second property defines the commutator bracket $[,]$ as an antisymmetric compositon law: $[X,Y] = -[Y,X]$, and the third imposes an integrability constraint called the Jacobi identity.

In principal, commutators are defined by the properties presented in eq. (39), whether or not composition of the operators X and Y is defined. If this composition is defined, then $[X,Y] = XY - YX$ and the commutator can be computed by applying the operators X and Y in different orders and subtracting the difference. For example, if $X = y\partial_z - z\partial_y$ (here $\partial_z = \frac{\partial}{\partial z}$) and if $Y = z\partial_x - x\partial_z$, then the operator XY can be applied to a general function whose second partial derivatives are continuous to give $XYf(x,y,z) = yf_x + yzf_{xz} - z^2 f_{xy} - xyf_{zz} + zxf_{yz}$. The value of $YXf(x,y,z)$ is computed similarly, and the difference is $(XY - YX)f = yf_x - xf_y = (y\partial_x - x\partial_y)f$. Since this holds for an arbitrary function $f(x,y,z)$ for which all second partial derivatives exist and are independent of the order taken, we find $[X,Y] = (XY - YX) = (y\partial_x - x\partial_y)$.

A. Structure Constants

When the underlying linear vector space for $\mathcal{L}$ has dimension n it is possible to choose a set of n basis vectors (matrices, operators) X_i. The commutation relations are encapsulated by a set of **structure constants** C_{ij}^k that are defined by

$$[X_i, X_j] = C_{ij}^k X_k \tag{40}$$

A Lie algebra is defined by its structure constants.

B. Constructing Lie Algebras by Linearization

Lie algebras are constructed for a Lie group by linearizing the constraints that define the Lie group in the neighborhood of the identity I. Matrix Lie algebras are obtained for $n \times n$ matrix Lie groups by linearizing the matrix group in the neighborhood of the unit matrix I_n. A Lie group and its Lie algebra have the same dimension.

In the neighborhood of the identity the groups $GL(n;R), GL(n;C), GL(n,Q)$ have the form

$$GL(n;F) \to I_n + \delta M \tag{41}$$

where δM is an $n \times n$ matrix, all of whose matrix elements are small. Over the real, complex, and quaternion fields the matrix elements are small real or complex numbers or small quaternions. Quaternions q can be expressed as 2×2 complex matrices using the Pauli spin matrices σ_μ:

$$q \to (c_0, c_1) = (r_0, r_1, r_2, r_3) = \sum_{\mu=0}^{3} r_\mu \sigma_\mu =$$

$$\begin{bmatrix} r_0 + ir_3 & r_1 - ir_2 \\ r_1 + ir_2 & r_0 - ir_3 \end{bmatrix} \tag{42}$$

The Lie algebras $\mathfrak{g}l(n;F)$ of $GL(n;F)$ have dimensions $dim(F) \times n^2$, with $dim(F) = 1, 2, 4 = 2^2$ for $F = R, C, Q$.

For the special linear groups, the determinant of a group element near the identity is

$$\det(I_n + \delta M) = 1 + \mathrm{tr}\delta M + \text{h.o.t.} \tag{43}$$

In order to ensure the unimodular condition, the Lie algebras of the special linear groups consist of traceless matrices. The Lie algebra $\mathfrak{sl}(n;R)$ of $SL(n;R)$ consists of real traceless $n \times n$ matrices. It has dimension $n^2 - 1$. The Lie algebra $\mathfrak{sl}(n;C)$ of $SL(n;C)$ consists of traceless complex $n \times n$ matrices. It has real dimension $2n^2 - 2$.

Many Lie groups are defined by a metric-preserving condition: $M^\dagger \mathbf{G} M = \mathbf{G}$, where $\mathbf{G}$ = is some suitable metric matrix (see the discussion of the Lorentz group $SO(3,1) \simeq SO(1,3)$ that preserves the metric $(+1,+1,+1,-1)$ in Subsec. **??** and the groups **O(p,q), U(p,q), Sp(p,q)** in Subsec. **??**). The linearization of this condition is

$$M^\dagger \mathbf{G} M = (I_n + \delta M)^\dagger \mathbf{G} (I_n + \delta M) = \mathbf{G}$$

so that from $M^\dagger \mathbf{G} M = \mathbf{G}$ it follows that, neglecting small terms of order two

$$\delta M^\dagger \mathbf{G} + \mathbf{G} \delta M = 0 \tag{44}$$

Thus the lie algebras $\mathfrak{so}(n;R), \mathfrak{su}(n;C), \mathfrak{sp}(n;Q)$, which correspond to the case $\mathbf{G} = I_n$, consist of real antisymmetric matrices $M^t = -M$, complex traceless antihermitian matrices $M^\dagger = -M$, and quaternion antihermitian matrices $M^\dagger = -M$, respectively.

The Lie algebra $\mathfrak{so}(3)$ of the rotation group $SO(3) = SO(3;R)$ consists of real 3×3 antisymmetric matrices. This group and its algebra are three dimensional. The Lie algebra (it is a linear vector space) is spanned by three "basis vectors". These are 3×3 antisymmetric matrices. A standard choice for these basis vectors is given in Eq. (27). Their commutation relations are given by

$$[L_i, L_j] = -\epsilon_{ijk} L_k \tag{45}$$

The structure constants for $\mathfrak{so}(3)$ are $C_{ij}^k = -\epsilon_{ijk}$, $1 \le i,j,k \le 3$. Here ϵ_{ijk} is the "sign symbol" (antisymmetric Levi-Civita 3-tensor) which is zero if any two symbols are the same, $+1$ for a cyclic permutation of inters (123), and -1 for a cyclic permutation of (321).

Two Lie algebras with the same set of structure constants are isomorphic. The Lie algebra of 2×2 matrices obtained from $\mathfrak{su}(2)$ is spanned by three operators that

can be chosen as proportional ($\frac{i}{2}$) to the Pauli spin matrices (c.f., Eq. (42)):

$$S_1 = \frac{i}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad S_2 = \frac{i}{2}\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} \quad S_3 = \frac{i}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{46}$$

These three operators satisfy the commutation relations

$$[S_i, S_j] = -\epsilon_{ijk} S_k \tag{47}$$

As a result, the Lie algebra for the group $\mathfrak{so}(3)$ of rotations in R^3 is isomorphic to the Lie algebra $\mathfrak{su}(2)$ for the group of unimodular metric-preserving rotations in a complex two dimensional space, $SU(2)$. Spin and orbital rotations are intimately connected.

C. Constructing Lie Groups by Exponentiation

The mapping of a Lie group, with a complicated nonlinear composition, down to a Lie algebra with a simple linear combinatorial structure plus a commutator, would not be so useful if it were not possible to undo this mapping. In effect, the linearization is "undone" by the exponential map. For an operator X the exponential is defined in the usual way:

$$EXP(X) = e^X = I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{X^k}{k!} \tag{48}$$

The radius of convergence of the exponential function is infinite. This means that we can map a Lie algebra back to its parent Lie group in an algorithmic way.

We illustrate with two important examples. For the first, we construct a simple parameterization of the group $SU(2)$ by exponentiating its Lie algebra. The Lie algebra is given in Eq. (46). Define $M = \frac{i}{2}\hat{\mathbf{n}} \cdot \sigma\theta$. Then $M^2 = -(\theta/2)^2 I_2$ is a diagonal matrix. The exponential expansion can be rearranged to contain even powers in one sum and odd powers in another:

$$e^M = I_2 \left(1 - \frac{(\theta/2)^2}{2!} + \frac{(\theta/2)^4}{4!} - \cdots\right) +$$

$$M \left(1 - \frac{(\theta/2)^2}{3!} + \frac{(\theta/2)^4}{5!} - \cdots\right) \tag{49}$$

The even terms sum to $\cos(\theta/2)$ and the odd terms sum to $\sin(\theta/2)/(\theta/2)$. The result is

$$EXP\left(\frac{i}{2}\hat{\mathbf{n}} \cdot \sigma\theta\right) = \cos\frac{\theta}{2} I_2 + i\hat{\mathbf{n}} \cdot \sigma \sin\frac{\theta}{2} \tag{50}$$

A similar power series expansion involving the angular momentum matrices in Eq. (27) leads to the parameterization of the rotation group operations given in Eq. (26). Specifically, $EXP\,(\hat{\mathbf{n}} \cdot \mathbf{L}\theta) =$

$$I_3 \cos\theta + \hat{\mathbf{n}} \cdot \mathbf{L} \sin\theta + \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \hat{n}_3 \end{bmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} (1 - \cos\theta) \tag{51}$$

The Lie groups $SO(3)$ and $SU(2)$ possess isomorphic Lie algebras. The Lie algebra is three-dimensional. The basis vectors in $\mathfrak{so}(3)$ can be chosen as the angular momentum matrices given in Eq. (27) and the basis vectors for $\mathfrak{su}(2)$ as $i/2$ times the Pauli spin matrices, as in Eq. (46). A point in the Lie algebra (e.g., R^3) can be identified by a unit vector $\hat{\mathbf{n}}$ and a radial distance from the origin θ. Under exponentiation, the point $(\hat{\mathbf{n}}, \theta)$ maps to the group operation given in Eq. (51) for $SO(3)$ and in Eq. (50) for $SU(2)$.

The simplest way to explore how the Lie algebra parameterizes the two groups is to look at how points along a straight line through the origin of the Lie algebra map to operations in the two groups. For simplicity we choose the z-axis. Then $(\hat{\mathbf{z}}, \theta)$ maps to

$$\begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \in SU(2), \quad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO(3) \tag{52}$$

As θ increases from 0 to 2π the $SU(2)$ group operation varies from $+I_2$ to $-I_2$ while the $SO(3)$ group operation starts at I_3 and returns to $+I_3$. The $SU(2)$ group operation returns to the identity $+I_2$ only after θ increases from 2π to 4π. The rotations by θ and $\theta + 2\pi$ give the same group operation in $SO(3)$ but they describe group operations in $SU(2)$ that differ by sign: $(\hat{\mathbf{z}}, 2\pi + \theta) = -I_2 \times (\hat{\mathbf{z}}, \theta)$. In short, two 2×2 matrices M and $-M$ in $SU(2)$ map to the same group operation in $SO(3)$. In words, $SU(2)$ is a **double cover** of $SO(3)$.

For $SU(2)$ all points inside a sphere of radius 2π in the Lie albegra map to different group operations, and all points on the sphere surface map to one group operation $-I_2$. The group $SU(2)$ is **simply connected**. Any path starting and ending at the same point (for example, the identity) can be continuously contracted to the identity.

By contrast, for $SO(3)$ all points inside a sphere of radius π in the Lie algebra map to different group operations, and two points $(\hat{\mathbf{n}}, \pi)$ and $-(\hat{\mathbf{n}}, \pi)$ at opposite ends of a straight line through the origin map to the same group operation. The group $SO(3)$ is *not* simply connected. Any closed path from the origin that cuts the surface $\theta = \pi$ once (or an odd number of times) cannot be continuously deformed to the identity. The group $SO(3)$ is **doubly connected**.

This is the simplest example of a strong theorem by Cartan. There is a 1 : 1 relation between Lie algebras and

simply connected Lie groups. Every Lie group with the same (isomorphic) Lie algebra is either simply connected or else the quotient (coset) of the simply connected Lie group by a discrete invariant subgroup.

For matrix Lie groups, discrete invariant subgroups consist of scalar multiples of the unit matrix. For the the isomorphic Lie algebras $\mathfrak{su}(2) = \mathfrak{so}(3)$ the Lie group $SU(2)$ is simply connected. Its discrete invariant subgroup consists of multiples of the identity matrix: $\{I_2, -I_2\}$. Cartan's theorem states $SO(3) = SU(2)/\{I_2, -I_2\}$. This makes explicit the $2 \downarrow 1$ nature of the relation between $SU(2)$ and $SO(3)$.

The group $SU(3)$ is simply connected. It discrete invariant subgroup consists of $\left\{I_3, \omega I_3, \omega^2 I_3\right\}$, with $\omega^3 = 1$. The only other Lie group with the Lie algebra $\mathfrak{su}(3)$ is the $3 \downarrow 1$ image $SU(3)/\left\{I_3, \omega I_3, \omega^2 I_3\right\}$. This group has a description in terms of real eight dimensional matrices ("the eightfold way").

D. Cartan Metric

The notation for the structure constants C_{ij}^k for a Lie algebra gives the appearance of being components of a tensor. In fact, they are: the tensor is first order contravariant (in k) and second order covariant, and antisymmetric, in i, j. It is possible to form a second order covariant tensor (Cartan-Killing metric) from the components of the structure constant by double contraction:

$$g_{ij} = \sum_{rs} C_{ir}^s C_{js}^r = g_{ji} \tag{53}$$

This real symmetric tensor "looks like" a metric tensor. In fact, it has very powerful properties. If g_{**} is nonsingular the Lie algebra, and its Lie group, is "semisimple" or "simple" (these are technical terms meaning that the matrices describing the Lie algebras are either fully reducible or irreducible). If g_{**} is negative definite, the group is compact. It is quite remarkable that an algebraic structure gives such powerful topological information.

As an example, for $SO(3)$ and $SU(2)$ the Cartan-Killing metric Eq. (53) is

$$g_{ij} = \sum_{r,s} (-\epsilon_{irs})(-\epsilon_{jsr}) = -\delta_{ij} \tag{54}$$

For the real forms $SO(2,1)$ of $SO(3)$ and $SU(1,1)$ of $SU(2)$ the Cartan-Killing metric tensor is

$$g\left(\mathfrak{so}(2,1)\right) = g\left(\mathfrak{su}(1,1)\right) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{55}$$

The structure of this metric tensor (two positive diagonal elements or eigenvalues, and one negative) tells us about the topology of the groups: they have two noncompact directions and one compact direction. The compact direction describes the compact subgroups $SO(2)$ and $U(1)$, respectively.

E. Operator Realizations of Lie Algebras

Each Lie algebra has three useful operator realizations. They are given in terms of boson operators, fermion operators, and differential operators.

