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LETTER TO THE EDITOR

Q and *P* representatives for spherical tensors

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Abstract. The Q and P representatives for the irreducible spherical tensor operators $\mathcal{Y}^L_M(\boldsymbol{J})$ are proportional to the corresponding spherical harmonics $Y^L_M(\theta, \phi)$. In the coherent state basis associated with the $(2j+1)$-dimensional unitary irreducible representation of SU(2), the proportionality factors are $(2j)!/(2j-L)!2^L$ and $(2j+1+L)!/(2j+1)!2^L$, respectively.

Let $\hat{F}$ be a symmetrized polynomial function of the angular momentum operators $\boldsymbol{J}$ for a single spin. Then $\hat{F}$ has a unique Q representative (Glauber 1963) $\hat{F} \xrightarrow{Q} f(\Omega)$ in the $(2j+1)$-dimensional space which carries the irreducible representation $D^j[\mathrm{SU}(2)]$:

$$f(\Omega) = \langle\Omega|\hat{F}|\Omega\rangle. \tag{1Q}$$

Here $\Omega = (\theta, \phi)$ represents a point on the surface of the Bloch sphere, with θ measured from the north pole, and $|\Omega\rangle$ is an atomic coherent state (Arecchi *et al* 1972):

$$|\Omega\rangle = \sum_m |{}^j_m\rangle D^j_{mj}(\Omega). \tag{2}$$

The operator $\hat{F}$ also possesses a non-unique (Arecchi *et al* 1972) P representative $\hat{F} \xrightarrow{P} F(\Omega)$ determined by

$$\hat{F} = \frac{2j+1}{4\pi} \int |\Omega\rangle F(\Omega) \langle\Omega| \, \mathrm{d}\mu(\Omega). \tag{1P}$$

The Q and P representatives of $\hat{F}$ are related by the convolution

$$f(\Omega) = \frac{2j+1}{4\pi} \int F(\Omega') |\langle\Omega'|\Omega\rangle|^2 \, \mathrm{d}\mu(\Omega'). \tag{3}$$

This convolution annihilates the non-unique part of the P representative, and provides a one-to-one correspondence between $f(\Omega)$ and the unique part of $F(\Omega)$.

The relation between $f(\Omega)$ and $F(\Omega)$ is made explicit by expanding these functions in terms of spherical harmonics as follows:

$$f(\Omega) = \sum_{L=0}^{2j} \sum_{M=-L}^{+L} f^L_M \frac{2L+1}{4\pi} Y^L_M(\Omega). \tag{4}$$

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If the Fourier coefficients of $F(\Omega)$ and the kernel

$$K(\Omega)=\frac{2j+1}{4\pi}|\langle\theta=0,\phi=0|\Omega\rangle|^2$$

are defined similarly, then (Gilmore 1972)

$$f^L_M=F^L_M K^L_0 \tag{5}$$

where K^L_0 is the square of a Clebsch–Gordan coefficient (Gilmore *et al* 1975) and

$$K^L_0=\frac{(2j+1)!(2j)!}{(2j+1+L)!(2j-L)!}. \tag{6}$$

The Q and P representatives of $\hat{F}$ are proportional if either is homogeneous of degree L.

The generating function (Arecchi *et al* 1972) for moments of angular momentum operators can be used to compute the Q representative for $(J_+)^L$:

$$\langle\Omega|(J_+)^L|\Omega\rangle=\frac{(2j)!}{(2j-L)!2^L}(\mathrm{e}^{\mathrm{i}\phi}\sin\theta)^L. \tag{7}$$

The operator $(J_+)^L$ is proportional to the spherical tensor $\mathcal{Y}^L_L(\boldsymbol{J})$ and $(\mathrm{e}^{\mathrm{i}\phi}\sin\theta)^L$ is proportional to the spherical harmonic $Y^L_L(\theta,\phi)$ with the same proportionality factor. Since $\mathcal{Y}^L_M(\boldsymbol{J})$ and $Y^L_M(\theta,\phi)$ transform identically under SU(2), we have a very simple expression for the Q representative of $\mathcal{Y}^L_M(\boldsymbol{J})$:

$$\mathcal{Y}^L_M(\boldsymbol{J})\overset{Q}{\rightarrow}\frac{(2j)!}{(2j-L)!2^L}Y^L_M(\Omega). \tag{8Q}$$

The unique part of the P representative follows directly from (5) and (6)

$$\mathcal{Y}^L_M(\boldsymbol{J})\overset{P}{\rightarrow}\frac{(2j+1+L)!}{(2j+1)!2^L}Y^L_M(\Omega). \tag{8P}$$

For example, for $L=1$ the Q representatives of $J_\pm, J_z$ are $j\,\mathrm{e}^{\pm\mathrm{i}\phi}\sin\theta$, $j\cos\theta$, while their P representatives are $(j+1)\,\mathrm{e}^{\pm\mathrm{i}\phi}\sin\theta$, $(j+1)\cos\theta$. For $L=2$, the Q and P representatives of $3J_z^2-\boldsymbol{J}^2$ are $j(j-\frac{1}{2})\,(3\cos^2\theta-1)$ and $(j+1)(j+\frac{3}{2})\,(3\cos^2\theta-1)$, as previously given by Lieb (1973).

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