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Lie Groups: general theory

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1 Introduction

Local continuous transformations were introduced by Lie as a tool for solving ordinary differential equations. In this program he followed the spirit of Galois, who used finite groups to develop algorithms for solving algebraic equations (the general quadratic, cubic, and quartic), or else to prove that some equations (the generic quintic) could not be solved by quadrature.

Lie's work lead eventually to the definition and study of Lie groups. Lie groups are beautiful in their own right — so beautiful that they have been studied independently of their origin as a tool for solving differential equations and studying the special functions determined by certain classes of these equations.

2 Lie Groups

Lie groups exist at the interface of the two great divisions of mathematics: Algebra and Topology. Their algebraic properties derive from the group axioms. Their geometric properties arise from the parameterization of the group elements by points in a differentiable manifold. The rigidity of these structures arises from the continuity requirements imposed on the group composition and inversion maps.

The algebraic axioms are standard.

Definition: A group G consists of a set $g_i, g_j, g_k, \ldots \in G$ together with a combinatorial operation $\circ$ that satisfy the four axioms

1. **Closure:** If $g_i \in G$, $g_j \in G$, then $g_i \circ g_j \in G$.
2. **Associativity:** If $g_i,\ g_j,\ g_k \in G$, then $(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$.
3. **Identity:** There is a unique operation $e \in G$ that satisfies $e \circ g_i = g_i = g_i \circ e$.
4. **Inverse:** Every group operation $g_i \in G$ has an inverse, denoted g_i^{-1}, that satisfies $g_i \circ g_i^{-1} = e = g_i^{-1} \circ g_i$.

Lie groups have more structure than groups. In particular, each $g_i \in G$ is a point in an n-dimensional manifold M^n. That is, the subscript i actually identifies a point $x \in M^n$, so that we can write $g_i = g(x)$ or most simply $g_i = x$. The group multiplication can be expressed in the form $g_i \circ g_j = g_k \to g(x) \circ g(y) = g(z)$, where $x \in M^n,\ y \in M^n,\ z = \phi(x, y) \in M^n$. The group inversion map can be expressed in the form $g(x) \to g(x)^{-1} = g(y)$, $y = \psi(x) \in M^n$. The topological axioms for Lie groups can be taken as

5. **Continuity of Compositon:** The mapping $z = \phi(x, y)$ defined by the group composition law is differentiable.

6. **Continuity of Inversion:** The mapping $y = \psi(x)$ defined by the group inversion law is differentiable.

The dimension of the Lie group is the dimension of the manifold that parameterizes the operations in the group.

The most familiar examples of Lie groups consist of $n \times n$ nonsingular matrices over the fields R, C, Q of real numbers, complex numbers, and quaternions. For example, the set of 2×2 real unimodular matrices $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$, $ad - bc = 1$, is a three-dimensional submanifold embedded in $\mathrm{R}^{2^2} = R^4$.

3 Matrix Lie Groups

Not every Lie group is a matrix group. Yet it is a surprising and useful result that almost every Lie group encountered in Physics is a matrix Lie group. These are all subgroups of the General Linear groups $GL(n;F)$ of $n \times n$ nonsingular matrices over the field F $(R,\ C,\ Q)$. These groups have real dimension $n^2 \times (1,\ 2,\ 4)$, respectively. The special linear subgroups $SL(n;F)$ are defined as the subgroups of $n \times n$ matrices with determinant $+1$: $M \in SL(n;F)$ if det $M = +1$. This definition is problematic for quaternions, as they do not commute. To avoid this problem, it is useful to map quaternions into 2×2 complex matrices in the same way complex numbers can be mapped into 2×2 real matrices:

$$a + ib \rightarrow \left[\begin{array}{cc} a & b \\ -b & a \end{array}\right] \qquad q_0 + \mathcal{I}q_1 + \mathcal{J}q_2 + \mathcal{K}q_3 \rightarrow \left[\begin{array}{cc} q_0 + iq_3 & iq_1 - q_2 \\ iq_1 + q_2 & q_0 - iq_3 \end{array}\right]$$

Here $(1, i)$ are basis vectors for C^1 considered as a real two-dimensional linear vector space, $(1, \mathcal{I}, \mathcal{J}, \mathcal{K})$ are basis vectors for Q^1 considered as a real four-dimensional linear vector space, and (a, b) and (q_0, q_1, q_2, q_3) are all real. The squares of the imaginary quantities i and $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are all -1: $i^2 = -1$; $\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 = -1$ and the imaginary quaternion basis elements anticommute: $\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{J}, \mathcal{K}\} = \{\mathcal{K}, \mathcal{I}\} = 0$. The unimodular subgroup $SL(n;Q)$ of $GL(n;Q)$ is obtained by replacing each quaternion matrix element by a 2×2 complex matrix, setting the determinant of the resulting $2n \times 2n$ matrix group to $+1$, and then mapping each of the n^2 complex 2×2 matrices back to quaternions.

Many other important groups are defined by imposing linear or quadratic constraints on the n^2 matrix elements of $GL(n;F)$ or $SL(n;F)$. The compact metric preserving groups $U(n;F)$ leave invariant lengths (preserve a positive-definite metric $g = I_n$) in linear vector spaces. The matrices $M \in U(n;F)$ satisfy $M^\dagger I_n M = I_n$. These conditions define the orthogonal groups $O(n) = U(n;R)$ and the unitary groups $U(n) = U(n;C)$. Their noncompact counterparts $O(p,q)$ and $U(p,q)$ leave invariant nonsingular indefinite metrics $g = I_{p,q} = \left[\begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array}\right]$ in real and complex $n = p + q$ dimensional linear vector spaces: $M^\dagger I_{p,q} M = I_{p,q}$.

Intersections of matrix Lie groups are also Lie groups. The special metric preserving groups are intersections of the special linear groups $SL(n;F) \subset GL(n;F)$ (with $F = Q$, $SL(n;Q)$ is defined as described above) and the metric preserving subgroups $U(n;F) \subset GL(n;F)$:

$$\begin{array}{lclcll} SL(n;R) & \cap & U(n;R) & = & SO(n) & n(n-1)/2 \\ SL(n;C) & \cap & U(n;C) & = & SU(n) & n^2 - 1 \\ SL(n;Q) & \cap & U(n;Q) & = & Sp(n) = USp(2n) & n(2n+1) \end{array}$$

The real dimensions of these groups are given in the right hand column. Under the replacement of quaternions by 2×2 complex matrices, the group of $n \times n$ metric preserving and unimodular matrices $Sp(n)$ over Q is identified as $USp(2n)$, an isomorphic group of $2n \times 2n$ matrices over C.

Noncompact forms $SO(p,q)$, $SU(p,q)$ and $Sp(p,q) = USp(2p,2q)$ are defined similarly.

The Lie group $SU(2)$ rotates spin states to spin states in a complex two-dimensional linear vector space. It leaves lengths, inner products, and probabilities invariant. If an interaction is spin independent only an invariant ("Casimir invariant") constructed from the spin operators can appear in the Hamiltonian. The same group can act in isospin space, rotating proton to neutron states. The Lie group $SU(3)$ similarly rotates quark states or color states into quark states or color states. The Lie group $SU(4)$ rotates spin-isospin states into themselves. The conformal group $SO(4,2)$ leaves angles but not lengths in space-time invariant. It is the largest group that leaves the source-free Maxwell equations invariant. It is also the largest group that transforms all the (bound, scattering, parabolic) hydrogen atom states into themselves.

Lie groups such as the Poincaré group (inhomogeneous Lorentz group) and the Galilei group have the matrix structures

Poincaré Group

$$\left[\begin{array}{cccc|c} & & & & t_1 \\ & O(3,1) & & & t_2 \\ & & & & t_3 \\ & & & & t_4 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \\ ct \\ 1 \end{array}\right]$$

Galilean Group

$$\left[\begin{array}{ccc|cc} & & & v_1 & t_1 \\ & O(3) & & v_2 & t_2 \\ & & & v_3 & t_3 \\ \hline 0 & 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} x \\ y \\ z \\ t \\ 1 \end{array}\right]$$

In these transformations $\mathbf{t} = (t_1, t_2, t_3)$ describes translations in the space (x-, y-, and z-) directions, $\mathbf{v} = (v_1, v_2, v_3)$ describes boosts, and t_4 resets clocks. The matrices in these defining matrix representations are reducible.

The Heisenberg group H_4 is a four dimensional Lie group with a simple 3×3 matrix structure

$$\text{Heisenberg Group} = H_4 = \left[\begin{array}{ccc} 1 & l & d \\ 0 & n & r \\ 0 & 0 & 1 \end{array}\right] \qquad n \neq 0.$$

This matrix representation of H_4 is faithful but nonunitary.

4 "Linearization" of a Lie Group

At the topological level a Lie group is homogeneous. That is, every point in a manifold that parameterizes a Lie group looks like every other point. At the algebraic level this is not true — the identity group operation e is singled out as an exceptional group element. At the analytic level the group composition law $z = \phi(x,y)$ is nonlinear, and can therefore be arbitrarily complicated.

The study of Lie groups is enormously simplified by exploiting these three observations. Specifically, it is useful to *linearize* the group multiplication law in the neighborhood of the identity. The linearization leads to a local Lie group. This is a linear vector space on which there is an additional structure. Once the local Lie group properties are known in the neighborhood of the identity, they are known everywhere else in the group, since the group is homogeneous.

A Lie group is linearized in the neighborhood of the identity by expressing an operator near the identity in the form $g(\epsilon) = I + \epsilon X$, where the local Lie group operator $\epsilon X = \delta x^i X_i$, the X_i are n linearly independent vector fields on the manifold M^n, and the small coordinates δx^i measure the distance (in some rough sense) of $g(\epsilon)$ from the point that parameterizes the identity group operation $e = g(0)$. For another group operation $g(\delta Y) = I + \delta Y$ in the neighborhood of the identity

(i) The product $g(\epsilon X)g(\delta Y) = (I + \epsilon X)(I + \delta Y) = I + (\epsilon X + \delta Y) +$(h.o.t) is in the local Lie group.

(ii) The commutator $g_i \circ g_j \circ g_i^{-1} \circ g_j^{-1}$ in the group leads to

$$g(\epsilon X)g(\delta Y)g(\epsilon X)^{-1}g(\delta Y)^{-1} = I + \frac{1}{2}\epsilon\delta\,(XY - YX) + \text{h.o.t} = I + \frac{1}{2}\epsilon\delta\,[X, Y] + \text{h.o.t}$$

in the local Lie group.

The first condition shows that the local Lie group is a linear vector space. The n vector fields X_i can be chosen as a set of basis vectors in this space.