Boson annihilation operators b_i and creation operators $b_j^\dagger$ for independent modes $i, j = 1, 2, \cdots$, their fermion counterparts $f_i, f_j^\dagger$, and the operators ∂_i, x_j satisfy the following commutation or anticommutation relations

$$\begin{array}{rcccl} \left[b_i, b_j^\dagger\right] & = & b_i b_j^\dagger - b_j^\dagger b_i & = & \delta_{ij} \\ \left\{f_i, f_j^\dagger\right\} & = & f_i f_j^\dagger + f_j^\dagger f_i & = & \delta_{ij} \\ \left[\partial_i, x_j\right] & = & \partial_i x_j - x_j \partial_i & = & \delta_{ij} \end{array} \tag{56}$$

In spite of the fact that bosons and differential operators satisfy commutation relations and fermion operators satisfy anticommutation (see the + sign in Eq. (56)) relations, bilinear combinations $Z_{ij} = b_i^\dagger b_j, f_i^\dagger f_j, x_i \partial_j$ of these operators satisfy commutation relations:

$$\left[Z_{ij}, Z_{rs}\right] = Z_{is}\delta_{jr} - Z_{rj}\delta_{si} \tag{57}$$

These commutation relations can be used to associate operator algebras to matrix Lie algebras. The procedure is simple. We illustrate for boson operators. Assume $A, B, C = [A, B]$ are $n \times n$ matrices in a matrix Lie algebra. Associate operator $\mathcal{A}$ to matrix A by means of

$$A \to \mathcal{A} = b_i^\dagger A_{ij} b_j \tag{58}$$

and similarly for other matrices. Then

$$[\mathcal{A}, \mathcal{B}] = \left[b_i^\dagger A_{ij} b_j, b_r^\dagger B_{rs} b_s\right] = b_i^\dagger \left[A, B\right]_{is} b_s = b_i^\dagger C_{is} b_s = \mathcal{C} \tag{59}$$

This result holds if the bilinear combinations of boson creation and annihilation operators are replaced by bilinear combinations of fermion creation and annihilation operators or products of multiplication (by x_i) and differentiation (by ∂_j) operators.

One consequence of this matrix Lie algebra to operator algebra isomorphism is that any Hamiltonian that can be expressed in terms of bilinear products of creation and annihilation operators for either bosons or fermions can be studied in a simpler matrix form.

The operator algebra constructed from the spin operators in Eq. (46) has been used by Schwinger for an elegant construction of all the irreducible representations of the Lie group $SU(2)$ (c.f., Sec. X E and Fig. 4).

We use the matrix-to-operator mapping now to construct a differential operator realization of the Heisenberg group, given in Eq. (24). Linearizing about the identify gives a three-dimensional Lie algebra of the form

$$\begin{bmatrix} 0 & l & d \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} = lL + rR + dD \tag{60}$$

Here L, R, D are 3×3 matrices. The only nonzero commutator is $[L, R] = D$. The corresponding differential operator algebra is

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & l & d \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} = l\mathcal{L} + r\mathcal{R} + d\mathcal{D} \tag{61}$$

The three differential operators are

$$\mathcal{L} = x\partial_y \quad \mathcal{R} = y\partial_z \quad \mathcal{D} = x\partial_z \tag{62}$$

Among these operators: none depends on z (so ∂_z has nothing to operate on) and none contains ∂_x, so that in essence x is an irrelevant variable. A more economical representation of this algebra is obtained by zeroing out the cyclic variables z, ∂_x and replacing their duals by ∂_z, x by $+1$ (duality is under the commutator $[\partial_i, x_j] = \delta_{ij}$)

$$\begin{bmatrix} 1 & y & 0 \end{bmatrix} \begin{bmatrix} 0 & l & d \\ 0 & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \partial_y \\ 1 \end{bmatrix} = l\mathcal{L}' + r\mathcal{R}' + d\mathcal{D}' \tag{63}$$

$$\mathcal{L}' = \partial_y \quad \mathcal{R}' = y \quad \mathcal{D}' = 1 \tag{64}$$

In essence, we have zeroed out the operators coupled to the vanishing rows and columns of the matrix Lie algebra and replaced their dual variables by 1. The representation given is essentially the Hiesenberg representation of the position (y) and conjugate momentum ($p_y \simeq \partial_y$) operators in Quantum Mechanics.

F. Disentangling Results

It happens surprisingly often in distantly related fields of physics that expressions of the form $e^{x+\partial_x}$ are encountered. Needless to say, these are not necessarily endearing to work with. One approach to simplifying computations involving such operators is to rewrite the operator in such a way that all differential operators ∂_x act first, and all multiplications by x act last. One way to effect this decomposition is to cross one's fingers and write this operator as $e^{ax+b\partial_x} \simeq e^{ax}e^{b\partial_x}$ and hope for the best. Of course this doesn't work, since the operators x and ∂_x do not commute.

Exponential operator rearrangements are called **disentangling theorems**. Since the exponential mapping is involved, powerful methods are available when the operators in the exponential belong to a finite-dimensional Lie algebra. Here is the algorithm:

1. Determine the Lie algebra.

2. Find a faithful finite-dimensional matrix representation of this Lie algebra.

3. Identify how you want the operators ordered in the final product of exponentials.

4. Compute this result in the faithful matrix representation.

5. Lift this result back to the operator form.

Here is how this algorithm works. The operators x and ∂_x have one nonzero commutator $[\partial_x, x] = 1$. These three operators close under commutation. They therefore form a Lie algebra. This is the algebra $\mathfrak{h}_3$ of the Heisenberg group, Eq. (64). We also have a faithful matrix representation of this Lie algebra, given in Eq. (63). We make the identification

$$e^{ax+b\partial_x} \to EXP \begin{bmatrix} 0 & b & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & b & \frac{ab}{2} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \tag{65}$$

Now we identify this matrix with

$$e^{rx}e^{dI}e^{l\partial_x} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{66}$$

By multiplying out the three matrices in Eq. (66) and comparing with the matrix elements of the 3×3 matrix in Eq. (65) we learn that $l = b, r = a, d = \frac{ab}{2}$. Porting the results of this matrix calculation back to the land of operator algebras, we find

$$e^{ax+b\partial_x} = e^{ax}e^{\frac{ab}{2}}e^{b\partial_x} \tag{67}$$

We will use this expression in Sec. XIV G below to construct a generating function for the Hermite polynomials.

VIII. RIEMANNIAN SYMMETRIC SPACES

A Riemannian symmetric space is a manifold on which a positive definite metric can be defined everywhere. In addition, at each point p there is an isometry (transformation that leaves distances between points unchanged) that: (1) leaves p fixed; (2) is not the identity; and (3) whose square is the identity. It was discovered by E. Cartan that Riemannian symmetric spaces are very closely related to Lie Groups. Specifically, they are quotients of

Lie groups by certain Lie subgroups. We illustrate with some examples.

The Lie algebra for the Lorentz group consists of 4×4 matrices:

$$\mathfrak{so}(1,3) = \left[\begin{array}{c|ccc} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha_1 & 0 & \theta_3 & -\theta_2 \\ \alpha_2 & -\theta_3 & 0 & \theta_1 \\ \alpha_3 & \theta_2 & -\theta_1 & 0 \end{array}\right] \tag{68}$$

The Cartan-Killing metric for $SO(1,3)$ is given by the trace of the product of this matrix with itself:

$$g(\mathfrak{so}(1,3)) = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 - \theta_1^2 - \theta_2^2 - \theta_3^2 \tag{69}$$

The subalgebra of rotations $\mathfrak{so}(3)$ describes the compact subgroup $SO(3)$. The remaining three infinitesimal generators parameterized by α_i span the noncompact part of this group, the coset $SO(1,3)/SO(3)$, and exponentiate to boost operations.

Cartan has pointed out that it is often possible to find a linear transformation, T, of a Lie algebra $\mathfrak{g}$ to itself whose square is the identity: $T \neq I, T^2 = I$. Such a T has two eigenspaces, $\mathfrak{k}$ and $\mathfrak{p}$, with $T\mathfrak{g} = T(\mathfrak{k} \oplus \mathfrak{p}) = (\mathfrak{k} \ominus \mathfrak{p})$. The two subspaces are orthogonal under the Cartan metric and satisfy the commutation relations:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \tag{70}$$

When this is possible, $EXP(\mathfrak{k}) = K$ is a subgroup of G and $EXP(\mathfrak{p}) = P = G/K$. Further, if the Cartan-Killing metric is negative definite on $\mathfrak{k}$, and positive definite on $\mathfrak{p}$, then K is a maximal compact subgroup of G and the coset $P = G/K$ is a Riemannian symmetric space. A Riemannian symmetric space is **homogeoeous**: every point looks like every other point. It is not necessarily **isotropic**: every direction looks like every other direction. Spheres are homogeneous *and* isotropic.

For the Lorentz group $SO(1,3)$, by Cartan's criterion $SO(3)$ is the maximal compact subgroup and the coset of boost transformations $B(\boldsymbol{\beta}) = SO(1,3)/SO(3)$ is a three-dimensional Riemannian space with positive-definite metric. In this case the space is a 3-hyperboloid $(ct)^2 - x^2 - y^2 - z^2 = cst.$ embedded in R^4. The metric on this space is obtained by moving the metric $(1,1,1)$ at the origin $(x,y,z) = (0,0,0)$ over the embedded space using the set of Lorenz group transformations in the quotient space $SO(1,3)/SO(3)$. Cartan also showed that all Riemannian symmetric spaces arise as quotients of (simple) Lie groups by maximal compact subgroups.

IX. APPLICATIONS IN CLASSICAL PHYSICS

Group Theory's first important role in physics came even before Quantum Mechanics was discovered. The two pillars of classical deterministic physics are Mechanics and Electrodynamics. Group Theory played a fundamental role in rectifying the difficulties in describing the interactions between these two fields. The principle tool used, besides Group Theory, was Galileo's Principle of Relativity and an assumption about the underlying elegance of physical theories.

A. Principle of Relativity

The Principle of Relativity posits: Two observers, S and S', observe the same physical system. Each knows how his coordinate system is related to the other's — that is, the transformation of coordinates that maps one coordinate system into the other. Assume both observers collect data on the same physical system. Given the data that S takes, and the coordinate transformation from S to S', S can predict the data that S' has recorded. *And he will be correct.*

Essentially, without this ability to communicate data among observers, there would be little point in pursuing the scientific method.

A second assumption that is used is usually not stated explicitly. This assumption is: The quantitative formulation of physical laws is simple and elegant (whatever that means).

B. Mechanics and Electrodynamics

The quantitative formulation of Mechanics for a single particle in an inertial frame is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{x}) \tag{71}$$

where $\mathbf{p}$ is defined by $\mathbf{p} = m\frac{d\mathbf{x}}{dt}$ for a particle with fixed mass m. The transformations from one inertial coordinate system to another consist of: displacements in space $\mathbf{d}$, displacements in time d, Rigid rotations R, and boosts with constant velocity $\mathbf{v}$ that keep the axis parallel. The space and time coordinates in S' are related to those in S by

$$\begin{aligned} \mathbf{x}' &= R\mathbf{x} + \mathbf{v}t + \mathbf{d} \\ t' &= t + d \end{aligned} \tag{72}$$

In inertial coordinate system S' Newton's equations are

$$\frac{d\mathbf{p}'}{dt} = \mathbf{F}'(\mathbf{x}') \qquad \begin{aligned} \mathbf{x}' &= R\mathbf{x} \\ \mathbf{p}' &= R\mathbf{p} \\ \mathbf{F}' &= R\mathbf{F} \end{aligned} \tag{73}$$

The equations of motion have the same vectorial form in both inertial coordinate systems (the simple and elegant assumption).

Newton's Laws were incredibly successful in describing planetary motion in our solar system, so when Maxwell developed his laws for electrodynamics, it was natural to assume that they also retained their form under the set of inertial coordinate transformations given in Eq. (72). Applying these transformations to Maxwell's equations creates a big mess. But if one looks only at signals propagating along or opposite the direction of the velocity $\mathbf{v}$, these assumptions predict

$$(c\ dt)' \to c'\ dt \qquad c' = c \pm |\mathbf{v}| \tag{74}$$

The round trip time for a light signal traveling in a cavity of length L as seen by S is $2L/c$ while in S' the time lapse was predicted to be $\frac{L}{c+v} + \frac{L}{c-v} = \frac{2L/c}{1-\beta^2}$, with $\beta = |\mathbf{v}|/c$.

This predicted difference in elapsed round trip times in 'rest' and 'moving' frames was thought to enable us to determine how fast the earth was moving in the Universe. As ever more precise measurements in the late nineteenth and early twentieth century lead to greater disappointment, more and more bizarre explanations were created to 'explain' this null result. Finally Einstein and Poincaré returned to the culprit Eq.(74) and asserted what the experiments showed: $c' = c$, so that

$$(c\ dt)' \to c\ dt' \qquad dt' = \text{linear comb. } dx, dy, dz, dt \tag{75}$$

The condition that the distance function

$$\begin{gathered}(c\ d\tau)^2 = (c\ dt)^2 - (dx^2 + dy^2 + dz^2) = \\ (c\ dt')^2 - (dx'^2 + dy'^2 + dz'^2)\end{gathered} \tag{76}$$

is invariant leads directly to the transformation law for infinitesimals $dx'^\mu = \Lambda^\mu_\nu x^\nu$, where the 4×4 matrices belong to the Lorentz group $\Lambda \in SO(3,1)$, $\Lambda^\mu_\nu = \partial x'^\mu / \partial x^\nu$. The transformation laws taking inertial frames S to inertial frames S' involves inhomogeneous coordinate transformations

$$\left[\begin{array}{c} x' \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} SO(3,1) & d \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ \hline 1 \end{array}\right] \tag{77}$$

While Maxwell's equations remain unchanged in form under this set of coordinate transformations (inhomogeneous Lorentz group), Newton's force law no longer preserves its form.

In order to find the proper form for the laws of classical mechanics under this new set of transformations the following two-step process was adopted:

1. Find an equation that has the proper transformation properties under the *inhomogeneous Lorentz group*;
2. If the equation reduces to Newton's equation of motion in the nonrelativistic limit $\beta \to 0$ it is the proper generalization of Newton's equation of motion.