The second condition shows that the commutator of two vectors in this linear vector space is also in this linear vector space. The commutator endows this linear vector space with an additional combinatorial operation ("vector multiplication") and provides it with the structure of an algebra, called a Lie algebra.

Definition: A Lie algebra $\mathfrak{la}$ consists of a set of operators $X,\ Y,\ Z, \ldots$, together with the operations of vector addition, scalar multiplication, and commutation $[X, Y]$ that satisfy the following three axioms:

1. **Closure (linear vector space):** If $X,\ Y \in \mathfrak{la}$, $\alpha X + \beta Y \in \mathfrak{la}$ and $[X, Y] \in \mathfrak{la}$.
2. **Antisymmetry:** $[X, Y] = -[Y, X]$.
3. **Jacobi Identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The structure of a Lie algebra, or local Lie group, is summarized by the structure constants, defined in terms of the basis vectors X_i, by

$$[X_i, X_j] = c_{ij}{}^k X_k \qquad \text{summation convention}$$

The structure constants $c_{ij}{}^k$ are components of a third order tensor, covariant and antisymmetric in two indices ($c_{ij}{}^k = -c_{ji}{}^k$) and contravariant in the third. These components obey the Jacobi identity, which places a quadratic constraint on them:

$$c_{ij}{}^s c_{sk}{}^t + c_{jk}{}^s c_{si}{}^t + c_{ki}{}^s c_{sj}{}^t = 0$$

Linearization of a Lie group generates a Lie algebra. A Lie group can be recovered by the inverse process. This is the exponential operation. A group operation a finite distance from the origin (the point identified with the identity group operation) of the manifold that parameterizes the Lie group can be obtained from the limiting procedure ($\epsilon = 1/K \to 0$):

$$g(X) = \lim_{K\to\infty} \prod \left(I + \frac{1}{K}X\right)^K = e^X = EXP(X)$$

The exponential operation is well defined for real numbers, complex numbers, quaternions, $n \times n$ matrices over these fields, and vector fields.

A 1:1 correspondence between Lie groups and Lie algebras does not exist. Isomorphic Lie groups have isomorphic Lie algebras. But nonisomorphic Lie groups may also possess isomorphic Lie algebras. The best known examples of nonisomorphic Lie groups and their isomorphic Lie algebras are

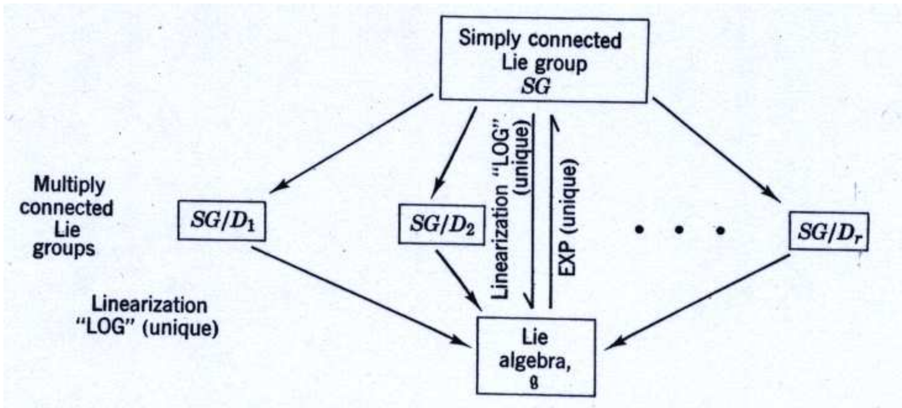

Figure 1: Cartan's theorem states that there is a 1:1 correspondence between Lie algebras and simply connected Lie groups. All other Lie groups with this Lie algebra are quotients of the covering group by one of its discrete invariant subgroups $D_j \subseteq D_{\text{Max}}$. There is a relation between the discrete invariant subgroup D_j and the homotopy group of SG/D_j. Reprinted with permission from R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, NY: Wiley, 1974.

$$\begin{array}{rclcrcl} SO(3) & \neq & SU(2) & \qquad & \mathfrak{so}(3) & = & \mathfrak{su}(2) \\ SO(4) & \neq & SU(2)\times SU(2) & & \mathfrak{so}(4) & = & \mathfrak{su}(2)+\mathfrak{su}(2) \\ SO(5) & \neq & Sp(2)=USp(4) & & \mathfrak{so}(5) & = & \mathfrak{sp}(2)=\mathfrak{usp}(4) \end{array}$$

There is a 1:1 correspondence between Lie algebras and *locally* isomorphic Lie groups. This has been extended to global Lie groups by a beautiful theorem due to E. Cartan.

Theorem (Cartan): There is a 1:1 correspondence between Lie algebras and simply connected Lie groups. Every Lie group with the same Lie algebra is either the simply connected ("universal covering") group or is the quotient of this universal covering group by one of its discrete invariant subgroups.

This relation is summarized in Fig. 1.

As a concrete example, the Lie algebra of $SO(3)$, which is the group of real 3×3 matrices satisfying $M^\dagger I_3 M = I_3$ and $\det(M) = +1$, is spanned by the three "angular momentum vector fields" $L_i(x) = \epsilon_{ijk}x^j\partial_k$ or the three angular momentum matrices

$$L_1 = L_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{bmatrix} \quad L_2 = L_{31} = -L_{13} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{bmatrix} \quad L_3 = L_{12} = \begin{bmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The Lie group $SU(2)$ is the group of complex 2×2 matrices satisfying $M^\dagger I_2 M = I_2$ and $\det(M) = +1$. Its Lie algebra is spanned by the three spin matrices $S_j = \frac{i}{2}\sigma_j$, which are multiples of the Pauli spin matrices σ_j

$$S_1 = \frac{i}{2}\begin{bmatrix} 0 & +1 \\ +1 & 0 \end{bmatrix} \qquad S_2 = \frac{i}{2}\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix} \qquad S_3 = \frac{i}{2}\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$$

The two Lie algebras are isomorphic as they share isomorphic commutation relations $[J_1, J_2] = -J_3$ (and cyclic), $J_j = L_j$ or $J_j = S_j$. The group $SU(2)$ is simply connected. Its maximal discrete invariant subgroup D consists of all multiples of the identity, αI_2, so that $\alpha = \pm 1$. According to Cartan's theorem $SO(3) = SU(2)/D_2$, $D_2 = \{I_2, -I_2\}$. The group $SO(3)$ is doubly connected, with a two element homotopy group.

5 Matrix Lie Algebras

A deep theorem of Ado guarantees that every Lie algebra is equivalent to a matrix Lie algebra, even though the same is not true of Lie groups.

Sets of $n \times n$ matrices that close under vector addition, scalar multiplication, and commutation ($M_1 \in \mathfrak{la}, M_2 \in \mathfrak{la} \Rightarrow [M_1, M_2] = M_1 M_2 - M_2 M_1 \in \mathfrak{la}$) form matrix Lie algebras. The antisymmetry properties and Jacobi identity are guaranteed by matrix multiplication.

Lie algebras for the general linear groups $GL(n; F)$ consist of $n \times n$ matrices over F. Lie algebras for the special linear groups $SL(n; F)$ consist of traceless $n \times n$ matrices. The Lie algebras of the unitary groups consist of antihermitian matrices. The Lie algebras of $U(p, q; F)$ consist of matrices that obey

$$M^\dagger I_{p,q} + I_{p,q} M = 0 \qquad M \in \mathfrak{u}(p, q; F)$$

The matrix Lie algebras of other matrix Lie groups are obtained by constructing the most general Lie group operation in the neighborhood of the identity by linearization. For example, the Lie algebra of the Heisenberg group H_4 is

$$\begin{bmatrix} 1 & l & d \\ 0 & n & r \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \delta l & \delta d \\ 0 & 1+\delta n & \delta r \\ 0 & 0 & 1 \end{bmatrix} \rightarrow I_3 + \delta n\ N + \delta r\ R + \delta l\ L + \delta d\ D$$

$$N \simeq a^\dagger a \qquad R \simeq a^\dagger \qquad L \simeq a \qquad D \simeq I = \left[a, a^\dagger\right]$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The four 3×3 matrices N, R, L, D that span the Lie algebra $\mathfrak{h}_4$ of H_4 satisfy commutation relations isomorphic with the commutation relations satisfied by the photon operators $(a^\dagger a, a^\dagger, a, I = \left[a, a^\dagger\right])$. The 3×3 matrix representations of the group H_4 and the algebra $\mathfrak{h}_4$ are faithful. The representation of H_4 is nonunitary and that of $\mathfrak{h}_4$ is nonhermitian.

There is a simple way to relate a large class of operator Lie algebras to matrix Lie algebras. If $A, B, C, \ldots$ belong to a Lie algebra of $n \times n$ matrices, the matrix-to-operator mapping

$$A \rightarrow \mathcal{A} = x^i A_i{}^j \partial_j$$

preserves commutation relations, for

$$[\mathcal{A}, \mathcal{B}] = \left[x^i A_i{}^j \partial_j, x^r B_r{}^s \partial_s\right] = x^i A_i{}^j \left[\partial_j, x^r\right] B_r{}^s \partial_s - x^r B_r{}^s \left[\partial_s, x^i\right] A_i{}^j \partial_j =$$

$$x^i A_i{}^j B_j{}^s \partial_s - x^r B_r{}^i A_i{}^j \partial_j = x^i \left[A, B\right]_i{}^j \partial_j = \mathcal{C}$$

This relation depends on the bilinear products $x^i \partial_j$ satisfying commutation relations

$$[x^i\partial_j, x^r\partial_s] = x^i\partial_s\delta_j{}^r - x^r\partial_j\delta_s{}^i$$

These commutation relations are satisfied by products of creation and annihilation operators $a_i^\dagger a_j$ for either bosons ($b_i^\dagger b_j$) or fermions ($f_i^\dagger f_j$). These matrix-to-operator mappings can be extended to include bilinear products such as $x^i x^j, x^i\partial_j, \partial_i\partial_j$ and their boson and fermion counterparts $a_i a_j, a_i^\dagger a_j, a_i^\dagger a_j^\dagger$. For example, the vector fields associated with the operator J_1 for $SO(3)$ and $SU(2)$ are $x^i(L_1)_i{}^j\partial_j = x^2\partial_3 - x^3\partial_2$ and $u^i(S_1)_i{}^j\partial_j = \frac{i}{2}(u^1\partial_2 + u^2\partial_1)$.

Boson and fermion bilinear products $a_i^\dagger a_j$ $(1 \le i, j \le n)$ are isomorphic to $\mathfrak{u}(n)$. Boson bilinear products $b_i b_j, b_i^\dagger b_j, b_i^\dagger b_j^\dagger$ are isomorphic to $\mathfrak{usp}(2n)$ while fermion bilinear products $f_i f_j, f_i^\dagger f_j, f_i^\dagger f_j^\dagger$ are isomorphic to $\mathfrak{so}(2n)$.

6 Structure of Lie Algebras

The study of Lie algebras is greatly facilitated by studying their structure. The structure is determined by the commutation properties of the Lie algebra.

Invariant subalgebra: If a Lie algebra has an invariant subalgebra, then the commutator of anything in the algebra with anything in the subalgebra is in the subalgebra. Suppose $\mathfrak{a}$ is a linear vector subspace of $\mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$ then $\mathfrak{a}$ is an invariant subspace of $\mathfrak{g}$. In particular $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$ and $\mathfrak{a}$ is therefore also subalgebra of $\mathfrak{g}$: it is an invariant subalgebra in $\mathfrak{g}$.

Example: The Lie algebra $\mathfrak{iso}(3)$ consists of the three rotation operators $L_{ij} = x^i\partial_j - x^j\partial_i$ and the three displacement operators $P_k = \partial_k$. The subset of displacement operators is an invariant subspace in $\mathfrak{iso}(3)$, since it is mapped into itself by all commutators. It is also a subalgebra in $\mathfrak{iso}(3)$. This particular invariant subalgebra is commutative.

Solvable algebra: If $\mathfrak{g}$ is a Lie algebra, the linear vector space obtained by taking all possible commutators of the operators in $\mathfrak{g}$ is called the *derived* algebra: $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)} \subseteq \mathfrak{g}$. If $\mathfrak{g}^{(1)} = \mathfrak{g}$ there is no point in continuing this process. If $\mathfrak{g}^{(1)} \subset \mathfrak{g}$, it is useful to define $\mathfrak{g} = \mathfrak{g}^{(0)}$ and to continue this process by defining $\mathfrak{g}^{(2)}$ as the derived algebra of $\mathfrak{g}^{(1)}$: $\mathfrak{g}^{(2)} = \left[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right]$. We can continue in this way, defining $\mathfrak{g}^{(n+1)}$ as the algebra derived from $\mathfrak{g}^{(n)}$. Ultimately (for finite dimensional Lie algebras) either $\mathfrak{g}^{(n+1)} = 0$ or $\mathfrak{g}^{(n+1)} = \mathfrak{g}^{(n)}$ for some n. If the former case occurs:

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \cdots \mathfrak{g}^{(n)} \supset \mathfrak{g}^{(n+1)} = 0$$

the Lie algebra $\mathfrak{g}^{(0)}$ is called solvable. Each algebra $\mathfrak{g}^{(i)}$ is an invariant subalgebra of $\mathfrak{g}^{(j)}$, $i > j$.

Example: The Lie algebra spanned by the boson number, creation, annihilation, and identity operators is solvable. The series of derived algebras has dimensions 4, 3, 1, 0.

$\mathfrak{g}^{(0)}$	$\mathfrak{g}^{(1)}$	$\mathfrak{g}^{(2)}$	$\mathfrak{g}^{(3)}$
$a^\dagger a$	–	–	–
$a^\dagger$	$a^\dagger$	–	–
a	a	–	–
I	I	I	–

Semidirect sum algebra: When a Lie algebra $\mathfrak{g}$ has an invariant subalgebra $\mathfrak{a}$, the linear vector space of the Lie algebra $\mathfrak{g}$ can be written as the direct sum of the linear vector subspace of the subalgebra $\mathfrak{a}$ plus a complementary subspace $\mathbf{b}$. The subspace $\mathbf{b}$ is generally not itself a Lie algebra. The Lie algebra $\mathfrak{g}$ is written as a semidirect sum of the two subspaces. The semidirect sum structure satisfies the commutation relations shown:

$$\mathfrak{g} = \mathbf{b} \wedge \mathfrak{a} \qquad \begin{array}{rcr} [\mathbf{b}, \mathbf{b}] & \subseteq & \mathbf{b} \wedge \mathfrak{a} \\ [\mathbf{b}, \mathfrak{a}] & \subseteq & \mathfrak{a} \\ [\mathfrak{a}, \mathfrak{a}] & \subseteq & \mathfrak{a} \end{array}$$

The subspace $\mathbf{b}$ can be given the structure of an algebra by modding out the component of the commutator in $\mathfrak{a}$: $\mathfrak{b} = \mathfrak{g} \bmod \mathfrak{a}$.

Example: The three dimensional Lie algebra spanned by the photon operators $a^\dagger, a, I$ has a semidirect sum decomposition where $\mathbf{b}$ is spanned by $a^\dagger, a$ and $\mathfrak{a}$ is spanned by I. The subspace $\mathbf{b}$ is not closed under commutation, and $\mathfrak{a}$ is commutative. The Lie algebra $\mathfrak{iso}(3)$ also has the structure of a semidirect sum, with $\mathbf{b} = \mathfrak{b} = \mathfrak{so}(3)$ and the invariant subalgebra $\mathfrak{a}$ is spanned by the three displacement operators P_k.

Nonsemisimple algebra: A Lie algebra is nonsemisimple if it has a solvable invariant subalgebra.

Example: The Lie algebra spanned by bilinear products of photon creation and annihilation operators $a_i^\dagger a_j$, creation operators $a_i^\dagger$, annihilation operators a_j, and the identity operator I $(1 \leq i, j \leq n)$ is nonsemisimple. The solvable invariant subalgebra is spanned by the $2n + 2$ operators consisting of the single photon operators $a_i^\dagger$, a_j, the identity operator I, and the total number operator $\hat{n} = \sum_{i=1}^n a_i^\dagger a_i$.

Semisimple algebra: A Lie algebra is semisimple if it has no solvable invariant subalgebras.

Example: The Lie algebra $\mathfrak{so}(4)$ is semisimple. This Lie algebra has two invariant subalgebras, both isomorphic to $\mathfrak{so}(3)$. The direct sum decomposition

$$\mathfrak{so}(4) = \mathfrak{so}(3) + \mathfrak{so}(3)$$

is well known to physical chemists and is responsible for the dualities that exist between rotating and laboratory frame descriptions of molecular systems.

Simple algebra: A Lie algebra is simple if it has no invariant subalgebras at all. The prettiest page in the theory of Lie groups is the classification theory of the simple Lie algebras. We turn to this subject now.

7 Lie Algebra Tools

Two powerful tools have been developed for studying the structure of a Lie algebra. These are the regular representation and the Cartan-Killing form.

7.1 Regular Representation

This representation assigns the structure constants to a set of n $n \times n$ matrices according to

$$X_\alpha \to R(X_\alpha)_\mu{}^\nu = c_{\alpha\mu}{}^\nu, \qquad [X_\alpha, X_\mu] = c_{\alpha\mu}{}^\nu X_\nu$$

The matrices of the regular representation contain exactly as much information as the components of the structure tensor. They can be studied by standard linear algebra methods. For example, a secular equation can be used to put the commutation relations into canonical form.

The structure of the matrices of the regular representation determines the structure of the Lie algebra. The identification is carried out according to the usual rules of representation theory, as

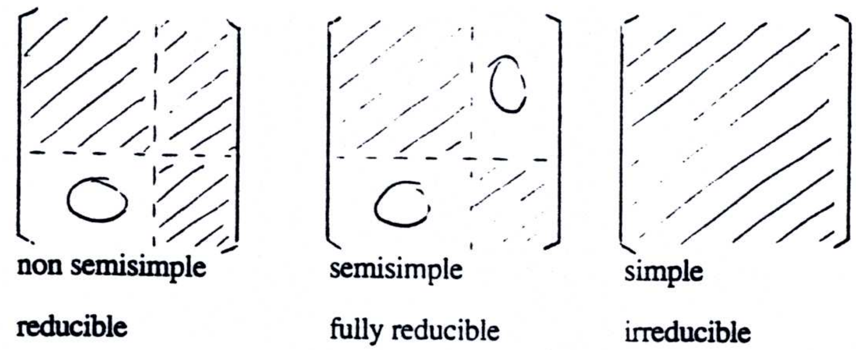

Figure 2: When the regular matrix representation of a Lie algebra is reducible, fully reducible, or irreducible the Lie algebra is nonsemisimple, semisimple, or simple.

shown in Fig. 2. If a basis X_α can be found in which all the matrices of the regular representation are simultaneously reducible, the algebra possesses an invariant subalgebra. If the representation is not fully reducible, the invariant subalgebra is solvable. If the regular representation is fully reducible, the algebra consists of the direct sum of two (or more) smaller, mutually commuting subalgebras. If the regular representation is irreducible, the algebra is simple.

If a Lie algebra is solvable ($\mathfrak{solv}$) all matrices in the regular representation can be transformed to upper triangular matrices. If the Lie algebra is nilpotent ($\mathfrak{nil} \subset \mathfrak{solv}$) the diagonal matrix elements in the upper triangular matrices are zero. The converses are also true.

7.2 Cartan-Killing Form

The Cartan-Killing form is a second order symmetric tensor that is constructed from the third order antisymmetric tensor $c_{\alpha\mu}^{\ \ \nu}$ by cross contraction

$$g_{\alpha\beta} = c_{\alpha\mu}^{\ \ \nu} c_{\beta\nu}^{\ \ \mu} = g_{\beta\alpha} = \mathrm{tr}\ R(X_\alpha)R(X_\beta) = (X_\alpha, X_\beta) = (X_\beta, X_\alpha)$$

The metric $g_{\alpha\beta}$ can be used to place an inner product (X_α, X_β) on this linear vector space. This inner product is not necessarily positive definite.

The matrix $g_{\alpha\beta}$ can also be treated by standard linear algebra methods. Since it is real and symmetric, it can be diagonalized. If there are n_- negative eigenvalues, n_+ positive eigenvalues, and n_0 vanishing eigenvalues ($n = n_- + n_+ + n_0$), the Lie algebra has a corresponding linear vector space decomposition of the form

$$\mathfrak{g} = \mathfrak{g}_- + \mathfrak{g}_+ + \mathfrak{g}_0$$

The inner product is positive definite on the subspace $\mathfrak{g}_+$ and negative definite on $\mathfrak{g}_-$. We call $\mathfrak{g}_0$ the singular subspace. The subspace $\mathfrak{g}_0$ is closed under commutation and in fact is a nilpotent invariant subalgebra of $\mathfrak{g}$.

8 Decomposition of Lie Algebras

The most general Lie algebra $\mathfrak{g}$ is the semidirect sum of a semisimple Lie algebra $\mathfrak{ss}$ and a solvable invariant subalgebra $\mathfrak{solv}$.

$$\mathfrak{g} = \mathfrak{ss} \wedge \mathfrak{solv} \qquad \begin{array}{rcl} [\mathfrak{ss}, \mathfrak{ss}] & = & \mathfrak{ss} \\ [\mathfrak{ss}, \mathfrak{solv}] & \subseteq & \mathfrak{solv} \\ [\mathfrak{solv}, \mathfrak{solv}] & \subset & \mathfrak{solv} \end{array}$$

The decomposition of $\mathfrak{g}$ into its component parts is accomplished by a simple two-step algorithm.