Appliction of the procedure leads to the relativistic equation for particle motion

$$\frac{dp^\mu}{d\tau} = f^\mu \tag{78}$$

where p^μ is defined by $p^\mu = m\frac{dx^\mu}{d\tau}$. The components of the relativistic four vector f^μ are related to the three-vector force $\mathbf{F}$ by

$$\begin{aligned} \mathbf{f} &= \mathbf{F} + (\gamma)\frac{\boldsymbol{\beta}\cdot\mathbf{F}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}\boldsymbol{\beta} \\ f^0 &= \gamma\boldsymbol{\beta}\cdot\mathbf{F} \end{aligned} \tag{79}$$

(c.f., Eq. (31)).

C. Gravitation

Einstein wondered how it could be possible to determine if you were in an inertial frame. He decided that the algorithm for responding to this question, Newton's First Law (*In an inertial frame*, an object at rest remains at rest and an object in motion remains in motion with the same velocity unless acted upon by external forces.) was circularly defined (How do you know there are no forces? When you are sufficiently far away from the fixed stars. How do you know you are sufficiently far away? When there are no forces.)

He therefore set out to formulate the laws of mechanics in such a way that they were invariant in form under an arbitrary coordinate transformation. While the Lorentz group is six dimensional, general coordinate transformations form an "infinite dimensional" group. The transformation properties at any point are defined by a Jacobian matrix $\left[\frac{\partial x'^\mu}{\partial x^\nu}(x)\right]$. Whereas for the Lorentz group this matrix is constant throughout space, for general coordinate transformations this 4×4 matrix is position dependent.

Nevertheless, he was able to modify the algorithm described above to formulate laws that are invariant under *all* coordinate transformations. This two-step process is a powerful formulation of the Equivalence Principle. It is called the Principle of General Covariance. It states that a law of physics holds in the presence of a gravitational field provided:

1. The equation is invariant in form under an *arbitrary coordinate transformation* $x \to x'(x)$.
2. In a locally free-falling coordinate system, or the absence of a gravitational field, the equation assumes the form of a law within the Special Theory of Relativity.

Using these arguments, he was able to show that the equation describing the trajectory of a particle is

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu,\kappa}\frac{dx^\nu}{d\tau}\frac{dx^\kappa}{d\tau} \tag{80}$$

The Christoffel symbols are defined in terms of the metric tensor $g^{\mu,\nu}(x)$ by

$$\Gamma^\mu_{\nu,\kappa} = \frac{1}{2}g^{\mu,\rho}\left(\frac{\partial g_{\nu,\rho}}{\partial x^\kappa} + \frac{\partial g_{\rho,\kappa}}{\partial x^\nu} - \frac{\partial g_{\nu,\kappa}}{\partial x^\rho}\right) \tag{81}$$

They are not components of a tensor, as coordinate systems can be found (freely falling, as in an elevator) in which they vanish and the metric tensor reduces to its form in Special Relativity form $g = \mathrm{diag}(1,-1,-1,-1)$.

Neither the left hand side nor the right hand side of Eq. (80) is invariant under arbitrary coordinate changes (extra terms creep in), but the following transformation law is valid (Weinberg, 1972)

$$\frac{d^2x'^\mu}{d\tau^2}+\Gamma'^\mu_{\nu,\kappa}\frac{dx'^\nu}{d\tau}\frac{dx'^\kappa}{d\tau} = \left(\frac{\partial x'^\mu}{\partial x^\lambda}\right)\left(\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\nu,\kappa}\frac{dx^\nu}{d\tau}\frac{dx^\kappa}{d\tau}\right) \tag{82}$$

This means that the set of terms on the left, or those within the brackets on the right, have the simple transformation properties of a four-vector. In a freely falling coordinate system the Christoffel symbols vanish and what remains is $\frac{d^2x^\lambda}{d\tau^2}$. This Special Relativity expression is zero in the absence of forces, so the equation that describes the trajectory of a particle in a gravitational field is

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\nu,\kappa}\frac{dx^\nu}{d\tau}\frac{dx^\kappa}{d\tau} = 0 \tag{83}$$

D. Reflections

Two lines of reasoning have entered the reconciliation of the two pillars of classical deterministic physics and the creation of a theory of gravitation. One is group theory and is motivated by Galileo's Principle of Relativity. The other is more vague. It is a Principle of Elegance: there is the mysterious assumption that the structure of the "real" equations of physics are simple, elegant, and invariant under a certain class of coordinate transformations. The groups are the 10 parameter Inhomogeneous Lorentz Group in the case of the Special Theory of Relativity and the much larger groups of general cordinate transformations in the case of the General Theory of Relativity. There is every likelihood that intergalactic travelers will recognize the Principle of Relativity but no guarantee that their sense of simplicity and elegance will be anything like our own.

X. LINEAR REPRESENTATIONS

The theory of representations of groups — more precisely the linear representations of groups by matrices — was actively studied by mathematicians while physicists actively ignored these results. This picture changed dramatically with the development of the Quantum Theory, the understanding that the appropriate "phase space" was the Hilbert space describing a quantum system, and that group operations acted in these spaces through their linear matrix representations.

A linear matrix representation is a mapping of group elements g to matrices $g \to \Gamma(g)$ that preserves the group operation:

$$g_i \circ g_j = g_k \Rightarrow \Gamma(g_i) \times \Gamma(g_j) = \Gamma(g_k) \tag{84}$$

Here $\circ$ is the composition in the group and $\times$ indicates matrix multiplication. Often the mapping is one-way: many different group elements can map to the same matrix. If the mapping is $1:1$ the mapping is an isomorphism and the representation is called **faithful**.

A. Maps to Matrices

We have already seen many matrix representations. We have seen representations of the two-element group Z_2 as reflection, rotation, and inversion matrices acting in R^3 (c.f. Eq. (13)).

So: how many representations does a group have? It is clear from the example of Z_2 that we can create an infinite number of representations. However, if we squint carefully at the three representations presented in Eq. (13) we see that all these representations are diagonal: direct sums of essentially two distinct one-dimensional matrix representations:

Z_2	e	f
Γ^1	$[1]$	$[1]$
Γ^2	$[1]$	$[-1]$

(85)

Each of the three matrix representations of Z_2 in Eq. (13) is a direct sum of these two **irreducible representations**:

$$\begin{aligned} \sigma_Z &= \Gamma^1 \oplus \Gamma^1 \oplus \Gamma^2 \\ R_Z(\pi) &= \Gamma^1 \oplus \Gamma^2 \oplus \Gamma^2 \\ \mathcal{P} &= \Gamma^2 \oplus \Gamma^2 \oplus \Gamma^2 \end{aligned} \tag{86}$$

A basic result of representation theory is that for large classes of groups (finite, discrete, compact Lie groups) every representation can be written as a direct sum of irreducible representations. The procedure for constructing this direct sum proceeds by matrix diagonalization.

Irreducible representations are those that cannot be further diagonalized. In particular, one dimensional representations cannot be further diagonalized. Rather than enumerating all possible representations of a group, it is sufficient to enumerate only the much smaller set of irreducible representations.

B. Group Element - Matrix Element Duality

The members of a group can be treated as a set of points. It then becomes possible to define a set of functions on this set of points. How many independent functions are needed to span this function space? A not too particularly convenient choice of basis functions are the delta functions $f_i(g) = \delta(g, g_i)$. For example, for C_{3v} there are six group elements and therefore six basis functions for the linear vector space of functions defined on this group.

Each matrix element in any representation is a function defined on the members of a group. It would seem reasonable that the number of matrix elements in all the irreducible representations of a group provide a set of basis functions for the function space defined on the set of group elements. This is true: it is a powerful theorem. There is a far-reaching duality between the elements in a group and the set of matrix elements in its set of irreducible representations. Therefore if $\Gamma^\alpha(g)$, $\alpha = 1, 2, \cdots$ are the irreducible representations of a group G and the dimension of $\Gamma^\alpha(g)$ is d_α (i.e., $\Gamma^\alpha(g)$ consists of $d_\alpha \times d_\alpha$ matrices), then the total number of matrix elements is the order of the group G:

$$\sum_{\alpha}^{\text{all irreps}} d_\alpha^2 = |G| \tag{87}$$

Further, the set of functions $\sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}(g)$ form a complete orthonormal set of functions on the group space. The **orthogonality relation** is

$$\sum_{g\in G} \sqrt{\frac{d_{\alpha'}}{|G|}}\Gamma^{\alpha'}_{r's'}{}^*(g) \sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}(g) = \delta(\alpha', \alpha)\delta(r's', rs) \tag{88}$$

and the **completeness relation** is

$$\sum_{\alpha}\sum_{rs} \sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}{}^*(g') \sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}(g) = \delta(g', g) \tag{89}$$

These complicated expressions can be considerably simplified when written in the Dirac notation. Define

$$\langle g| \begin{matrix}\alpha\\ rs\end{matrix} \rangle = \sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}(g), \quad \langle \begin{matrix}\alpha\\ rs\end{matrix} |g\rangle = \sqrt{\frac{d_\alpha}{|G|}}\Gamma^\alpha_{rs}{}^*(g) \tag{90}$$

For convenience, we have assumed that the irreducible representations are unitary: $\Gamma^\dagger(g) = \Gamma(g^{-1})$ and $\dagger = t\,*$.

In Dirac notation, the orthogonality and completeness relations are

$$\begin{aligned} \text{Orthogonality}: \quad \langle \begin{matrix}\alpha'\\ r's'\end{matrix} |g\rangle\langle g| \begin{matrix}\alpha\\ rs\end{matrix} \rangle &= \langle \begin{matrix}\alpha'\\ r's'\end{matrix} | \begin{matrix}\alpha\\ rs\end{matrix} \rangle \\ \text{Completeness}: \quad \langle g'| \begin{matrix}\alpha\\ rs\end{matrix} \rangle\langle \begin{matrix}\alpha\\ rs\end{matrix} |g\rangle &= \langle g'|g\rangle \end{aligned} \tag{91}$$

As usual, doubled dummy indices are summed over.

C. Classes and Characters

The group element - matrix element duality is elegant and powerful. It leads to yet another duality, somewhat less elegant but, in compensation, even more powerful. This is the **character - class duality**.

We have already encountered classes in Eq. (17). Two elements c_1, c_2 are in the same class if there is a group element, g, for which $gc_1g^{-1} = c_2$. The *character* of a matrix is its trace. All elements in the same class have the same character in any representation, for

$$\text{Tr } \Gamma(c_2) = \text{Tr } \Gamma(gc_1g^{-1}) = \text{Tr } \Gamma(g)\Gamma(c_1)\Gamma(g^{-1}) = \text{Tr } \Gamma(c_1) \tag{92}$$

The last result comes from invariance of the trace under cyclic permutation of the argument matrices.

With relatively little work, the powerful orthogonality and completeness relations for the group elements - matrix elements can be transformed to corresponding orthogonality and completeness relations for classes and characters. If $\chi^\alpha(i)$ is the character for elements in class i in irreducible representation α and n_i is the number of group elements in that class, the character-class duality is described by the following relations:

$$\text{Orthogonality}: \quad \sum_i n_i \chi^{\alpha'*}(i)\chi^\alpha(i) = |G|\delta(\alpha', \alpha) \tag{93}$$

$$\text{Completeness}: \quad \sum_\alpha n_i \chi^{\alpha*}(i)\chi^\alpha(i') = |G|\delta(i', i) \tag{94}$$

D. Fourier Analysis on Groups

The group C_{3v} has six elements. Its set of irreducible representations has a total of six matrix elements. Therefore $d_1^2 + d_2^2 + \cdots = 6$. This group has three classes. By the character-class duality, it has three irreducible representations. As a result, $d_1 = d_2 = 1$ and $d_3 = 2$. The matrices of the six group elements in the three irreducible representations are:

	Γ^1	Γ^2	Γ^3
e	$[1]$	$[1]$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
C_3^+	$[1]$	$[1]$	$\begin{bmatrix} -a & b \\ -b & -a \end{bmatrix}$
C_3^-	$[1]$	$[1]$	$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}$
σ_1	$[1]$	$[-1]$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
σ_2	$[1]$	$[-1]$	$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$
σ_3	$[1]$	$[-1]$	$\begin{bmatrix} a & -b \\ -b & -a \end{bmatrix}$

$$a = \tfrac{1}{2} \quad b = \tfrac{\sqrt{3}}{2} \tag{95}$$

The character table for this group is

	$\{e\}$	$\{C_3^+, C_3^-\}$	$\{\sigma_1, \sigma_2, \sigma_3\}$
	1	2	3
χ^1	1	1	1
χ^2	1	1	-1
χ^3	2	-1	0

(96)

The first line shows how the group elements are apportioned to the three classes. The second shows the number of group operations in each class. The remaining lines show the trace of the matrix representatives of the elements in each class in each representation. For example, the -1 in the middle of the last line is $-1 = -\frac{1}{2} - \frac{1}{2}$. The character of the identity group element e is the dimension of the matrix representation, d_α.

We use this character table to perform a Fourier analysis on representations of this group. For example, the representation of $C_{3v} = S_3$ in terms of 3×3 permutation matrices is not irreducible (c.f., Eq. (15)). For various reasons we might like to know which irreducible representations of C_{3v} are contained in this reducible representation. The characters of the matrices describing each class are:

	$\{e\}$	$\{C_3^+, C_3^-\}$	$\{\sigma_1, \sigma_2, \sigma_3\}$
$\chi^{3\times 3}$	3	0	1

(97)

To determine the irreducible content of this representation we take the inner product of Eq. (97) with the rows of Eq. (96) using Eq. (93) with the results

$$\begin{aligned} \langle \chi^{3\times 3} | \chi^1 \rangle &= 1 \times 3 \times 1 + 2 \times 0 \times 1 + 3 \times 1 \times 1 &&= 6 \\ \langle \chi^{3\times 3} | \chi^2 \rangle &= 1 \times 3 \times 1 + 2 \times 0 \times 1 + 3 \times 1 \times -1 &&= 0 \\ \langle \chi^{3\times 3} | \chi^3 \rangle &= 1 \times 3 \times 2 + 2 \times 0 \times -1 + 3 \times 1 \times 0 &&= 6 \end{aligned} \tag{98}$$

As a result, the permutation representation is reducible and $\chi^{3\times 3} \simeq \chi^1 \oplus \chi^3$.