1. Compute the Cartan-Killing metric for $\mathfrak{g}$ and determine the singular subspace. If there is none, stop. If the dimension of $\mathfrak{g}_0$ is greater than 0, $\mathfrak{nil} = \mathfrak{g}_0$ is the maximal nilpotent invariant subalgebra of $\mathfrak{g}$.

2. Compute the structure constants of the Lie algebra $\mathfrak{g}' = \mathfrak{g} - \mathfrak{nil} = \mathfrak{g}$ mod $\mathfrak{nil} = \mathfrak{g}/\mathfrak{nil}$, the Cartan-Killing metric tensor on $\mathfrak{g}'$, and the decomposition $\mathfrak{g}' = \mathfrak{g}_-^{'} + \mathfrak{g}_+^{'} + \mathfrak{g}_0^{'}$. Then $\mathfrak{a} = \mathfrak{g}_0^{'}$ is abelian and invariant in $\mathfrak{g}'$. In fact, $\mathfrak{a}$ is the largest abelian invariant subalgebra in $\mathfrak{g}'$.

The algorithm stops here, for the algebra $\mathfrak{g}^{''} = \mathfrak{g}^{'}$ mod $\mathfrak{a} = \mathfrak{g}'/\mathfrak{a} = \mathfrak{g}_-^{'} + \mathfrak{g}_+^{'}$ has no singular subspace under its Cartan-Killing metric.

Under this algorithm the decomposition of $\mathfrak{g}$ into its semisimple part and its maximal solvable invariant subalgebra is

$$\mathfrak{g} = \left(\mathfrak{g}_-^{'} + \mathfrak{g}_+^{'}\right) \wedge \left(\mathfrak{g}_0^{'} \wedge \mathfrak{g}_0\right)$$

The maximum solvable invariant subalgebra $\mathfrak{solv}$ in $\mathfrak{g}$ is the semidirect sum of $\mathfrak{a}$ and $\mathfrak{nil}$: $\mathfrak{solv} = \mathfrak{g}_0^{'} \wedge \mathfrak{g}_0 = \mathfrak{a} \wedge \mathfrak{nil}$. In addition, $\mathfrak{ss} = \mathfrak{g}$ mod $\mathfrak{solv} = \mathfrak{g}/\mathfrak{solv} = \mathfrak{g}_-^{'} + \mathfrak{g}_+^{'}$. The subspace $\mathfrak{g}_-^{'}$ is closed under commutation and exponentiates into a compact subgroup of $G^{'}$. The subspace $\mathfrak{g}_+^{'}$ exponentiates to a noncompact coset in $G^{'}$ that is simply connected.

Every element in a semisimple Lie algebra can be expressed as the commutator of two elements in the Lie algebra. In this sense a semisimple algebra reproduces itself under commutation.

To illustrate this algorithm we tear apart the eight dimensional Lie algebra spanned by the photon operators $a_i^\dagger a_j$, $1 \le i,j \le 2$ and $a_3^\dagger a_3, a_3^\dagger, a_3, I$, where the photon operators obey $\left[a_i, a_j^\dagger\right] = \delta_{ij} I$. The regular representative of the general linear combination

$$X = \sum_{ij} m_{ij} a_i^\dagger a_j + n a_3^\dagger a_3 + r a_3^\dagger + l a_3 + \delta I \qquad \text{is}$$

$$R(X) = \left[\begin{array}{cccc|cccc} 0 & & -m_{12} & m_{21} & & & & \\ & 0 & m_{12} & -m_{21} & & & & \\ -m_{21} & m_{21} & +m_{11}-m_{22} & 0 & & & & \\ m_{12} & -m_{12} & 0 & -m_{11}+m_{22} & & & & \\ \hline & & & & 0 & & & \\ & & & & & n & & l \\ & & & & & & -n & -r \\ & & & & & & & 0 \end{array}\right] \begin{array}{c} a_1^\dagger a_1 \\ a_2^\dagger a_2 \\ a_1^\dagger a_2 \\ a_2^\dagger a_1 \\ a_3^\dagger a_3 \\ a_3^\dagger \\ a_3 \\ I \end{array}$$

The Cartan-Killing inner product is the trace of the square of this matrix:

$$(X, X) = \text{tr } R(X)^2 = 2(m_{11} - m_{22})^2 + 8 m_{12} m_{21} + 2n^2$$

The subspace $\mathfrak{g}_0$ is spanned by $a_1^\dagger a_1 + a_2^\dagger a_2, a_3^\dagger, a_3, I$, leaving the four operators $a_1^\dagger a_1 - a_2^\dagger a_2, a_1^\dagger a_2, a_2^\dagger a_1, a_3^\dagger a_3$ to span $\mathfrak{g}^{'}$. A simple calculation shows that $\mathfrak{g}_0^{'}$ is spanned by $a_3^\dagger a_3$. As a result

Subspace	Spanned by
$\mathfrak{g}_{+}^{'}$	$a_1^\dagger a_1 - a_2^\dagger a_2, \frac{1}{\sqrt{2}}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right)$
$\mathfrak{g}_{-}^{'}$	$\frac{1}{\sqrt{2}}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right)$
$\mathfrak{g}_{0}^{'}$	$a_3^\dagger a_3$
$\mathfrak{g}_0$	$a_1^\dagger a_1 + a_2^\dagger a_2, a_3^\dagger, a_3, I$

The Lie algebra is the direct sum $\mathfrak{g} = \mathfrak{u}(2) + \mathfrak{h}_4 = \mathfrak{su}(2) + \mathfrak{u}(1) + \mathfrak{h}_4$.

9 Structure of Semisimple Lie Algebras

The Cartan-Killing metric $g_{\alpha\beta}$ is nonsingular on a semisimple Lie algebra. It, and its inverse $g^{\alpha\beta}$, can be used to raise and lower indices. In particular, the tensor whose components are $c_{\alpha\beta\gamma} = c_{\alpha\beta}^{\ \ \mu} g_{\mu\gamma}$ is third order antisymmetric: $c_{\alpha\beta\gamma} = c_{\beta\gamma\alpha} = c_{\gamma\alpha\beta} = -c_{\beta\alpha\gamma} \cdots$. Classification of semisimple Lie algebras is equivalent to classifying such tensors.

Another useful way to describe semisimple Lie algebras is to search for a canonical structure for the commutation relations. A useful canonical form is an eigenvalue form

$$[X, Y] = \lambda Y$$

In a basis X_i, with $X = x^i X_i$ and $Y = y^j X_j$ this equation reduces to a standard eigenvalue equation for the regular representation

$$\sum_j \sum_k y^j \left(R(x^i X_i)_j^{\ k} - \lambda \delta_j^{\ k}\right) X_k = 0$$

Thus the search for a standard form for the commutation relations reduces to a study of the secular equation

$$\det \ (R(X) - \lambda I) = \sum_{j=0}^{n} (-\lambda)^{n-j} \phi_j(X) = 0 \tag{1}$$

The coefficients $\phi_j(X)$ are homogeneous polynomials of degree j in the coefficients x^i of $X = x^i X_i$.

In order to extract maximum information from this secular equation a generic vector $X \in \mathfrak{g}$ is chosen. Such a choice minimizes all degeneracies. With a generic choice of $X \in \mathfrak{g}$ it is useful to define the rank, l, of the Lie algebra $\mathfrak{g}$ as:

1. The number of functionally independent coefficients $\phi_j(X)$ in the secular equation.

2. The number of independent roots, $\alpha_1, \alpha_2, \ldots, \alpha_l$ of the secular equation.

3. The dimension of the subspace $H \subset \mathfrak{g}$ that commutes with X.

4. The number of independent (Casimir) operators that commute with all X_i: $\mathcal{C}_j(X) = \phi_j(x^i \rightarrow X_i)$: $[\mathcal{C}_j(X), X_i] = 0$.

For example, for $\mathfrak{so}(3)$ or $\mathfrak{su}(2)$ the secular equation for $X = x^i X_i$ is

$$\det \left[\begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} - \lambda I_3 \right] = (-\lambda)^3 + (-\lambda)\phi_2(x) = 0$$

where $\phi_2(x) = x_1^2 + x_2^2 + x_3^2$. The rank is $l = 1$. There is one independent coefficient $\phi_2(x)$ and one independent root of this equation, $\alpha_1 = \sqrt{-\delta_{ij} x^i x^j} = i\sqrt{x \cdot x}$. The only linear operators that commute with X are scalar multiples of X. There is one independent homogeneous operator that commutes with all generators X_i, obtained by the substitutions $x^i \to L_i$ (for $\mathfrak{so}(3)$) or $x^i \to S_i$ (for $\mathfrak{su}(2)$)

$$\mathcal{C}^2(\mathbf{L}) = \phi_2(x_i \to L_i) = L_1^2 + L_2^2 + L_3^2$$

The secular equation (1) is over the field of real numbers. This is not an algebraically closed field. There is no guarantee that the number of independent functions $\phi_j(x)$ in the secular equation is equal to the number of (real) roots of this equation until we extend the field from R to C, which is algebraically closed. As a result, the classification of semisimple Lie algebras is done over complex numbers. After the complex extensions of the simple Lie algebras have been classified, their different inequivalent real forms can be determined.

10 Root Spaces

When the secular equation for the regular representation of a generic element in a Lie algebra is solved, the commutation relations can be put into a simple and elegant canonical form. This canonical form depends on the rank, l, of the Lie algebra, not the dimension, n, of the Lie algebra. This provides a very useful simplification, as $n \sim l^2$.

For this canonical form, the independent roots $\alpha_1(x), \alpha_2(x), \ldots, \alpha_l(x)$ are gathered into a single vector $\boldsymbol{\alpha}$ with l components. The vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ are called root vectors. The root vectors exist in an l-dimensional space on which a positive-definite inner product can be defined. The root vectors for a rank-l semisimple Lie algebra $\mathfrak{g}$ span this Euclidean space. The basis vectors of $\mathfrak{g}$ can be identified with the roots in the root space.

The roots in a root space have the following properties:

1. A positive definite metric can be placed on the root space.

2. The vector $\mathbf{0}$ is a root.

3. The root $\mathbf{0}$ is l-fold degenerate.

4. If $\boldsymbol{\alpha}$ is a root and $c\boldsymbol{\alpha}$ is a root, $c = \pm 1, 0$.

5. If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are roots

$$\boldsymbol{\beta}' = \boldsymbol{\beta} - \frac{2\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}} \boldsymbol{\alpha}$$

   is also a root and $2\boldsymbol{\alpha} \cdot \boldsymbol{\beta} / \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$ is an integer, n_1. In fact, $\boldsymbol{\beta}'$ is the root obtained by reflecting $\boldsymbol{\beta}$ in the hyperplane orthogonal to $\boldsymbol{\alpha}$.