Remark on Terminology: The cyclic group C_n has n group elements g_k, $k = 0, 1, 2 \cdots, n-1$ that can be identified with rotations through an angle $\theta_k = 2\pi k/n$. This group is is abelian. It therefore has n one-dimensional irreducible matrix representations Γ^m, $m = 0, 1, 2, \cdots n-1$ whose matrix elements are $\Gamma^m(\theta_k) = \left[e^{2\pi i k m/n}\right]$. Any function defined at the n equally spaced points at angles θ_k around the circle can be expressed in terms of the matrix elements of the unitary irreducible representations of C_n. The study of such functions, and their transforms, is the study of Fourier Series. This analysis method can be applied to functions define along the real line R^1 using the unitary irreducible representations $\Gamma^k(x) = \left[e^{ikx}\right]$ of the commutative group of translations T_x along the real line through the distance x. This is Fourier Analysis on the real line. This idea generalizes to groups and their complete set of unitary irreducible representations.

E. Irreps of $SU(2)$

The unitary irreducible representations ("UIR" or "irreps") of Lie groups can be constructed following two routes. One route begins with the group. The second begins with its Lie algebra. The second method is simpler to implement, so we use it here to construct the hermitian irreps of $\mathfrak{su}(2)$ and then exponentiate them to the unitary irreps of $SU(2)$.

The first step is to construct shift operators from the basis vectors in $\mathfrak{su}(2)$:

$$\begin{aligned} S_+ &= S_x + iS_y &&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ S_- &= S_x - iS_y &&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad [S_z, S_\pm] = \pm S_\pm \\ S_z &= \tfrac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad [S_+, S_-] = 2S_z \end{aligned} \tag{99}$$

Next, we use the matrix algebra to operator algebra mapping (c.f., Sec. VII E) to construct a useful boson operator realization of this Lie algebra:

$$\begin{aligned} S_+ \to \mathcal{S}_+ &= b_1^\dagger b_2 \\ S_- \to \mathcal{S}_- &= b_2^\dagger b_1 \end{aligned} \qquad S_z \to \mathcal{S}_z = \frac{1}{2}\left(b_1^\dagger b_1 - b_2^\dagger b_2\right) \tag{100}$$

The next step introduces representations. Introduce a state space on which the boson operators $b_1, b_1^\dagger$ act, with basis vectors $|n_1\rangle$, $n_1 = 0, 1, 2, \cdots$ with the action given as usual by

$$b_1^\dagger |n_1\rangle = |n_1+1\rangle\sqrt{n_1+1} \qquad b_1|n_1\rangle = |n_1-1\rangle\sqrt{n_1} \tag{101}$$

Introduce a second state space for the operators $b_2, b_2^\dagger$ and basis vectors $|n_2\rangle$, $n_2 = 0, 1, 2, \cdots$. In order to construct the irreducible representations of $\mathfrak{su}(2)$ we introduce a grid, or lattice, of states $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. The operators $\mathcal{S}_\pm, \mathcal{S}_z$ are number-conserving and move along the diagonal $n_1 + n_2 =$ cnst. (c.f., Fig. 4). It is very useful to relabel the basis vectors in this lattice by two integers. One (j) identifies the diagonal, the other (m) specifies position along a diagonal:

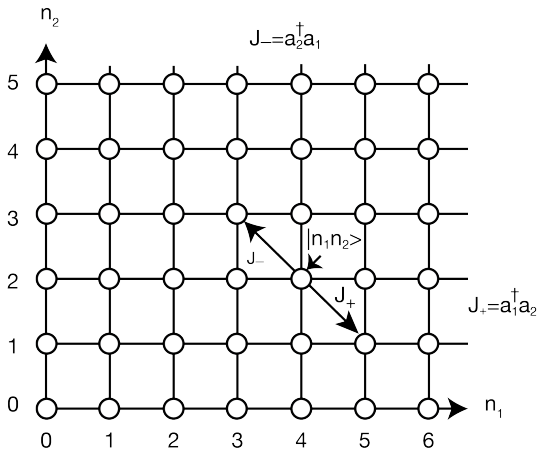

FIG. 4: Angular momentum operators $\mathbf{J}$ have isomorphic commutation relations with specific biliner combinations $b_i^\dagger b_j$ of boson creation and annihilation operators for two modes. The occupation number for the first mode is plotted along the x axis and that for the second mode is plotted along the y axis. The number-conserving operators act along diagonals like the operators J_+, J_-, J_z to easily provide states and matrix elements for the $\mathfrak{su}(2)$ operators.

$$\begin{aligned} 2j &= n_1 + n_2 \quad & n_1 &= j + m \\ 2m &= n_1 - n_2 \quad & n_2 &= j - m \end{aligned} \qquad |n_1, n_2\rangle \leftrightarrow |\begin{array}{c} j \\ m \end{array}\rangle \tag{102}$$

The spectrum of allowed values of the quantum number j is $2j = 0, 1, 2, \cdots$ and $m = -j, -j+1, \cdots, +j$.

The matrix elements of the operators $\mathcal{S}$ with respect to the basis $|\begin{array}{c} j \\ m \end{array}\rangle$ are constructed from the matrix elements of the operators $b_i^\dagger b_j$ on the basis vectors $|n_1, n_2\rangle$. For $\mathcal{S}_z$ we find

$$\begin{aligned} \mathcal{S}_z|\begin{array}{c} j \\ m \end{array}\rangle = \tfrac{1}{2}(b_1^\dagger b_1 - b_2^\dagger b_2)|n_1, n_2\rangle = \\ |n_1, n_2\rangle\tfrac{1}{2}(n_1 - n_2) = |\begin{array}{c} j \\ m \end{array}\rangle m \end{aligned} \tag{103}$$

For the shift-up operator

$$\begin{aligned} \mathcal{S}_+|\begin{array}{c} j \\ m \end{array}\rangle = b_1^\dagger b_2|n_1, n_2\rangle = |n_1+1, n_2-1\rangle\sqrt{n_1+1}\sqrt{n_2} = \\ |\begin{array}{c} j \\ m+1 \end{array}\rangle\sqrt{(j+m+1)(j-m)} \end{aligned} \tag{104}$$

and similarly for the shift-down operator

$$\mathcal{S}_-|\begin{array}{c} j \\ m \end{array}\rangle = |\begin{array}{c} j \\ m-1 \end{array}\rangle\sqrt{(j-m+1)(j+m)} \tag{105}$$

In this representation of the (spin) angular momentum algebra $\mathfrak{su}(2)$, $\mathcal{S}_z = J_z$ is diagonal and $\mathcal{S}_\pm = J_\pm$ have one nonzero diagonal row just above (below) the main diagonal. The hermitian irreducible representations of $\mathfrak{su}(2)$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \cdots$ form a complete set of irreducible representations for this Lie algebra.

The unitary irreducible representations of $SU(2)$ are obtained by exponentiating i times the hermitian representations of $\mathfrak{su}(2)$:

$$\mathcal{D}^J\left[SU(2)\right] = EXPi\hat{\mathbf{n}} \cdot \mathbf{J}\theta \tag{106}$$

with $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/2i$, and J_* are the $(2j+1) \times (2j+1)$ matrices whose matrix elements are given in Eqs. (103-105). The $(2j+1) \times (2j+1)$ matrices $\mathcal{D}^J$ are traditionally called **Wigner matrices**. For many purposes only the character of an irreducible representation is needed. The character depends only on the class and the class is uniquely determined by the rotation angle θ (rotations by angle θ about any axis $\hat{\mathbf{n}}$ are geometrically equivalent). It is sufficient to compute the trace of any rotation, for example the rotation about the z axis. This matrix is diagonal: $\left(e^{iJ_z\theta}\right)_{m',m} = e^{im\theta}\delta_{m',m}$ and its trace is

$$\chi^j(\theta) = \sum_{m=-j}^{+j} e^{im\theta} = \frac{\sin(j+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} \tag{107}$$

These characters are orthonormal with respect to the weight $w(\theta) = \frac{1}{\pi}\sin^2\left(\frac{\theta}{2}\right)$.

TABLE II: (top) Character table for the cubic group O^h. The functions in the right-hand column are some of the basis vectors that "carry" the corresponding representation. (bottom) Characters for rotations through the indicated angle in the irreducible representations of the rotation group.

O^h	E	$8C_3$	$3C_4^2$	$6C_2$	$6C_4$	Basis
A_1	1	1	1	1	1	$r^2 = x^2 + y^2 + z^2$
A_2	1	1	1	−1	−1	
E	2	−1	2	0	0	$(x^2 - y^2, 3z^2 - r^2)$
T_1	3	0	−1	−1	1	$(x, y, z), (L_x, L_y, L_z)$
T_2	3	0	−1	1	−1	(yz, zx, xy)
$L:\theta$	0	$\frac{2\pi}{3}$	$\frac{2\pi}{2}$	$\frac{2\pi}{2}$	$\frac{2\pi}{4}$	Reduction
0 S	1	1	1	1	1	A_1
1 P	3	0	−1	−1	1	T_1
2 D	5	−1	1	1	−1	$E \oplus T_2$
3 F	7	1	−1	−1	−1	$A_2 \oplus T_1 \oplus T_2$
4 G	9	0	1	1	1	$A_1 \oplus E \oplus T_1 \oplus T_2$
5 H	11	−1	−1	−1	1	$E \oplus 2T_1 \oplus T_2$

F. Crystal Field Theory

The type of Fourier analysis outlined above has found a useful role in Crystal (or Ligand) Field Theory. This theory was created to describe the behavior of charged particles (electrons, ions, atoms) in the presence of an electric field that has some symmetry, usually the symmetry of a host crystal. We illustrate with a simple example.

A many-electron atom with total angular momentum L is placed in a crystal field with cubic symmetry. How do the $2L+1$-fold degenerate levels split?

Before immersion in the crystal field, the atom has spherical symmetry. Its symmetry group is the rotation group, the irreducible representations $\mathcal{D}^L$ have dimension $2L+1$, the classes are rotations through angle θ, and the character for the class θ in representation $\mathcal{D}^L$ is given in Eq. (107) with $j \to L$ (integer). When the atom is placed in an electric field with cubic symmetry O^h, the irreducible representations of $SO(3)$ become reducible. The irreducible content is obtained through a character analysis.

The group O^h has 24 operations partitioned into five classes. These include the identity E, eight rotations C_3 by $2\pi/3$ radians about the diagonals through the opposite vertices of the cube, six rotations C_4 by $2\pi/4$ radians about the midpoints of opposite faces, three rotations C_4^2 by $2\pi/2$ radians about the same midpoints of opposite faces, and six rotations C_2 about the midpoints of opposite edges. The characters for these five classes in the five irreducible representations are collected in the **character table** for O^h. This is shown at the top in Table II. At the bottom of the table are the characters of the irreducible representations of the rotation group $SO(3)$ in the irreducible representations of dimension $2L+1$. These are obtained from Eq. (107). A **character analysis** (c.f., Eq. (98)) leads to the O^h irreducible content of each of the lowest six irreducible representations of $SO(3)$.

XI. SYMMETRY GROUPS

Groups first appeared in the Quantum Theory as a tool for labelling eigenstates of a Hamiltonian with useful quantum numbers. If a Hamiltonian $\mathcal{H}$ is invariant under the action of a group G, then $g\mathcal{H}g^{-1} = \mathcal{H}$, $g \in G$. If $|\psi_\mu^\alpha\rangle$ satisfies Schrödinger's time-independent equation $\mathcal{H}|\psi_\mu^\alpha\rangle - E|\psi_\mu^\alpha\rangle = 0$, so that

$$g(\mathcal{H}-E)|\psi_\mu^\alpha\rangle = \left\{g(\mathcal{H}-E)g^{-1}\right\} g|\psi_\mu^\alpha\rangle = (\mathcal{H}-E)|\psi_\nu^\alpha\rangle\langle\psi_\nu^\alpha|g|\psi_\mu^\alpha\rangle = (\mathcal{H}-E)|\psi_\nu^\alpha\rangle\mathcal{D}^\alpha_{\nu,\mu}(g) \tag{108}$$

All states $|\psi_\nu^\alpha\rangle$ related to each other by a group transformation $g \in G$ (more precisely, a group represenattion $\mathcal{D}^\alpha(g)$) have the same energy eigenvalue. The existence of a symmetry group G for a Hamiltonian $\mathcal{H}$ provides representation labels for the quantum states *and also* describes the degeneracy patterns that can be observed. If the symmetry group G is a Lie group, so that $g = e^X$, then $e^X\mathcal{H}e^{-X} = \mathcal{H} \Rightarrow [X, \mathcal{H}] = 0$. The existence of operators X that commute with the Hamiltonian $\mathcal{H}$ is a clear signal that the physics described by the Hamiltonian is invariant under a Lie group.

For example, for a particle in a spherically symmetric potential $V(r)$ Schrödinger's time-independent equation is

$$\left(\frac{\mathbf{p}\cdot\mathbf{p}}{2m} + V(r)\right)\psi = E\psi \tag{109}$$

with $\mathbf{p} = (\hbar/i)\nabla$. The Hamiltonian operator is invariant under rotations. Equivalently, it commutes with the angular momentum operators $\mathbf{L} = \mathbf{r} \times \mathbf{p}$: $[\mathbf{L}, \mathcal{H}] = 0$. The wavefunctions can be partly labeled by rotation group quantum numbers, l and m: $\psi \to \psi^l_m(r, \theta, \phi)$. In fact, by standard separation of variables arguments this description can be made more precise: $\psi(r, \theta, \phi) = \frac{1}{r}R_{nl}(r)Y^l_m(\theta, \phi)$. Here $R_{nl}(r)$ are radial wavefunctions that depend on the potential $V(r)$ but the angular function $Y^l_m(\theta, \phi)$ is "a piece of geometry": it depends only on the existence of rotation symmetry. It is the same no matter what the potential is. In fact, these functions can be constructed from the matrix representations of the group $SO(3)$. The action of a rotation group operation g on the angular functions is

$$gY^l_m(\theta, \phi) = Y^l_{m'}(\theta, \phi)\mathcal{D}^l_{m'm}(g) \tag{110}$$

where the construction of the Wigner $\mathcal{D}$ matrices has been described in Sec. X E.