6. The set of reflections generated by nonzero roots itself forms a group, the Weyl group of the Lie algebra.

7. The angle between roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is determined by

$$\cos^2(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}} = \frac{n_1}{2}\frac{n_2}{2} = 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1$$

The integers n_1, n_2 for noncolinear roots are constrained by $|n_1 n_2| < 4$.

8. The relative lengths of the roots are determined by the angles between them:

$\cos^2(\theta(\boldsymbol{\alpha},\boldsymbol{\beta}))$	$\theta(\boldsymbol{\alpha},\boldsymbol{\beta})$	$\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}/\boldsymbol{\beta}\cdot\boldsymbol{\beta}$
3/4	$30^o, 150^o$	$3^{\pm 1}$
2/4	$45^o, 135^o$	$2^{\pm 1}$
1/4	$60^o, 120^o$	1

9. When the roots are normalized so that

$$\sum_{\boldsymbol{\alpha}\neq\mathbf{0}} \alpha_i\alpha_j = \delta_{ij} \qquad \text{or} \qquad \sum_{\boldsymbol{\alpha}\neq\mathbf{0}} \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = l$$

the commutation relations can be placed in the canonical form presented in the next section.

It is possible to build up all possible root space diagrams using an "Aufbau" construction. We start with a rank-one root space. This consists of three roots in R^1: $\boldsymbol{\alpha}$, $\mathbf{0}$, $-\boldsymbol{\alpha}$.

To construct rank two root spaces, a new noncolinear root $\boldsymbol{\beta}$ is adjoined to the two nonzero roots. The new root and the old roots span R^2. The new root can only have a limited set of angles with the roots already present. The set of roots $\boldsymbol{\alpha},\boldsymbol{\beta}$ is completed by reflection in hyperplanes orthogonal to all roots present. If any pair of roots violates the angle conditions, the result is not a root space. In this way the rank two root spaces G_2 (30^o), $B_2 = C_2$ (45^o), A_2 (60^o), and $D_2 = A_1 + A_1$ (90^o) are constructed from A_1. Proceeding in this way it is possible to construct rank three root spaces ($B_3, C_3, A_3 = D_3$) from the rank two root spaces, the rank four root spaces from the rank three root spaces, and so forth. Ultimately, there are four unending chains A_n, B_n, C_n, D_n and five exceptional root spaces G_2, F_4, E_6, E_7, E_8. The rank-two root spaces are shown in Fig. 3 and the rank-three root spaces are shown in Fig. 4. The normalization factors (cf., point 9 above) are shown for the rank-two root spaces in Fig. 3.

11 Canonical Commutation Relations

The canonical commutation relations are expressed in terms of root vectors. The l operators in $\mathfrak{g}$ with the l-fold degenerate root vector $\mathbf{0}$ are $H_1, H_2, \ldots, H_l$. These l operators mutually commute. In a matrix Lie algebra they can be taken as simultaneously commuting diagonal matrices. Associated with each nonzero root $\boldsymbol{\alpha}\neq\mathbf{0}$ there is exactly one basis vector, $E_{\boldsymbol{\alpha}}$, in $\mathfrak{g}$. The canonical commutation relations are expressed in terms of the roots as follows

$$\begin{array}{lcll} [H_i, H_j] & = & 0 & \qquad 1 \leq i,j \leq l \\ [H_i, E_{\boldsymbol{\alpha}}] & = & \alpha_i E_{\boldsymbol{\alpha}} & \\ \left[E_{\boldsymbol{\alpha}}, E_{-\boldsymbol{\alpha}}\right] & = & \boldsymbol{\alpha}\cdot\mathbf{H} & \\ \left[E_{\boldsymbol{\alpha}}, E_{\boldsymbol{\beta}}\right] & = & N_{\boldsymbol{\alpha}\boldsymbol{\beta}} E_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ a root} \\ & & 0 & \boldsymbol{\alpha}+\boldsymbol{\beta} \text{ not a root} \end{array}$$

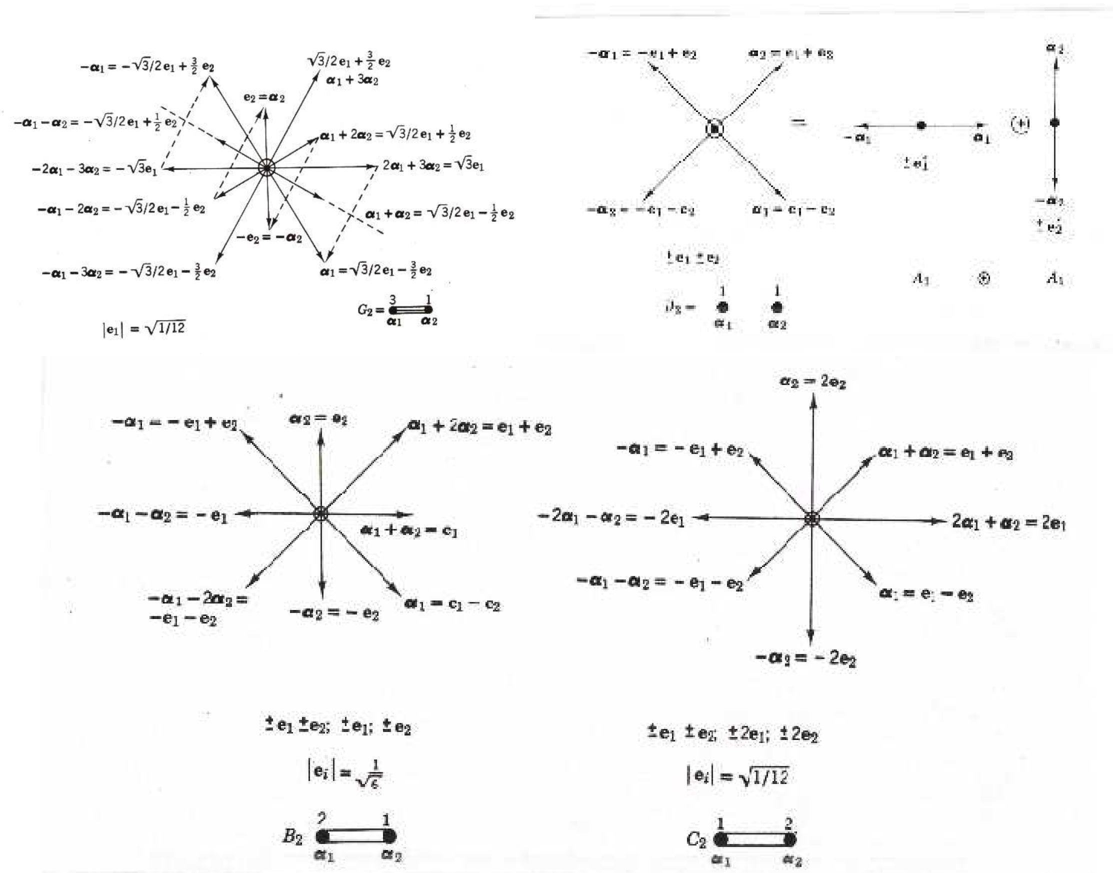

Figure 3: Rank-two root spaces. G_2 30°, $B_2 = C_2$ 45°, A_2 60°, $D_2 = A_1 + A_1$ 90°.

The structure constants $N_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ are determined from a recursion relation derived from a chain of roots $\boldsymbol{\beta}-m\ \boldsymbol{\alpha}$, $\boldsymbol{\beta}-(m-1)\ \boldsymbol{\alpha}$, ..., $\boldsymbol{\beta}+(n-1)\ \boldsymbol{\alpha}$, $\boldsymbol{\beta}+n\ \boldsymbol{\alpha}$, where $\boldsymbol{\beta}-(m+1)\ \boldsymbol{\alpha}$ and $\boldsymbol{\beta}+(n+1)\ \boldsymbol{\alpha}$ are not roots (cf. Fig. 5). The structure constants are

$$N^2_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \frac{1}{2} n(1+m)\,(\boldsymbol{\alpha}\cdot\boldsymbol{\alpha})$$

The operators $\mathbf{H}$ and $E_{\boldsymbol{\alpha}}$ are often called diagonal and shift operators, respectively. They are generalizations of the shift operators J_3 and $J_\pm$ of angular momentum theory. The general idea is as follows. Since the operators H_i mutually commute, the matrices $\Gamma(H_i)$ representing these operators can be chosen as diagonal in any matrix representation. The action of any of these operators on a basis vector in this representation is $H_i|\mathbf{m}\rangle = m_i|\mathbf{m}\rangle$. The operator $E_{\boldsymbol{\alpha}}$ shifts the eigenvalue of $\mathbf{H}$ according to

$$\mathbf{H}\,(E_{\boldsymbol{\alpha}}|\mathbf{m}\rangle) = ([\mathbf{H}, E_{\boldsymbol{\alpha}}] + E_{\boldsymbol{\alpha}}\mathbf{H})\,|\mathbf{m}\rangle = (\boldsymbol{\alpha}+\mathbf{m})\,(E_{\boldsymbol{\alpha}}|\mathbf{m}\rangle)$$

In this sense the operators $E_{\boldsymbol{\alpha}}$ act on basis vectors $|\mathbf{m}\rangle$ in such a way that the eigenvalue $\mathbf{m}$ is shifted by $\boldsymbol{\alpha}$ to $\mathbf{m}+\boldsymbol{\alpha}$.