If the symmetry group is reduced, as in the case of $SO(3) \downarrow O^h$ described in Sec. X F, the eigenstates are identified by the labels of the irreducible representations of O^h: A_1, A_2, E, T_1, T_2.

Once the states have been labeled, computations must be done. At this point the power of group theory becomes apparent. Matrices must be computed — for example, matrix elements of a Hamiltonian. Typically, most matrix elements vanish (by group-theoretic selection rules). Of the small number that do not vanish, many are simply related to a small number of the others. In short, using group theory as a guide, only a small number of computations must actually be done.

This feature of group theory is illustrated by computing the eigenstates and their energy eigenvalues for an electron in the $N = 4$ multiplet of the hydrogen atom under the influence of a constant external field $\boldsymbol{\mathcal{E}}$. The Hamiltonian to be diagonalized is

$$\langle N' \begin{array}{c} L' \\ M' \end{array} | \frac{\mathbf{p}\cdot\mathbf{p}}{2m} - \frac{e^2}{r} + e\boldsymbol{\mathcal{E}}\cdot\mathbf{r} | N \begin{array}{c} L \\ M \end{array} \rangle \tag{111}$$

The first two terms in the Hamiltonian describe the electron in a Coulomb potential, the last is the Stark perturbation, which describes the interaction of a dipole $\mathbf{d} = -e\mathbf{r}$ with a constant external electric field: $\mathcal{H}_{\rm St.} = -\mathbf{d}\cdot\boldsymbol{\mathcal{E}}$. In the $N = 4$ multiplet we set $N' = N = 4$, so that $L', L = 0, 1, 2, 3$ and M ranges from $-L$ to $+L$ and $-L' \le M' \le +L'$. The matrix elements of the Coulomb Hamiltonian are $E_N \delta_{N'N}\delta_{L'L}\delta_{M'M}$, with $E_4 = -13.6/4^2 eV$.

There are $\sum_{L=0}^{3=4-1}(2L+1) = 16$ states in the $N = 4$ multiplet, so 16^2 matrix elements of the 16×16 matrix must be computed. We simplify the computation by choosing the z axis in the direction of the applied uniform electric field, so that $e\boldsymbol{\mathcal{E}}\cdot\mathbf{r} \to e\mathcal{E}z$ ($\mathcal{E} = |\boldsymbol{\mathcal{E}}|$). In addition we write $z = \sqrt{\frac{4\pi}{3}} r Y_0^1(\theta,\phi)$. The matrix elements factor (separation of variables) into a radial part and an angular part, as follows:

$$\begin{array}{l} \langle 4L'M'|e\mathcal{E}z|4LM\rangle \to e\mathcal{E} \times \text{Radial} \times \text{Angular} \\ \\ \text{Radial} = \int_0^\infty R_{4L'}(r) r^1 R_{4L}(r) dr \\ \text{Angular} = \sqrt{\frac{4\pi}{3}} \int Y_{M'}^{L'*}(\Omega) Y_0^1(\Omega) Y_M^L(\Omega) d\Omega \end{array} \tag{112}$$

where $\Omega = (\theta,\phi)$ and $d\Omega = \sin\theta d\theta d\phi$.

Selection rules derived from $SO(3)$ simplify the angular integral. First, the integral vanishes unless $\Delta M = M' - M = 0$. It also vanishes unless $\Delta L = \pm 1, 0$. By parity, it vanishes if $\Delta L = 0$, and by time reversal its value for M and $-M$ are the same. The nonzero angular integrals are

$$\mathcal{A}(L,M) = \sqrt{\frac{4\pi}{3}} \int_\Omega Y_{M'}^{L*}(\Omega) Y_0^1(\Omega) Y_M^{L-1}(\Omega) d\Omega = \delta_{M'M} \sqrt{\frac{(L+M)(L-M)}{(2L+1)(2L-1)}} \tag{113}$$

The radial integrals also satisfy selection rules: they vanish unless $\Delta L = \pm 1$. The nonzero integrals are all related:

$$\mathcal{R}(N,L) = \int_0^\infty R_{N,L}(r) r R_{N,L-1} dr = \frac{N\sqrt{N^2 - L^2}}{2\sqrt{3}} \times \mathcal{R}(2,1) \tag{114}$$

with $1 \le L \le N-1$. All integrals are proportional to the single integral $\mathcal{R}(2,1)$. This comes from yet another symmetry that the Coulomb potential exhibits (c.f. Sec. XII below), not shared by other spherically symmetric potentials. The single integral to be evaluated is

$$\mathcal{R}(2,1) = -3\sqrt{3} a_0 \tag{115}$$

This integral is proportional to the Bohr radius a_0 of the hydrogen atom, whose value was estimated in Eq. (6).

The arguments above show drastic simplifications in the computational load for computing the energy eigenfunctions and eigenvalues of a many electron atom in a uniform external electric field (Stark problem).

Of the $256 = 16^2$ matrix elements to compute only 18 are nonzero. All are real. Since the Hamiltonian is hermitian (symmetric if real) there are in fact only 9 nonzero matrix elements to construct. Each is a product of two factors, so only 6 (angular) plus 1 (radial) quantities need be computed. These numbers must be stuffed into a 16×16 matrix to be diagonalized. But there are no nonzero matrix elements between states with $M' \ne M$. This means that by organizing the row and columns appropriately the matrix can be written in block diagonal form. The block diagonal form consists of a 1×1 matrix for $M = 3$, a 2×2 matrix for $M = 2$, a 3×3 matrix for $M = 1$, a 4×4 matrix for $M = 0$, a 3×3 matrix for $M = -1$, etc. The 1×1 matrices are already diagonal. The 2×2 matrices are identical, so only one needs to be diagonalized. Similarly for the two 3×3 matrices. There is only one 4×4 matrix. The computational load for diagonalizaing this matrix has been reduced from $T \simeq 16^2 \log 16$ to $T \simeq 2^2 \log 2 + 3^2 \log 3 + 4^2 \log 4$, a factor of 20 (assuming the effort required for diagonalizing an $n \times n$ matrix goes like $n^2 \log n$)!

It gets even better. For the $N = 5$ multiplet the $1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4$ matrices are all proportional to the matrices of the corresponding size for $N = 4$. The proportionality factor is $5/4$. Only *one* new matrix needs to be constructed — the 5×5 matrix. This symmetry extends to all values of N.

This is a rather simple example that can be carried out by hand. This was done when Quantum Mechanics was first developed, when the fastest computer was a greased abacus. Today time savings of a factor of 20 on such a simple problem would hardly be noticed. But calculations have also inflated in size. Reducing a $10^6 \times 10^6$ matrix to about 1000 $10^3 \times 10^3$ reduces the computational effort by a factor of 2000. For example, a computation that would take 6 years without such methods could be done in a day with these methods.

Symmetry groups play several roles in Quantum Mechanics.

- They provide group representation labels to identify the energy eigenstates of a Hamiltonian with symmetry.
- They provide selection rules that save us the effort of computing matrix elements whose values are zero (by symmetry!).
- And they allow transformation of a Hamiltonian matrix to block diagonal form, so that the computational load can be drastically reduced.

XII. DYNAMICAL GROUPS

A widely accepted bit of wisdom among Physicists is that symmetry implies degeneracy, and the larger the symmetry, the larger the degeneracy. What works forward ought to work backward ("Newton's Third Law"): if the degeneracy is greater than expected, the symmetry is greater than apparent.

A. Conformal Symmetry

The hydrogen atom has rotational symmetry $SO(3)$, and this requires $2L+1$ fold degeneracy. But the states with the same principal quantum number N are all degenerate in the absence of spin and other relativistic effects, and nearly degenerate in the presence of these effects. It would make sense to look for a larger than apparent symmetry. It exists in the form of the **Runge-Lenz vector** $\mathbf{M} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - e^2\mathbf{r}/r$, where $\mathbf{r}, \mathbf{p}, \mathbf{L} = \mathbf{r} \times \mathbf{p}$ are the position, momentum, and orbital angular momentum operators for the electron. The three orbital angular momentum operators L_i and three components of the Runge-Lenz vector close under commutation to form a Lie algebra. The six operators commute with the Hamiltonian, so the "hidden" symmetry group is larger than the obvious symmetry group $SO(3)$. On the bound states this Lie algebra describes the Lie group $SO(4)$. The irreducible representation labels for the quantum states are N, $N = 1, 2, 3..., \infty$. The three nested groups $SO(2) \subset SO(3) \subset SO(4)$ and their representation labels and branching rules are:

Group	Rep.Label	Degeneracy	Branching Rules
$SO(4)$	N	N^2	
$SO(3)$	L	$2L+1$	$0, 1, 2, \cdots N-1$
$SO(2)$	M	1	$-L \leq M \leq +L$

(116)

Branching rules identify the irreducible representations of a subgroup that any representation of a larger group branches to under group - subgroup reduction. We have seen branching rules in Table II.

One advantage of using the larger group is that there are more shift operators in the Lie algebra. The shift operators, acting on one state, moves it to another (c.f. $|LM\rangle \xrightarrow{L_+} |L, M+1\rangle$). This means that there are well-defined algebraic relations among states that belong to the same N multiplet. This means that more of any computation can be pushed from the physical domain to the geometric domain, and simplifications accrete.

Why stop there? In the hydrogen atom the energy difference between the most tightly bound state, the ground state, and the most weekly bound state ($N \to \infty$) is 13.6 eV. When this difference is compared with the electron rest energy of 511 KeV, the **symmetry-breaking** is about 13.6/511000 $\simeq$ 0.000027 or $2.7 \times 10^{-3}\%$. This suggests that there is a yet larger group that accounts for this near degeneracy. Searches eventually lead to the noncompact conformal group $SO(4,2) \supset SO(4) \cdots$ as the all-inclusive "symmetry group" of the hydrogen atom. The virtue of using this larger group is that states in different multiplets $N, N \pm 1$ can be connected by shift operators within the algebra $\mathfrak{so}(4,2)$, and ultimately there is only one number to compute. Including this larger group in Eq. (116) would include inserting it in the row above $SO(4)$, showing there is only one representation label for bound states, indicating its degeneracy is "∞", and adding branching rules $N = 1, 2 \cdots, \infty$ to the $SO(4)$ row.

B. Atomic Shell Structure

Broken symmetry beautifully accounts for the systematics of the chemical elements. It accounts for the filling scheme as electrons enter a screened Coulomb potential around a nuclear charge $+Ze$ as the nuclear charge increases from $Z = 1$ to $Z > 92$. The screening is caused by "inner electrons". The filling scheme accounts for the "magic numbers" among the chemical elements: these are the nuclear charges of exceptionally stable chemical elements He, Ne, Ar, Kr, Xe, Rn with atomic numbers $2, 10, 18, 36, 54, 86$.

When more than one electron is present around a nuclear charge $+Ze$ then the outer electrons "see" a screened central charge and the $SO(4)$ symmetry arising from the Coulomb nature of the potential is lost. There is a reduction in symmetry, a "broken symmetry":

$SO(4) \downarrow SO(3)$. The quantum numbers (N, L) can be used to label states and energies, $E_{N,L}$ and these energy levels are $(2L+1)$-fold degenerate. The $SO(4)$ multiplet with quantum number N splits into orbital angular momentum multiplets with L values ranging from $L = 0$ to a maximum of $L = N - 1$.

Each additional electron must enter an orbital that is not already occupied by the Pauli Exclusion Principle. This Principle is enforced by the requirement that the total electron wavefunction transform under the unique antisymmetric representation $\Gamma^{\text{anti.}}(S_k)$ on the permutation group S_k for k electrons.

Generally, the larger the L value the further the outer electron is from the central charge, on average. And the further it is, the larger is the negative charge density contributed by inner electrons that reduces the strength of the central nuclear attraction. As a result $E_{N,0} < E_{N,1} < \cdots < E_{N,L=N-1}$. There is mixing among levels with different values of N and L. The following energy ordering scheme, ultimately justified by detailed calculations, accounts for the systematics of the chemical elements, including the magic numbers:

$$1S|2S\ 2P|3S\ 3P|4S\ 3D\ 4P|5S\ 4D\ 5P|6S\ 4F\ 5D\ 6P|7S \tag{117}$$

Each level can hold $2(2L+1)$ electrons. The first factor of $2 = (2s+1)$ with $s = \frac{1}{2}$ is due to electron spin. The vertical bar | indicates a large energy gap. The cumulative occupancy reproduces the magic numbers of the chemical elements: $2, 10, 18, 36, 54, 86$. The filling order is shown in Fig. 5. Broken symmetry is consistent with Mendeleev's periodic table of the chemical elements.

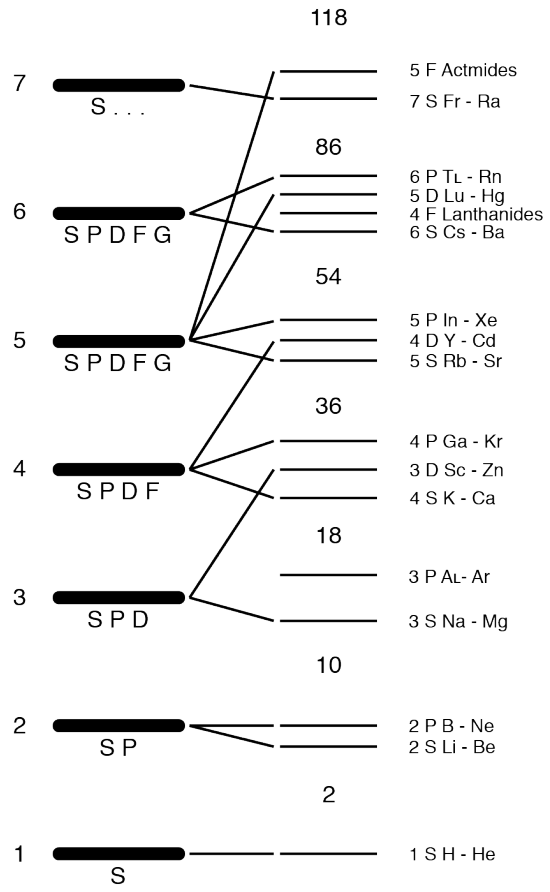

FIG. 5: Broken $SO(4)$ dynamical symmetry due to screening of the central Coulomb potential by inner electrons successfully accounts for the known properties of the chemical elements, as reflected in Mendeleev's Periodic Table of the Chemical Elements.

C. Nuclear Shell Structure

Magic numbers among nuclei suggested that, here also, one could possibly describe many different nuclei with a single simple organizational structure. The magic numbers are: $2, 8, 20, 28, 40, 50, 82, 126$, both for protons and for neutrons. The following model was used to organize this information.

Assume that the effective nuclear potential for protons (or neutrons) is that of a a three-dimensional isotropic harmonic oscillator. The basis states can be described by $|n_1, n_2, n_3\rangle$. One excitation would be three-fold degenerate: $|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle$, two excitations would be 6-fold degenerate, and states with N excitations would have a degeneracy $(N+2)(N+1)/2$. Under a spherically symmetric perturbation these highly degenerate multiplets would split into mutliplets identified by an angular momentum index. A character analysis gives this branching result

N	L Values	Spectroscopic
0	0	S
1	1	P
2	2, 0	D, S
3	3, 1	F, P
4	4, 2, 0	G, D, S

(118)

For example, the $N = 4$ harmonic oscillator multiplet splits into an $L = 4$ multiplet, an $L = 2$ multiplet, and an $L = 0$ multiplet. The larger the angular momentum, the lower the energy. After this splitting, the spin of the proton (or neutron) is coupled to the orbital angular momentum to give values of the total angular momentum $J = L \pm \frac{1}{2}$, except that for S states only the $J = \frac{1}{2}$ state occurs. Again, the larger angular momentum occurs at a lower energy than the smaller angular momentum. The resulting filling order, analogous to Eq. (117), is

$$0S_{1/2}|1P_{3/2}\ 1P_{1/2}|2D_{5/2}\ 2S_{1/2}\ 2D_{3/2}|3F_{7/2}|$$
$$3P_{3/2}\ 3F_{5/2}\ P_{1/2}\ 4G_{9/2}|4D_{5/2}\ 4G_{7/2}\ 4S_{1/2}\ 4D_{3/2}\ 5H_{11/2}|$$
$$5H_{9/2}\ 5F_{7/2}\ 5F_{5/2}\ 5P_{3/2}\ 5P_{1/2}\ 6I_{13/2}| \tag{119}$$

Each shell with angular momentum j can hold up to $2j+1$ nucleons. Broken symmetry is also consistent with the "periodic table" associated with nuclear shell models. The filling order is shown in Fig. 6.

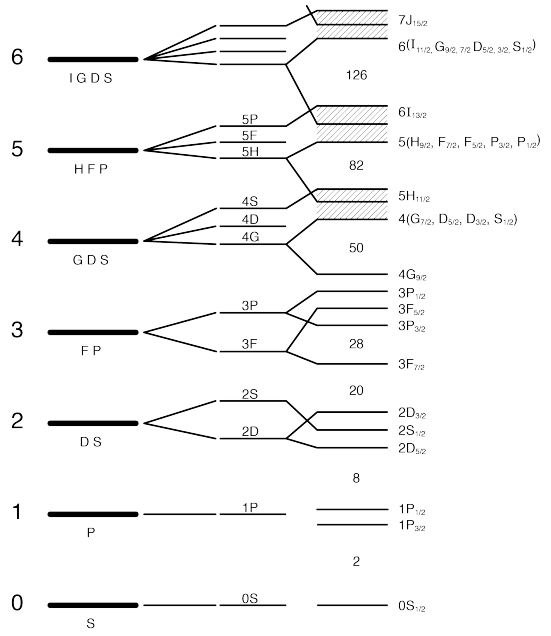

FIG. 6: The filling order describing very many properties of nuclear ground states is described by the levels of an isotropic harmonic oscillator potential with multiplets having N excitations and degeneracy $(N+1)(N+2)/2$. The degeneracy is broken by a spherically symmetric perturbation and broken further by spin-orbit coupling. For both perturbations energy increases as angular momentum decreases. The filling order shown successfully accounts for the known properties of the ground states of most even-even nuclei, including the magic numbers. In the higher levels the largest spin angular momentum state (e.g., $5H_{11/2}$) is pushed down into the next lower multiplet, containing all the remaining $N=4$ states, with the exception of the $4G_{9/2}$.

At a group theoretical level, our starting point has been the Lie algebra $\mathfrak{u}(3)$ with basis vectors $b_i^\dagger b_j$ $(1 \le i,j \le 3)$ whose representations are labeled by an integer index N, the number of excitations present. This algebra can be embedded in a larger Lie algebra containing in addition shift up operators $b_i^\dagger$, their counterpart annihilation operators b_j, and the identity operator I. The Lie algebra is $9+2\cdot 3+1=16=4^2$ dimensional, and is closely related to the noncompact Lie algebra $\mathfrak{u}(3,1)$. The embedding $\mathfrak{u}(3) \subset \mathfrak{u}(3,1)$ is analogous to the inclusion $SO(4) \subset SO(4,2)$ for the hydrogen atom.

D. Dynamical Models

In this Section so far we have described the hydrogen atom using a very large group $SO(4,2)$ and breaking down the symmetry to $SO(4)$ and further to $SO(3)$ when there are Coulomb-breaking perturbations that maintain their spherical symmetry. We have also introduced a sequence of groups and subgroups $U(3,1) \downarrow U(3) \downarrow SO(3)$ to provide a basis for the nuclear shell model.

Nuclear computations are very difficult because there is "no nuclear force". The force acting between nucleons is a residual force from the quark-quark interaction. This is analogous to the absence of a "molecular force". There is none - the force that binds together atoms in molecules is the residual electromagnetic force after exchange and other interactions have been taken into account.

For this reason it would be very useful to develop a systematic way for making nuclear models and carrying out calculations within the context of these models. Group theory to the rescue!

The first step in creating a simple environment for quantitative nuclear models is to assume that pairs of nucleons bind tightly into boson-like excitations. The leading assumption is that of all the nuclear-pair degrees of freedom, the most important are those with scalar $(S, L=0)$ and quadrupole $(D, L=2)$ transformation properties under the rotation group $SO(3)$. States in a Hilbert space describing 2 protons (neutrons, nucleons) can be produced by creation operators $s^\dagger, d_m^\dagger$ acting on the vacuum $|0;0,0,0,0,0\rangle$. For n pairs of nucleons, n creation operators act to produce states $|n_s; n_{-2}, n_{-1}, n_0, n_1, n_2\rangle$ with $n_s + \sum_m n_m = n$. There are $(n+6-1)!/n!(6-1)!$ states in this Hilbert space. For computational convenience they can be arranged by their transformation properties under rotations $SO(3)$. For example, the two boson Hilbert space has 21 states consisting of an $L=0$ state from $s^\dagger s^\dagger$, an $L=2$ multiplet from $s^\dagger d_m^\dagger$, and multiplets with $L=0,2,4$ from $d_{m'}^\dagger d_m^\dagger$.

The Hamiltonian acts within the space with a fixed number of bosons. It must therefore be constructed from number-conserving operators: $b_i^\dagger b_j$,where the boson operators include the s and d excitations. These operators must be rotationally invariant. At the linear level only two such operators exist: $s^\dagger s$ and $d_m^\dagger d_m$. At the quadratic level there are a small number of additional rotationally invariant operators. The n boson Hamiltonian can therefore be systematically parameterized by a relatively small number of terms. The parameters can be varied in attempts to fit models to nuclear spectra and transition rates. In the two boson example with 21 states, it is sufficient to diagonalize this Hamiltonian in the two-dimensional subspace of $L=0$ multiplets, in another two dimensional subspace with the two states with $L=2$ and $M_L=2$ (all other M_L values will give the same result), and the one-dimensional suspace with $L=4, M_L=4$.

The Interacting Boson Model (IMB) outlined above has deeply extended our understanding of nuclear physics

(Arima and Iachello, 1976). In fact, some Hamiltonians can be solved "by hand". These involve a group-subgroup chain. The chain of groups is shown in Fig. 7. This model incorporates in a magnificent way the use of groups in their capacity as symmetry groups, implying degeneracy, and dynamical groups, implying relations among multiplets of different energies.

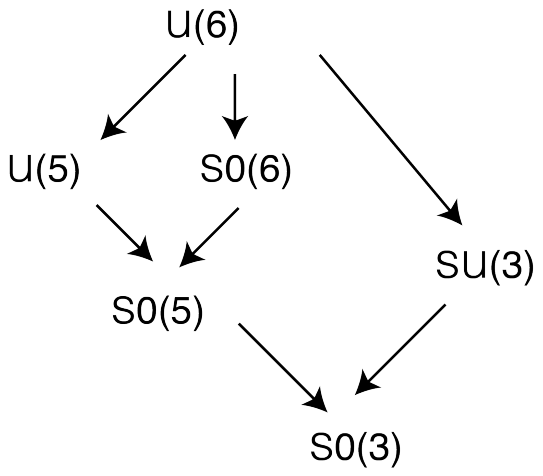

FIG. 7: States with $2N$ nucleons outside a closed shell are described by N bosons in the Interating Boson Model. The basis states carry a symmetric representation of the Lie group $U(6)$. Various limiting Hamiltonians that exhibit a group-subgroup symmetry can be diagonalized by hand. The three group-subgroup chains for which this is possible are shown here.

XIII. GAUGE THEORY

Gauge transformations were introduced by Weyl following Einstein's development (1916) of the Theory of General Relativity. In crude terms, Weyl's original idea was to introduce a ruler (the "gauge" of gauge theory) whose length was an arbitrary function of position. His original objective was to unify the two then-known forces of Nature: gravitation and electromagnetism. His theory is quite beautiful but Einstein raised serious objections, and Weyl eventually relinquished it. Einstein's objection was that if Weyl's theory were correct then the results of laboratory experiments would depend on the history of the material being investigated.

Weyl came back to this general idea following Schrödinger's development (1926) of Wave Mechanics. In this case a modified objective was achieved: he succeeded in describing how light interacts with charged matter.

The original theory (GR) involved a *real* scaling transformation that was space-time dependent. As a result, it is in the same spirit as the discussion about scaling in Sec. II C, but more general. His modified theory (QM) involved a *complex* phase transformation. In some sense this would be an analytic continuation of the scaling arguments, but the spirit of the discussion given in Sec. II C does not in any sense suggest phase changes.

The starting point of this work is the observation that if $\psi(x,t)$ satisfies Schrödinger's time-dependent equation, so also does $e^{i\phi}\psi(x,t)$, for

$$\left(\mathcal{H} - i\hbar\frac{\partial}{\partial t}\right) e^{i\phi}\psi(x,t) = e^{i\phi}\left(\mathcal{H} - i\hbar\frac{\partial}{\partial t}\right)\psi(x,t) = 0 \tag{120}$$

This fails to be true if the phase ϕ depends on space-time coordinates, for then the derivative terms act on this phase when we try to pull it through the Hamiltonian and time-derivative operators:

$$\left(\left(\frac{\mathbf{p}\cdot\mathbf{p}}{2m}\right) + q\Phi(x,t) - i\hbar\frac{\partial}{\partial t}\right) e^{i\phi(x,t)}\psi(x,t) =$$

$$e^{i\phi(x,t)}\left(\frac{(\mathbf{p}+\hbar\nabla\phi)^2}{2m} + q\Phi(x,t) + \hbar\frac{\partial\phi}{\partial t} - i\hbar\frac{\partial}{\partial t}\right)\psi(x,t) \tag{121}$$

Symmetry is not preserved! What to do?

It had long been known that the electric and magnetic fields $\mathbf{E},\mathbf{B}$ could be represented by "fictitious" potentials that served to simplify Maxwell's equations but were otherwise "not real". The vector potential $\mathbf{A}$ and scalar potential Φ are related to the "real" fields by

$$\begin{aligned} \mathbf{B} &= \nabla\times\mathbf{A} \\ \mathbf{E} &= -\nabla\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} \end{aligned} \tag{122}$$

This simplification is not unique. The vector potential can be changed by the addition of the gradient of a scalar field $\chi(x,t)$, and the scalar potential correspondingly changed:

$$\begin{aligned} \mathbf{A} \;\rightarrow\; \mathbf{A}' &= \mathbf{A} + \nabla\chi \;\;\Rightarrow\; \mathbf{B}' = \mathbf{B} \\ \Phi \;\rightarrow\; \Phi' &= \Phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \;\;\Rightarrow\; \mathbf{E}' = \mathbf{E} \end{aligned} \tag{123}$$

The resolution of the difficulty is to assume that the electrostatic part of the interaction is described by the term $q\Phi(x,t)$ in the Hamiltonian and the magnetic part is represented by replacing $\mathbf{p}$ by $\mathbf{p} - \frac{q}{c}\mathbf{A}(x,t)$ wherever it appears in the Hamiltonian. Under these conditions

$$\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(x,t)\right) e^{i\phi(x,t)} = e^{i\phi(x,t)}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}(x,t) + \hbar\nabla\phi(x,t)\right) \tag{124}$$

and

$$\left(q\Phi - i\hbar\frac{\partial}{\partial t}\right) e^{i\phi(x,t)} = e^{i\phi(x,t)}\left(q\Phi + \hbar\frac{\partial\phi}{\partial t} - i\hbar\frac{\partial}{\partial t}\right) \tag{125}$$

If we choose $\phi(x,t) = -\frac{q}{\hbar c}\chi(x,t)$, then the added terms on the right in Eq. (124) are

$$\mathbf{p} - \frac{q}{c}\mathbf{A}(x,t) - \frac{q}{c}\nabla\chi(x,t) = \mathbf{p} - \frac{q}{c}\mathbf{A}'(x,t) \qquad (126)$$

and those on the right in Eq. (125) are

$$q\Phi(x,t) - \frac{q}{c}\frac{\partial\chi(x,t)}{\partial t} - i\hbar\frac{\partial}{\partial t} = q\Phi'(x,t) - i\hbar\frac{\partial}{\partial t} \qquad (127)$$

The result is that the structure of the interaction between the electromagnetic fields and charged particles is *invariant* provided the interaction is given in terms of the "fictitious" fields $\mathbf{A}, \Phi$ by

$$\mathbf{p} \to \mathbf{p} - \frac{q}{c}\mathbf{A}(x,t) \quad -i\hbar\frac{\partial}{\partial t} \to -i\hbar\frac{\partial}{\partial t} + q\Phi(x,t) \qquad (128)$$

There are several other ways to couple the electromagnetic field with charged particles that are allowed by symmetry (Bethe and Salpeter, 1957). But the structure of the interaction described by Eq. (128) is sufficient to account for all known measurements. It turns out that Maxwell's equations are also a consequence of the structure of this interaction.