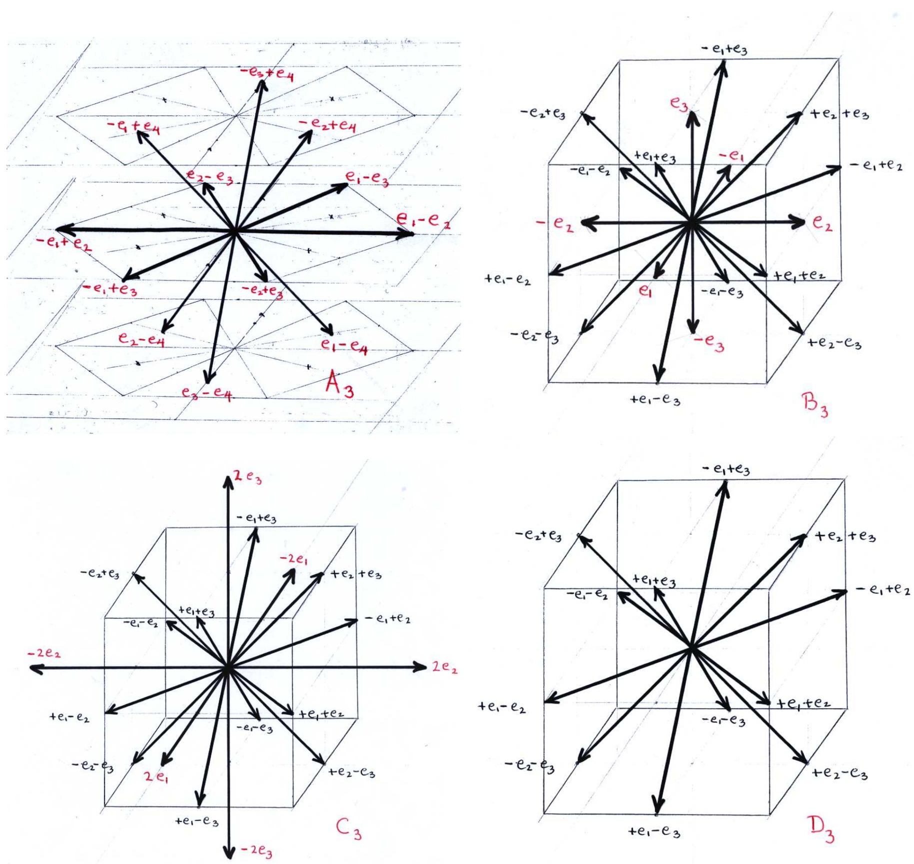

Figure 4: Rank-three root spaces. A_3, B_3, C_3, $D_3 = A_3$.

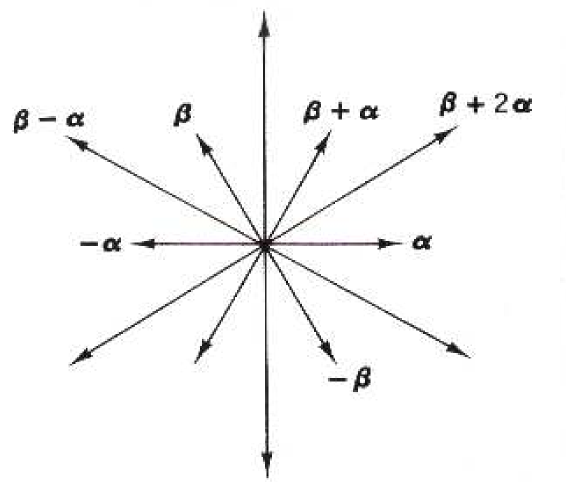

Figure 5: An $\boldsymbol{\alpha}$ chain containing $\boldsymbol{\beta}$.

Table 1: Roots for the simple classical Lie groups and algebras.

Group	Algebra	Root Space	Rank	Roots	Conditions
$SU(l)$	$\mathfrak{su}(n)$	A_{l-1}	$l-1$	$+\mathbf{e}_i - \mathbf{e}_j$	$1 \leq i \neq j \leq l$
$SO(2l)$	$\mathfrak{so}(2l)$	D_l	l	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$1 \leq i < j \leq l$
$SO(2l+1)$	$\mathfrak{so}(2l+1)$	B_l	l	$\pm\mathbf{e}_i \pm \mathbf{e}_j, \pm\mathbf{e}_k$	$1 \leq i < j, k \leq l$
$Sp(l) = USp(2l)$	$\mathfrak{sp}(l) = \mathfrak{usp}(2l)$	C_l	l	$\pm\mathbf{e}_i \pm \mathbf{e}_j, \pm 2\mathbf{e}_k$	$1 \leq i < j, k \leq l$

For the simple classical Lie algebras the roots can be expressed in terms of an orthogonal Euclidean basis set as shown in Table 1 and Figs. 3 and 4 for the rank-two and rank-three root spaces. The roots for the five remaining inequivalent simple Lie algebras ("exceptional" algebras) are shown in Table 2.

The diagonal and shift operators for several of the classical Lie algebras can be related to bilinear products of boson or fermion creation and annihilation operators. For $\mathfrak{u}(n)$ the bilinear products $a_i^\dagger a_j$ are related to $E_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = \mathbf{e}_i - \mathbf{e}_j$, $1 \leq i \neq j \leq n$, and $H_i = a_i^\dagger a_i$. This holds for either boson or fermion operators. For $\mathfrak{sp}(2n; R)$ we have the identifications with bilinear products of boson operators as follows: $+\mathbf{e}_i + \mathbf{e}_j \leftrightarrow b_i^\dagger b_j^\dagger$, $+\mathbf{e}_i - \mathbf{e}_j \leftrightarrow b_i^\dagger b_j$, $-\mathbf{e}_i - \mathbf{e}_j \leftrightarrow b_i b_j$, and $H_i = b_i^\dagger b_i$. In particular, $+2\mathbf{e}_i \leftrightarrow b_i^{\dagger 2}$ and $-2\mathbf{e}_i \leftrightarrow b_i^2$. For $\mathfrak{so}(2n)$ we have the identifications with bilinear products of fermion operators as follows: $+\mathbf{e}_i + \mathbf{e}_j \leftrightarrow f_i^\dagger f_j^\dagger$, $+\mathbf{e}_i - \mathbf{e}_j \leftrightarrow f_i^\dagger f_j$, $-\mathbf{e}_i - \mathbf{e}_j \leftrightarrow f_i f_j$, and $H_i = f_i^\dagger f_i$. In particular $f_i^\dagger f_i^\dagger = f_i^2 = 0$. These identifications make it a relatively simple matter to construct unitary matrix representations of the compact Lie groups $SU(n)$ that are symmetric or antisymmetric, of $USp(2n)$ that are symmetric, and of $SO(2n)$ that are antisymmetric (bosons $\leftrightarrow$ symmetric, fermions $\leftrightarrow$ antisymmetric).

Table 2: Roots for the simple exceptional Lie algebras.

Root Space	Rank	Dimension	Roots	Conditions
G_2	2	14	$+\mathbf{e}_i - \mathbf{e}_j$	$1 \leq i \neq j \neq k \leq 3$
			$\pm[(\mathbf{e}_i + \mathbf{e}_j) - 2\mathbf{e}_k]$	
F_4	4	52	$\pm\mathbf{e}_i \pm \mathbf{e}_j, \pm 2\mathbf{e}_i$	$1 \leq i \neq j \leq 4$
			$\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4$	
E_6	6	78	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$1 \leq i \neq j \leq 5$
			$\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5) \pm \frac{\sqrt{3}}{4}\mathbf{e}_6$	$^{(a)}$
E_7	7	133	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$1 \leq i \neq j \leq 6$
			$\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6) \pm \frac{\sqrt{2}}{4}\mathbf{e}_7$	$^{(b)}$
E_8	8	248	$\pm\mathbf{e}_i \pm \mathbf{e}_j$	$1 \leq i \neq j \leq 8$
			$\frac{1}{2}(\pm\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3 \pm \mathbf{e}_4 \pm \mathbf{e}_5 \pm \mathbf{e}_6 \pm \mathbf{e}_7 \pm \mathbf{e}_8)$	$^{(a)}$

$^{(a)}$ Even number of + signs.
$^{(b)}$ Even number of + signs within bracket.

12 Dynkin Diagrams

Every root in a rank l root space can be represented as a linear combination of l "basis roots." These basis roots can be chosen in such a way that all coefficients are integers. In fact, the basis roots can be chosen so that all linear combinations that are roots involve only positive integers (and zero) or only negative integers and zero. This comes about because every shift operator $E_{\boldsymbol{\delta}}$ can be written as a multiple commutator

$$E_{\boldsymbol{\delta}} \sim \left[E_{\boldsymbol{\alpha}}, \left[E_{\boldsymbol{\beta}}, E_{\boldsymbol{\gamma}}\right]\right] \qquad \boldsymbol{\delta} = \boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}$$

One simple way to construct such a basis set of fundamental roots is to construct an $l-1$ dimensional plane through the origin of the root space that contains no nonzero roots, and choose as l fundamental roots the l roots on one side of this hyperplane that are closest to it. For the classical simple Lie algebras the fundamental roots are

Root Space	$\boldsymbol{\alpha}_1$	$\boldsymbol{\alpha}_2$	$\boldsymbol{\alpha}_{l-1}$	$\boldsymbol{\alpha}_l$
A_{l-1}	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_2 - \mathbf{e}_3$	$\mathbf{e}_{l-1} - \mathbf{e}_l$	
D_l	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_2 - \mathbf{e}_3$	$\mathbf{e}_{l-1} - \mathbf{e}_l$	$\mathbf{e}_{l-1} + \mathbf{e}_l$
B_l	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_2 - \mathbf{e}_3$	$\mathbf{e}_{l-1} - \mathbf{e}_l$	$+1\mathbf{e}_l$
D_l	$\mathbf{e}_1 - \mathbf{e}_2$	$\mathbf{e}_2 - \mathbf{e}_3$	$\mathbf{e}_{l-1} - \mathbf{e}_l$	$+2\mathbf{e}_l$

All roots in the rank-two root spaces have been expressed in terms of both two orthogonal vectors and two fundamental roots in Fig. 3.

If $\boldsymbol{\alpha}_i$ and $\boldsymbol{\alpha}_j$ are fundamental roots their inner product is zero or negative

$$\cos(\boldsymbol{\alpha_i}, \boldsymbol{\alpha_j}) = 0, -\sqrt{\frac{1}{4}}, -\sqrt{\frac{2}{4}}, -\sqrt{\frac{3}{4}}$$

This information has been used to classify the root spaces of the inequivalent simple Lie algebras (over C). The procedure is as follows. Each of the l fundamental roots in a rank l root space is represented by a dot in a plane. Dots representing roots $\boldsymbol{\alpha_i}$ and $\boldsymbol{\alpha_j}$ are connected by n_{ij} lines, where $\cos(\boldsymbol{\alpha_i},\, \boldsymbol{\alpha_j}) = -\sqrt{n_{ij}/4}$. Orthogonal roots are not connected by any lines.

Such diagrams are called Dynkin diagrams. Disconnected Dynkin diagrams describe semisimple Lie algebras. Connected Dynkin diagrams classify simple Lie algebras.

The properties of Dynkin diagrams arise from two simple observations.

O1: The root space is positive definite.

O2: If $\mathbf{u}$ is a unit vector and $\mathbf{v}_i$ are an orthonormal set of vectors,

$$\sum (\mathbf{u} \cdot \mathbf{v}_i)^2 \leq 1$$

These two observations lead to three important properties of Dynkin diagrams.