This principle is called the Principle of Minimal Electromagnetic Coupling.

The phase transformation introduced in Eq. (120) belongs to the Lie group $U(1)$. Its generalization to position-dependent phase $e^{i\phi(x,t)}$ does not belong to a Lie group.

Questions soon surfaced if the same process could be used to describe the interaction between more complicated "charged" particles and the fields that cause interactions among them. It seemed that the proton-neutron pair was a good candidate for such a treatment. These two particles seemed to be essentially the same, except that one was charged and the other not. Neglecting charge, these two particles could be treated as an *iso*spin doublet. The nucleon wavefunction ϕ could be treated as a two-state system: $|\phi\rangle = |\begin{matrix}\psi_p \\ \psi_n\end{matrix}\rangle$ and the Hamiltonian describing nuclear interactions should be invariant under a *global* $SU(2)$ transformation, analogous to a global $U(1)$ transformation $e^{i\phi}$ in Eq. (120). If the $SU(2)$ rotation were allowed to vary with position, perhaps it would be possible to determine the nature of the interaction between the nucleons (fermions) and the bosons ($\pi^{\pm}, \pi^0$, analogous to photons that carry the electromagnetic interaction) responsible for the interaction among the fermions.

This program was carried out by Yang and Mills. They succeeded in determining the nature of the interaction. But we now understand that nuclear interactions are residual forces left over from the strong interactions among the quarks.

Nevertheless, the program persisted. The gauge program can be phrased as follows.

1. Suppose there is a set of n fermion fields that are invariant under a g-parameter Lie group.

2. Assume that the Hamiltonian (Lagrangian, Action Integral) for these fields, without any interaction, is known.

3. Now assume that the Lie group parameters are allowed to be functions on spacetime. What additional terms occur in the Hamiltonian (c.f., Eq. (121) above).

4. How many boson fields must be introduced in order to leave the structure of the Hamiltonian invariant?

5. How must they be introduced into the Hamiltonian. That is: what is the structure of the "Minimal Coupling" in terms of the Lie algebra parameters (its structure constants)?

6. How do these new fields transform under the Lie group and its spacetime extension?

7. What field equations do the new fields satisfy?

These questions have all been answered (Utiyama 1956). The number of new fields required is exactly the number of generators of the Lie group (i.e., its dimension). Each field is a four component field. Their dynamical equations are a consequence of this theory. All new fields are massless.

This theory has been applied to describe the electroweak interaction $U(2) \simeq U(1) \times SU(2)$ to predict the massless electromagnetic field and three boson fields called $W^{\pm}, Z^0$ that transmit the weak interaction. This theory was also applied to describe three quarks. The Lie group used was $SU(3)$ and the theory predicted the existence of eight (that's the dimension of the Lie group $SU(3)$) gluon fields, all massless. The gluon fields transmit the strong interaction. In the case of the gluons, the mass seems to be small enough to be consistent with "zero" but that is definitely not the case of the very massive weak gauge bosons $W^{\pm}, Z^0$. A new mechanism was called for, and proposed, to describe how these "massless" particles acquire such a heavy mass. This mechanism was proposed by Higgs, among others, and is called the Higgs mechanism. The discovery of the Higgs boson has recently been announced.

XIV. GROUP THEORY AND SPECIAL FUNCTIONS

A. Summary of Some Properties

The classical special functions of mathematical physics were developed in the 19^{th} century in response to a variety of specific physical problems. They include the Legendre and associated Legendre functions, the Laguerre and associated Laguerre functions, the Gegenbauer, Chebyshev, Hermite, and Bessel functions. They are for the most part orthogonal polynomials. They are constructed by choosing a basis set $f_0, f_1, f_2, \cdots$ that are

monomials in the position representation (Dirac notation): $\langle x|f_0\rangle = x^0, \langle x|f_1\rangle = x^1, \langle x|f_2\rangle = x^2, \cdots$ and then creating an orthogonal set by successive Gramm-Schmidt orthogonalization by means of an inner product $\langle f|g\rangle = \int_a^b f^*(x)g(x)w(x)dx$ with various weights $w(x)$ for the different functions:

$$|\phi_0\rangle = |f_0\rangle, \qquad |\phi_1\rangle = |f_1\rangle - \frac{|\phi_0\rangle\langle\phi_0|}{\langle\phi_0|\phi_0\rangle}|f_0\rangle$$
$$|\phi_j\rangle = |f_j\rangle - \sum_{k=0}^{j-1}\frac{|\phi_k\rangle\langle\phi_k|}{\langle\phi_k|\phi_k\rangle}|f_j\rangle \tag{129}$$

The Bessel functions are the exception to this rule, as they are not polynomials.

These functions obey a common variety of properties

Differential Equation:

$$g_2(x)y'' + g_1(x)y' + g_0(x)y = 0 \tag{130a}$$

Recurrence Relations:

$$a_{1n}f_{n+1}(x) = (a_{2n} + a_{3n}x)f_n(x) - a_{4n}f_{n-1}(x) \tag{130b}$$

Differential Relations:

$$g_2(x)\frac{df_n(x)}{dx} = g_1(x)f_n(x) + g_0(x)f_{n-1}(x) \tag{130c}$$

Generating Functions:

$$g(x,z) = \sum_{n=0}^{\infty} a_n f_n(x) z^n \tag{130d}$$

Rodrigues' Formula:

$$f_n(x) = \frac{1}{a_n\rho(x)}\frac{d^n}{dx^n}\{\rho(x)(g(x))^n\} \tag{130e}$$

The coefficients and functions can be found in standard tabulations (e.g., Abramowicz and Stegun). The Bessel functions have similar properties.

B. Relation with Lie Groups

A Lie group lives on a manifold $\mathcal{M}^n$ of dimension n. Each group operation is a function of position in the manifold: $g = g(x), x \in \mathcal{M}$. The product of two group operations is defined by an analytic composition law on the manifold:

$$g(x) \circ g(y) = g(z) \qquad z = z(x,y) \tag{131}$$

It is not until we construct representations for the group, or on top of the manifold, that really interesting things begin to happen. Representations

$$g(x) \to \Gamma^\alpha_{ij}(g(x)) \tag{132}$$

are functions defined on the manifold. Suitably normalized, the set of matrix elements for the complete set of UIR (unitary irreducible representations) form a complete orthonormal set of functions on the manifold $\mathcal{M}^n$. By duality (the miracles of Hilbert space theory), the triplet of indices α, i, j is described by as many integers as the dimension of $\mathcal{M}^n$. For example, for three dimensional Lie groups, such as $SO(3), SU(2), SO(2,1), ISO(2), H_3$ the matrix elements are indexed by three integers and can be represented in the form $\Gamma^\alpha_{ij}(g(x)) = \langle \begin{smallmatrix}\alpha\\ i\end{smallmatrix} |g(x)| \begin{smallmatrix}\alpha\\ j\end{smallmatrix} \rangle$. Including the appropriate normalization factor, they can be expressed as

$$\sqrt{\frac{dim(\alpha)}{Vol(G)}}\,\Gamma^\alpha_{ij}(g(x)) = \langle g(x)| \begin{smallmatrix}\alpha\\ i,j\end{smallmatrix} \rangle \tag{133}$$

For noncompact groups $Vol(G)$ is not finite, but $dim(\alpha)$ is also not finite, so the ratio under the radical needs tobe taken with care.

Representations are powerful because they lie in two worlds: geometric and algebraic. They have one foot in the manifold ($\langle g(x)| \simeq \langle \mathbf{x}|$ above) and the other in algebra ($| \begin{smallmatrix}\alpha\\ ij\end{smallmatrix} \rangle \simeq |\mathbf{n}\rangle$ above).

All classical special functions are specific matrix elements, evaluated on specific submanifolds, of specific irreducible representations of some Lie group.

We illustrate these ideas with a few examples without pretending we've even scratched the surface of this vast and fascinating field.

C. Spherical Harmonics and $SO(3)$

For the group $SU(2)$ the underlying manifold is a solid three-dimensional sphere. There are many ways to parameterize an operation in this group. We use an Euler-angle like parameterization introduced by Wigner:

$$\mathcal{D}^j_{mk}(\phi,\theta,\psi) = \langle \begin{smallmatrix}j\\ m\end{smallmatrix} |e^{-i\phi J_z}e^{-i\theta J_y}e^{-i\psi J_z}| \begin{smallmatrix}j\\ k\end{smallmatrix} \rangle = \tag{134}$$
$$e^{-im\phi}d^j_{mk}(\theta)e^{-ik\psi}$$

The orthogonality properties of the matrix elements are

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\psi\ \mathcal{D}^{j'*}_{m'k'}(\phi,\theta,\psi)\mathcal{D}^j_{mk}(\phi,\theta,\psi)$$
$$= \frac{8\pi^2}{2j+1}\delta^{j'j}\delta_{m'm}\delta_{k'k} \tag{135}$$

The volume of the group in this parameterization is $8\pi^2$. The normalization factor, converting the matrix elements to a complete orthonormal set, is $\sqrt{(2j+1)/8\pi^2}$.

In order to find a complete set of functions on the sphere (θ, ϕ) we search for those matrix elements above that are independent of the angle ψ. These only occur for $k = 0$, which occurs only among the subset of irreducible representations with $j = l$ (integer). Integrating out the $d\psi$ dependence in Eq. (134) leads to a definition of the spherical harmonics in terms of some Wigner $\mathcal{D}$ matrix elements (c.f., Eq. (135):

$$Y^l_m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^{l*}_{m0}(\phi,\theta,-) \tag{136}$$

These functions on the two-dimensional unit sphere surface (θ, ϕ) inherit their orthogonality and completeness properties from the corresponding properties of the UIR matrix elements $\mathcal{D}^j_{mk}$ on the three-dimensional solid sphere of radius 2π.

Other special functions are similarly related to these matrix elements. The associated Legendre polynomials are

$$P^m_l(\cos\theta) = \sqrt{\frac{(l+m)!}{(l-m)!}}\; d^l_{0,0}(\theta) \tag{137}$$

and the Legendre polynomials are

$$P_l(\cos\theta) = \mathcal{D}^l_{0,0}(-,\theta,-) = d^l_{0,0}(\theta) \tag{138}$$

These functions inherit the their measure $w(\theta)$ from the measure on $SU(2)$ and their orthogonality and completeness properties from those of the Wigner rotation matrix elements $\mathcal{D}^j_{mk}\,[SU(2)]$.

We emphasize again that these functions are specific matrix elements $\mathcal{D}^j_{mk}$, evaluated on specific submanifolds (sphere, line), of specific irreducible representations ($j = l$) of $SU(2)$.

D. Differential and Recursion Relations

We can understand the wide variety of relations that exist among the special functions (e.g., recursion relations, etc.) in terms of group theory/representation theory as follows. It is possible to compute the matrix elements of an operator $\mathcal{O}$ in either the continuous basis $\langle \mathbf{x}'|\mathcal{O}|\mathbf{x}\rangle$ or the discrete basis $\langle \mathbf{n}'|\mathcal{O}|\mathbf{n}\rangle$. In the first basis the coordinates $\mathbf{x}$ describe a submanifold in the group manifold $\mathcal{M}^n$, and the operator is a differential operator. In the second basis the indices $\mathbf{n}$ are an appropriate subset of the group representation α and row/column (i, j) index set and operator is a matrix with entries in the real or complex field.

It is also possible to compute the matrix elements in a *mixed* basis $\langle \mathbf{x}|\mathcal{O}|\mathbf{n}\rangle$. It is in this basis that really exciting things happen, for

$$\begin{array}{c} \langle \mathbf{x}|\mathcal{O}|\mathbf{n}\rangle \\ \swarrow \qquad\qquad \searrow \\ \langle \mathbf{x}|\mathcal{O}|\mathbf{x}'\rangle\langle \mathbf{x}'|\mathbf{n}\rangle = \langle \mathbf{x}|\mathbf{n}'\rangle\langle \mathbf{n}'|\mathcal{O}|\mathbf{n}\rangle \end{array} \tag{139}$$

On the left hand side a differential operator $\langle \mathbf{x}|\mathcal{O}|\mathbf{x}'\rangle$ acts on the special function $\langle \mathbf{x}'|\mathbf{n}\rangle$, while on the right-hand side a matrix $\langle \mathbf{n}'|\mathcal{O}|\mathbf{n}\rangle$ multiplies the special functions $\langle \mathbf{x}|\mathbf{n}'\rangle$.

For the rotation group acting on the sphere surface (θ, ϕ) and the choice $\mathcal{O} = L_\pm$ we find for $\langle \theta\phi|L_\pm|\begin{smallmatrix} l \\ m \end{smallmatrix}\rangle$ computed as on the left in Eq.(139)

$$\begin{aligned} & e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)\delta(\cos\theta'-\cos\theta)\delta(\phi'-\phi)Y^l_m(\theta',\phi') \\ & = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right)Y^l_m(\theta,\phi) \end{aligned} \tag{140}$$

and as computed on the right

$$\langle \theta\phi|\begin{smallmatrix} l' \\ m' \end{smallmatrix}\rangle\langle\begin{smallmatrix} l' \\ m' \end{smallmatrix}|L_\pm|\begin{smallmatrix} l \\ m \end{smallmatrix}\rangle = Y^l_{m\pm1}(\theta,\phi)\sqrt{(l\pm m+1)(l\mp m)} \tag{141}$$

There are a number of Lie groups that can be defined to act on a one dimensional space. In such cases the infinitesimal generators take the form of functions of the coordinate x and the derivative d/dx. We illustrate the ideas behind differential and recursion relations in the context of the Hiesenberg group H_3. Its algebra $\mathfrak{h}_3$ is spanned by three operators, universally identified as $a, a^\dagger, I$ with commutation relations $\left[a, a^\dagger\right] = I, [a, I] = \left[a^\dagger, I\right] = 0$. These operators have matrix elements as follows in the continuous basis (geometric) representation:

$$\begin{aligned} \langle x'|a|x\rangle &= \delta(x'-x)\;\tfrac{1}{\sqrt{2}}(x+D) \\ \langle x'|a^\dagger|x\rangle &= \delta(x'-x)\;\tfrac{1}{\sqrt{2}}(x-D) \\ \langle x'|I|x\rangle &= \delta(x'-x) \end{aligned} \tag{142}$$

and discrete basis (algebraic) representation:

$$\begin{aligned} \langle n'|a|n\rangle &= \delta_{n',n-1}\;\sqrt{n} \\ \langle n'|a^\dagger|n\rangle &= \delta_{n',n+1}\;\sqrt{n'} \\ \langle n'|I|n\rangle &= \delta_{n',n} \end{aligned} \tag{143}$$

Here $D = \frac{d}{dx}$.

The special functions are the mixed basis matrix elements $\langle x|n\rangle$. We can compute these starting with the ground, or lowest, state $|0\rangle$.

$$\begin{array}{ccc} & \langle x|a|0\rangle & \\ & \swarrow \quad \searrow & \\ \langle x|a|0\rangle & = & \langle x|n\rangle\langle n|a|0\rangle \\ \frac{1}{\sqrt{2}}(x+D)\langle x|0\rangle & = & 0 \end{array} \tag{144}$$

This equation has a unique solution $N\langle x|0\rangle = e^{-x^2/2}$ up to scale factor, $N = 1/\sqrt[4]{\pi}$.

The remaining normalized basis states are constructed by applying the raising operator:

$$\begin{aligned} \langle x|n\rangle &= \langle x|\frac{(a^\dagger)^n}{n!}|x'\rangle\langle x'|0\rangle = \frac{(x-D)^n}{\sqrt{2^n n!\sqrt{\pi}}}\ e^{-x^2/2} \\ &= \frac{H_n(x)e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}} \end{aligned} \tag{145}$$

The Hermite polynomials in Eq. (145) are defined by

$$H_n(x) = e^{+x^2/2}(x-D)^n e^{-x^2/2} \tag{146}$$

The states $\langle x|n\rangle$ are normalized to $+1$.

In order to construct the recursion relations for the Hermite polynomials, choose $\mathcal{O} = x = (a+a^\dagger)/\sqrt{2}$ in Eq. (139). Then

$$\langle x|\mathcal{O}|n\rangle = x\frac{H_n(x)e^{-x^2/2}}{\sqrt{2^n n!\sqrt{\pi}}} = \frac{1}{\sqrt{2}}\langle x|n'\rangle\langle n'|(a+a^\dagger)|n\rangle \tag{147}$$

The two nonzero matrix elements on the right are given in Eq. (143). They couple $xH_n(x)$ on the left with $H_{n\pm1}(x)$ on the right. When the expression is cleaned up the standard recursion relation is obtained:

$$2x\ H_n(x) = H_{n+1}(x) + 2n\ H_{n-1}(x) \tag{148}$$

The differential relation is obtained in the same way, replacing $x = (a+a^\dagger)/\sqrt{2}$ by $D = (a-a^\dagger)/\sqrt{2}$ in Eq. (147). On the left hand side we find the derivative of $H_n(x)$ as well as the derivative of $e^{-x^2/2}$, and on the right hand side a linear combination of $H_{n\pm1}(x)$. When the expression is cleaned up there results the standard differential relation

$$H_n^{'}(x) = 2n\ H_{n-1}(x) \tag{149}$$

E. Differential Equation

It happens often that an operator can be formed that is quadratic in the basis vectors of the Lie algebra and it also commutes with every element in the Lie algebra. Such operators can always be constructed for semisimple Lie algebras where the Cartan metric g_{ij} (c.f., Eq. (53)) is nonsingular. The operator $g^{ij}X_iX_j$ has this property. The construction of nontrivial quadratic operators with this property is even possible for many Lie algebras that are not semisimple. When it is possible, the left-hand side of Eq. (139) is a second order differential operator and the right hand side is a constant. This constant is the eigenvalue in the differential equation (first property listed above).

For the three-dimensional nonsemisimple group $ISO(2)$ of length-preserving translations and rotations of the plane to itself, the three infinitesimal generators are L^3, which generates rotations around the z axis, and T_1, T_2, which generate displacements in the x and y directions. The operators T_1 and T_2 commute. The operators $L^3, T^\pm = T_1 \pm iT_2$ satisfy commutation relations

$$\left[L^3, T^\pm\right] = \pm T^\pm \qquad \left[T^+, T^-\right] = 0 \tag{150}$$

When acting on the plane, the three can be expressed in terms of a radial (r) and angular (ϕ) variable.

$$L^3 = \frac{1}{i}\frac{\partial}{\partial\phi} \qquad T^\pm = e^{\pm i\phi}\left(\pm\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\right) \tag{151}$$

Basis vectors $|m\rangle$ are introduced that satisfy the condition

$$L^3|m\rangle = m|m\rangle \Rightarrow \langle r\phi|m\rangle = g_m(r)e^{im\phi} \tag{152}$$

Single-valuedness requires m is an integer. Adjacent basis vectors are defined by

$$T^\pm|m\rangle = -|m\pm1\rangle \Rightarrow \left(\pm\frac{d}{dr} - \frac{m}{r}\right)g_m(r) = -g_{m\pm1}(r) \tag{153}$$

Finally, the identity $T^+T^-|m\rangle = |m\rangle$ gives Bessel's equation

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + 1 - \frac{m^2}{r^2}\right)g_m(r) = 0 \tag{154}$$

F. Addition Theorems

Addition theorems reflect the group composition property through the matrix multiplication property of representations:

$$\begin{array}{c} \langle \mathbf{n}|g(\mathbf{x})g(\mathbf{y})|\mathbf{n}'\rangle \\ \swarrow \qquad \searrow \\ \sum_{\mathbf{k}}\langle \mathbf{n}|g(\mathbf{x})|\mathbf{k}\rangle\langle \mathbf{k}|g(\mathbf{y})|\mathbf{n}'\rangle = \langle \mathbf{n}|g\left[\mathbf{z}(\mathbf{x},\mathbf{y})\right]|\mathbf{n}'\rangle \end{array} \tag{155}$$

The special function at argument $\mathbf{z}$ is expressed as a pairwise product of special functions evaluated at the group

elements $g(\mathbf{x})$ and $g(\mathbf{y})$ for which $\mathbf{x} \circ \mathbf{y} = \mathbf{z}$. The best-known of these addition results is

$$\begin{aligned} \mathcal{D}^l_{00}(\Theta) &= \mathcal{D}^l_{0m}(g_1^{-1})\mathcal{D}^l_{m0}(g_2) \\ \mathcal{D}^l_{00}(\Theta) &= \mathcal{D}^{l*}_{m0}(g_1)\mathcal{D}^l_{m0}(g_2) \\ \frac{2l+1}{4\pi} P_l(\cos\Theta) &= \sum_m Y^l_m(\theta_1,\phi_1) Y^{l*}_m(\theta_2,\phi_2) \end{aligned} \tag{156}$$

Here we have taken $g_1 = (\theta_1, \phi_1, -)$ and $g_2 = (\theta_2, \phi_2, -)$, and Θ is the angle between these two points on the sphere surface, defined by

$$\cos\Theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1) \tag{157}$$

G. Generating Functions

Generating functions are constructed by computing the exponential of an operator $\mathcal{O}$ in the Lie algebra in two different ways and then equating the results. We illustrate for H_3 by computing $\langle x|e^{\sqrt{2}ta^\dagger}|0\rangle$. We first compute the brute strength Taylor series expansion of the exponential:

$$\langle x|e^{t(x-D)}|x'\rangle\langle x'|0\rangle = e^{-x^2/2}\frac{1}{\sqrt[4]{\pi}}\sum_{n=0}^{\infty}\frac{t^n\ H_n(x)}{n!} \tag{158}$$

Here $\langle x|0\rangle = e^{-x^2/2}/\sqrt[4]{\pi}$. Eq. (146) was used to obtain this result.

Next, we observe that exponentials of differential operators are closely related to Taylor series expansions, for instance $e^{-t\frac{d}{dx}} f(x) = f(x-t)$. To exploit this we use the result of the disentangling theorem Eq. (67) to write

$$e^{t(x-D)} = e^{tx}e^{-t^2/2}e^{-tD} \tag{159}$$

Then

$$\begin{aligned} \langle x|e^{t(x-D)}|x'\rangle\langle x'|0\rangle = e^{tx}e^{-t^2/2}e^{-tD}\langle x|0\rangle = \\ \frac{1}{\sqrt[4]{\pi}}e^{tx}e^{-t^2/2}e^{-(x-t)^2/2} \end{aligned} \tag{160}$$

By comparing the two calculations, Eq. (158) with Eq. (160), we find the standard generating function for the Hermite polynomials.

$$e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{t^n H_n(x)}{n!} \tag{161}$$

XV. SUMMARY

The study of symmetry has had a profound influence on the development of the natural sciences. Group Theory has been used in constructive ways before groups even existed. We have given a flavor of what can be done with symmetry and related arguments in Sec. II, which describes three types of arguments that live in the same ballpark as Group Theory. Groups were formally introduced in Sec. III and a number of examples given, ranging from finite groups to Lie groups. Transformation groups played a big role in the development of classical physics: mechanics and electrodynamics. In fact, it was the need to formulate these two theories so that their structure remained unchanged under transformations from the same group that led to the Theory of Special Relativity. The group at hand was the Inhomogeneous Lorentz Group, the 10-parameter Lie group of Lorentz transformations and translations acting on fields defined over space-time. Sec. IX describes how group theory played a role in the development of Special Relativity. The next step beyond requiring invariance under the *same* Lorentz transformation at every space-time point involved allowing the Lorentz transformation to vary from point to point in a continuous way and still requiring some kind of invariance (c.f., Gauge Theories as discussed in Sec. XIII). This extended aesthetic led to the Theory of General Relativity.

Up to this point Physicists could have done without all the particular intricacies of Group Theory. The next step in the growth of this subject was the intensive study of the linear representations of groups. The growth was indispensible when Quantum Theory was developed, because groups acted in Hilbert spaces through their linear matrix representations. We provided an overview of representation theory in Sec. X. At first, groups were applied in the Quantum Theory as symmetry groups (c.f., Sec. XI. In this capacity they were used to describe the degeneracies in energy levels that were required by symmetry. Shortly afterward they were used in a more daring way to describe nondegenerate levels related to each other either by a broken symmetry or simply by operators that had little to do with symmetry but had the good sense to close under commutation with each other. Some applications of Dynamical Groups are described in Sec. XII.

Gauge theories were briefly treated in Sec. XIII. In such theories one begins with a symmetry group and requires that a Hamiltonina, Lagrangian, or Action remain "invariant" under the transformation when the parameters of the transformation group are allowed to be functions over space-time. It is remarkable that this requirement leads to the prediction of new fields, the nature of the interaction of the new fields with the original fields, the structure of the equations of the new fields, and the mass spectrum of these new fields: all new masses are zero. This problem was overcome by proposing that a new particle, now called the Higgs boson, exists. It's

discovery was announced in 2012.

As a closing tribute to the theory of groups and their linear matrix representations, we hint how the entire theory of the Special Functions of Mathematical Physics, which was created long before the Lie groups were invented, is a study of the properties of specific matrix elements of specific matrix representations of particular Lie groups acting over special submanifolds of the differentiable manifold that parameterizes the Lie group. These ideas are sketched by simple examples in Sec. XIV.

Group theory has progressed from the outer fringes of theoretical physics in 1928, when it was referred to as the *Gruppenpest* (1928 Weyl to Dirac at Princeton), through the mainstream of modern physics, to wind up playing the central role in the development of physical theory. Theoretical Physicists now believe that if a theory of fundamental interactions is not a gauge theory it doesn't have the right to be considered a theory of interactions at all. Gauge theory is the new version of 'simple' and 'elegant'.

We learn that Nature was not tamed until Adam was able to give names to all the animals. Just so, we can't even give names to particles and their states without knowing at least a little bit about group theory. Group theory has migrated from the outer fringes of physics (Gruppenpest, 1928) to the central player, even the lingua franca, of modern physics.

Further Reading

Arima, A. and F. Iachello (1976), *Interacting Boson Model of collective states, Part I (the vibrational limit)*, Annals of Physics (NY), **99**, 253-317.

Barenblatt, G. 2003 *Scaling*, Cambridge: Cambridge University Press.

Bethe, H. A., and E. E. Salpeter (1957), *Quantum Mechanics of One- and Two-Electron Atoms*, New zyork, Academic Press.

Coxeter, H. S. M., and W. O. J. Moser (1980) *Generators and Relations for Discrete Groups, 4^{th} Ed.*, Berlin: Springer-Verlag.

Gilmore, R. (2008), *Lie Groups, Physics, and Geometry, An Introduction for Physicists, Engineers, and Chemists*, Cambridge: Cambridge University Press.

Griffiths, D. (2008), *Introduction to Elementary Particles*, Weinheim, Germany: Wiley-VCH.

Hamermesh, M. (1962), *Group Theory and its Application to Physical Problems*, Reading, MA: Addison-Wesley; reprint (1989), New York: Dover.

Landau, L. D. and E. M. Lifshitz, (1960)*Mechanics*, Reading, MA: Addison-Wesley.

Sternberg, S. (1994), *Group Theory and Physics*, Cambridge: University Press.

Talman, J. D. (1968), *Special Functions: A Group Theoretic Approach (Based on Lectures by Eugene P. Wigner)*, New York: Benjamin.

Tinkham, M. (1964), *Group Theory and Quantum Mechanics*, New York: McGraw Hill.

Utiyama, R. (1956), *Invariant theoretical interpretation of interaction*, Phys. Rev. **101**, 1597-1607.

Weinberg, S. (1972), *Gravitation and Cosmology, Principles and Applications of the GeneralTheory of Relativity*, New York: Wiley.

Wigner, E. P. (1959),*Group Theory, And its Application to the Quantum Mechanics of Atomic Spectra*, (J. J. Griffin, translator), New York: Academic Press.