D1: There are no loops. If $\boldsymbol{\alpha_i}$ $(i = 1, 2, \ldots k)$ are in a loop, then there are at least as many lines as vertices. With $\mathbf{u}_i = \boldsymbol{\alpha_i}/\ |\boldsymbol{\alpha_i}|$,

$$\left(\sum_{i=1}^{k} \mathbf{u}_i, \sum_{j=1}^{k} \mathbf{u}_j \right) = k + 2 \sum_{i<j}^{k} \mathbf{u}_i \cdot \mathbf{u}_j > 0$$

Since $2\mathbf{u}_i \cdot \mathbf{u}_j \leq -1$ if $\mathbf{u}_i \cdot \mathbf{u}_j \neq 0$, there cannot be as many lines as vertices.

D2: The number of lines connected to any node is less than four. If $\boldsymbol{\alpha_i}$ are connected to $\mathbf{v}$, then with $\mathbf{u}_i = \boldsymbol{\alpha_i}/|\boldsymbol{\alpha_i}|$,

$$\sum (\mathbf{v} \cdot \mathbf{u}_i)^2 = \sum n_i/4 < 1$$

since $\mathbf{v}$ is linearly independent of the $\boldsymbol{\alpha_i}$.

D3: A simple chain connecting any two nodes can be shrunk. If the original diagram is allowed, the shrunk diagram is also allowed, and conversely.

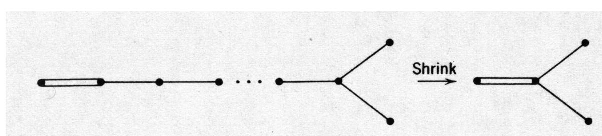

Figure 6: A chain with single links can be removed from a diagram. If the original is an allowed Dynkin diagram, the shrunk diagram is also allowed, and conversely.

Since the shrunk diagram in Fig. 6 violates **D2** the original is not an allowed Dynkin diagram.

According to these results, the maximum number of lines that can be attached to a vertex is three. If a vertex is attached to three lines, it can be connected to three (one line each) other vertices, two (two plus one) other vertices, or only one other vertex (all three lines). This last case describes Dynkin diagram G_2 (cf. Figs. 3, 5).

The only remaining possibilities are shown in Fig. 7.

For diagrams of type (B, C, F) we define vectors

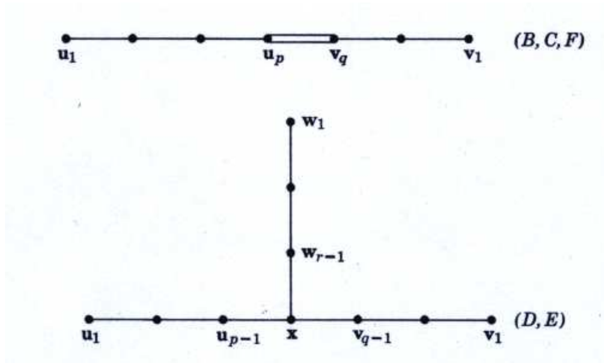

Figure 7: The only remaining candidate Dynkin diagrams have either two vertices (B, C, F) or one vertex (D, E) connected to three lines.

$$\mathbf{u} = \sum_{i=1}^{p} i\mathbf{u}_i \qquad \mathbf{v} = \sum_{j=1}^{q} j\mathbf{v}_j$$

where as usual $\mathbf{u}_i$, $\mathbf{v}_j$ are unit vectors $\boldsymbol{\alpha_k}/|\boldsymbol{\alpha_k}|$. The Schwartz inequality applied to $\mathbf{u}$ and $\mathbf{v}$ leads to the inequality

$$\left(1+\frac{1}{p}\right)\left(1+\frac{1}{q}\right) > 2$$

The solutions with $p \geq q$ are

p	q	Root Space	Constraint
arbitrary	1	B_l, C_l	$l = p + 1$
2	2	F_4	

For diagrams of type (D, E) we define vectors

$$\mathbf{u} = \sum_{i=1}^{p-1} i\mathbf{u}_i \qquad \mathbf{v} = \sum_{j=1}^{q-1} j\mathbf{v}_j \qquad \mathbf{w} = \sum_{k=1}^{r-1} k\mathbf{w}_k$$

where as usual $\mathbf{u}_i$, $\mathbf{v}_j$, $\mathbf{w}_k$ are unit vectors $\boldsymbol{\alpha_m}/|\boldsymbol{\alpha_m}|$. With similar arguments, we obtain the inequality

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 2$$

The solutions with $p \geq q \geq r$ are

p	q	r	Root Space	Regular Euclidean Solid
arbitrary	2	2	D_{p+2}	
3	3	2	E_6	Tetrahedron
4	3	2	E_7	Cube – Octahedron
5	3	2	E_8	Icosahedron – Duodecahedron

All allowed Dynkin diagrams are shown in Fig. 8. In these diagrams roots making an angle of 120° with each other (joined by single lines) have equal length. Roots joined by double lines or triple lines have different lengths. The arrows on double lines indicated the shorter and longer roots. Arrows point to longer roots. The root space G_2 is self dual, so it doesn't matter which way the arrow points.

Coxeter-Dynkin diagrams also appear in classical geometry and catastrophe theory.

13 Real Forms

The metric tensor $g_{\mu\nu}$ for a simple Lie algebra (over C) in the canonical basis $\mathbf{H}, E_{\boldsymbol{\alpha}}$ is

$$g \rightarrow \left[\begin{array}{cccc|ccc} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ \hline & & & & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & & \\ & & & & & \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} & \\ & & & & & & \ddots \end{array}\right] \begin{array}{c} H_1 \\ H_2 \\ \vdots \\ H_l \\ E_{+\boldsymbol{\alpha}} \\ E_{-\boldsymbol{\alpha}} \\ E_{+\boldsymbol{\beta}} \\ E_{-\boldsymbol{\beta}} \\ \vdots \end{array} \tag{2}$$

In this basis the Lie algebra decomposes into positive- and negative-definite subspaces according to

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$$

$$\begin{array}{lll} \mathfrak{g}_+ & \text{spanned by} & H_i \ , \ \left(E_{+\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}\right)/\sqrt{2} \\ \mathfrak{g}_- & \text{spanned by} & \qquad \left(E_{+\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}\right)/\sqrt{2} \end{array}$$

The choice of basis suggested above diagonalizes the Cartan-Killing form in Equ (2): $g \rightarrow I_{p,q}$, with $p = l + \frac{1}{2}(n-l)$ positive values $+1$ on the diagonal and $q = \frac{1}{2}(n-l)$ values -1 on the diagonal. The trace of this matrix is the trace of g: $+l$.

An arbitrary element in this (complex) Lie algebra is a linear superposition of the form

$$X = \sum_i h^i H_i + \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} e^{\boldsymbol{\alpha}} E_{\boldsymbol{\alpha}} \tag{3}$$

where all n coefficients h^i, $e^{\boldsymbol{\alpha}}$ are complex. If all these coefficients are taken real the resulting Lie algebra closes under commutation and describes a noncompact Lie group. The subalgebra describing the maximal compact subgroup is spanned by the linear combinations $\left(E_{+\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}}\right)/\sqrt{2}$. The remaining operators exponentiate to a noncompact coset

$$EXP\left\{h^i H_i + e_+^{\boldsymbol{\alpha}}\left(E_{+\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}}\right)/\sqrt{2}\right\}$$

which is topologically equivalent to R^K, $K = l + \frac{1}{2}(n-l) = \frac{1}{2}(n+l)$. Of all the real forms of the complex Lie algebra described by this set of canonical commutation relations (or root space, or Dynkin diagram), this is the least compact real form.

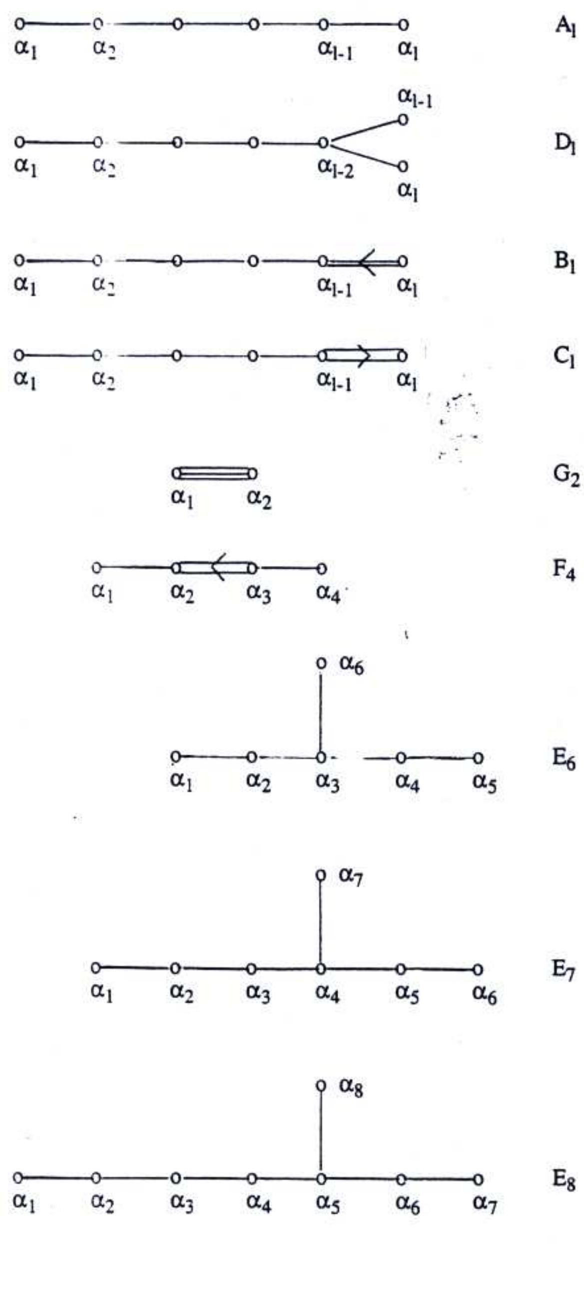

Figure 8: Four infinite series (A_l, D_l, B_l, C_l) of Dynkin diagrams exist and correspond to the classical simple Lie groups $(SU(l+1), SO(2l), SO(2l+1), USp(2l))$. The five exceptional Dynkin diagrams include a short finite series $(E_l,\ l = 6, 7, 8)$, F_4 and G_2.

The compact real form is obtained from (3) by taking linear combinations

$$X = \sum_i ih^i H_i + \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} ie_+^{\boldsymbol{\alpha}} \left(E_{+\boldsymbol{\alpha}} + E_{-\boldsymbol{\alpha}} \right) / \sqrt{2} + \sum_{\boldsymbol{\alpha} \neq \mathbf{0}} e_-^{\boldsymbol{\alpha}} \left(E_{+\boldsymbol{\alpha}} - E_{-\boldsymbol{\alpha}} \right) / \sqrt{2}$$

where $h^i, e_+^{\boldsymbol{\alpha}}, e_-^{\boldsymbol{\alpha}}$ are real. The compact real forms of the simple Lie algebras are

Root Space	Group
A_{l-1}	$SU(l)$
D_l	$SO(2l)$
B_l	$SO(2l+1)$
C_l	$USp(2l) = Sp(l)$

If the imaginary factor i is absorbed into the Cartan-Killing metric, this metric is diagonal, all matrix elements are -1, the trace of this form is $-n$, and the linear combinations for X are real.

Every complex simple Lie algebra (i.e., simple Lie algebra over C) has a spectum of inequivalent real forms. These can all be obtained from the compact real form by an analog of Minkowski's "rotation trick," derived by Cartan. Cartan introduced a metric preserving linear mapping ("involutive automorphism") $T : \mathfrak{g} \to \mathfrak{g}$ with the property $T^2 = I$ and $(TX, TY) = (X, Y)$, with $X, Y \in \mathfrak{g}$. The operator T has eigenvalues ± 1 and induces a decomposition ("Cartan decomposition") in $\mathfrak{g}$ as follows

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \qquad \begin{array}{ccccc} T(\mathfrak{g}) & = & T(\mathfrak{k}) & + & T(\mathfrak{p}) \\ & & \downarrow & & \downarrow \\ & & \mathfrak{k} & - & \mathfrak{p} \end{array}$$

As a result, the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal. The subspaces obey the following commutation and inner product properties

$$\begin{array}{ccc} [\mathfrak{k}, \mathfrak{k}] & \subseteq & \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{p}] & \subseteq & \mathfrak{p} \\ [\mathfrak{p}, \mathfrak{p}] & \subseteq & \mathfrak{k} \end{array} \qquad \begin{array}{ccc} (\mathfrak{k}, \mathfrak{k}) & < & 0 \\ (\mathfrak{k}, \mathfrak{p}) & = & 0 \\ (\mathfrak{p}, \mathfrak{p}) & < & 0 \end{array}$$

Under the analytic continuation $\mathfrak{p} \to i\mathfrak{p}$ the compact Lie algebra $\mathfrak{g}$ is rotated to a noncompact Lie algebra $\mathfrak{g}'$ whose commutation relations and inner product properties are

$$\begin{array}{ccccccc} \mathfrak{g} & = & \mathfrak{k} + \mathfrak{p} & \to & \mathfrak{g}' & = & \mathfrak{k} + \mathfrak{p}' \\ & & & & & & \\ [\mathfrak{k}, \mathfrak{k}] & \subseteq & \mathfrak{k} & & (\mathfrak{k}, \mathfrak{k}) & < & 0 \\ [\mathfrak{k}, \mathfrak{p}'] & \subseteq & \mathfrak{p}' & & (\mathfrak{k}, \mathfrak{p}') & = & 0 \\ [\mathfrak{p}', \mathfrak{p}'] & \subseteq & \mathfrak{k} & & (\mathfrak{p}', \mathfrak{p}') & > & 0 \end{array}$$

The maximal compact subalgebra of $\mathfrak{g}'$ is $\mathfrak{k}$. The subspace $\mathfrak{p}'$ exponentiates to a simply connected submanifold on which the Cartan-Killing metric is positive definite. This manifold is topologically equivalent to R^K, $K = \dim \mathfrak{p}$. It is not geometrically equivalent to R^K once an invariant metric is placed on it.

Three linear mappings that satisfy $T^2 = I$ suffice to generate all real forms of all the simple classical Lie algebras.

13.1 Block Matrix Decomposition

The compact Lie algebra $\mathfrak{u}(n;F)$ has a block submatrix decomposition $(n = p + q)$

$$\mathfrak{u}(n;F) = \begin{bmatrix} A_p & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} 0 & +B \\ -B^\dagger & 0 \end{bmatrix}$$

where $A_p^\dagger = -A_p, A_q^\dagger = -A_q$ and B is an arbitrary $p \times q$ matrix over F. Under the map $T(\mathfrak{g}) = I_{p,q}\mathfrak{g}I_{p,q}$, $I_{p,q} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, the diagonal subspace $\begin{bmatrix} A_p & 0 \\ 0 & A_q \end{bmatrix}$ has eigenvalue $+1$ and the off-diagonal subspace $\begin{bmatrix} 0 & +B \\ -B^\dagger & 0 \end{bmatrix}$ has eigenvalue -1. Under the Cartan rotation

$$\mathfrak{u}(n;F) \to \mathfrak{u}(p,q;F) = \begin{bmatrix} A_p & 0 \\ 0 & A_q \end{bmatrix} + \begin{bmatrix} 0 & +B \\ +B^\dagger & 0 \end{bmatrix}$$

The real forms of the classical Lie groups obtained in this way are

	D_n, B_n		A_{n-1}	C_n
$SO(2n)$ $SO(2n+1)$	$\to$	$SO(p,q)$	$SU(n) \to SU(p,q)$	$Sp(n) \to Sp(p,q)$ $USp(2n) \to USp(2p,2q)$

13.2 Subfield Restriction

The Lie algebra $\mathfrak{su}$(n) of complex traceless antihermitian matrices has a subalgebra $\mathfrak{so}(n)$ of real antisymmetric matrices. The algebra $\mathfrak{su}(n)$ can be expressed in terms of real $n \times n$ antisymmetric matrices A_n and traceless symmetric matrices S_n

$$\mathfrak{su}(n) = \mathfrak{so}(n) + [\mathfrak{su}(n) - \mathfrak{so}(n)] = A_n + iS_n$$

The Cartan rotation is

$$\mathfrak{su}(n) \to \mathfrak{sl}(n;R) = \mathfrak{so}(n) + i\,[\mathfrak{su}(n) - \mathfrak{so}(n)] = A_n + S_n$$

The classical Lie group generated by this transformation is $SL(n;R)$.

A similar rotation can be carried out on unitary matrices over the quaternion field, $\mathfrak{u}(n;Q) = \mathfrak{sp}(n)$. This algebra contains the subalgebra $\mathfrak{u}(n)$, in which quaternions $q = q_0 + \mathcal{I}q_1 + \mathcal{J}q_2 + \mathcal{K}q_3$ are restricted to complex numbers $q = q_0 + iq_1$. There is a natural decomposition

$$\mathfrak{sp}(n) = \mathfrak{u}(n) + [\mathfrak{sp}(n) - \mathfrak{u}(n)]$$

It is useful at this point to replace each quaternion matrix element by a 2×2 complex matrix: $\mathfrak{sp}(n) \to \mathfrak{usp}(2n)$. This is a unitary representation of the symplectic algebra. Replacing the complex matrix elements in $\mathfrak{u}(n)$ by 2×2 real matrices simultaneously generates a real matrix representation of $\mathfrak{u}(n)$ named $\mathfrak{ou}(2n)$. This is an orthogonal representation of the unitary algebra. The decomposition above is

$$\mathfrak{sp}(n) \to \mathfrak{u}(n) + [\mathfrak{sp}(n) - \mathfrak{u}(n)] \to \mathfrak{ou}(2n) + [\mathfrak{usp}(2n) - \mathfrak{ou}(2n)] = A_{2n} + iS_{2n}$$

where as before A_{2n} and S_{2n} are $2n \times 2n$ antisymmetric and symmetric matrices. The Cartan rotation maps this to $\mathfrak{sp}(2n;R)$

$$\mathfrak{usp}(2n) \to \mathfrak{sp}(2n;R) = A_{2n} + S_{2n}$$

The classical Lie group generated in this way is $Sp(2n;R)$. Matrices in this group satisfy the quadratic constraint $M^tGM = G$, $G^t = -G$, $\det(G) \neq 0$. The real symplectic groups leave invariant Hamilton's equations of motion: $dp_i/dt = -\partial H/\partial q_i$, $dq_i/dt = +\partial H/\partial p_i$.

Table 3: Real forms of the simple classical Lie algebras.

Mapping	Real Form	Maximal Compact Subalgebra	Root Space	Condition
Block Submatrix	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p)+\mathfrak{so}(q)$	D_n	$p+q=2n$
	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p)+\mathfrak{so}(q)$	B_n	$p+q=2n+1$
	$\mathfrak{su}(p,q)$	$\mathfrak{u}(1)+\mathfrak{su}(p)+\mathfrak{su}(q)$	A_{n-1}	$p+q=n$
	$\mathfrak{sp}(p,q)=\mathfrak{usp}(2p,2q)$	$\mathfrak{usp}(2p)+\mathfrak{usp}(2p)$	C_n	$p+q=n$
Subfield Restriction	$\mathfrak{sl}(n;R)$	$\mathfrak{so}(n)$	A_{n-1}	
	$\mathfrak{sp}(2n;R)$	$\mathfrak{u}(n)$	C_n	
Field Embedding	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	D_n	
	$\mathfrak{su}^*(2n)$	$\mathfrak{sp}(n)=\mathfrak{usp}(2n)$	A_{2n-1}	

13.3 Field Embeddings

The image of $\mathfrak{u}(n)\rightarrow\mathfrak{ou}(2n)$ consists of a set of $2n\times 2n$ antisymmetric matrices of dimension n^2. These matrices form a subset of $\mathfrak{so}(2n)$, which consists of $2n\times 2n$ antisymmetric matrices of dimension $2n(2n-1)/2$. As a result $\mathfrak{ou}(2n)$ is a subalgebra in $\mathfrak{so}(2n)$. Thus $\mathfrak{ou}(2n)\sim\mathfrak{k}$ and $\mathfrak{so}(2n)\sim\mathfrak{g}$ and we have a Cartan decomposition

$$\begin{array}{ccccccc} \mathfrak{so}(2n) & = & \mathfrak{ou}(2n) & + & [\mathfrak{so}(2n)-\mathfrak{ou}(2n)] & & \\ & & \downarrow & & \downarrow & & \\ & & \mathfrak{ou}(2n) & + & i\,[\mathfrak{so}(2n)-\mathfrak{ou}(2n)] & = & \mathfrak{so}^*(2n) \end{array}$$

In the same way the image of $\mathfrak{sp}(2n)\rightarrow\mathfrak{usp}(2n)$ consists of an $n(2n+1)$ dimensional set of $2n\times 2n$ antihermitian matrices. This is a subset of $\mathfrak{su}(2n)$, which has dimension $(2n)^2-1$. It is also a subalgebra of $\mathfrak{su}(2n)$. Thus $\mathfrak{usp}(2n)\sim\mathfrak{k}$ and $\mathfrak{su}(2n)\sim\mathfrak{g}$, so we have a Cartan decomposition

$$\begin{array}{ccccccc} \mathfrak{su}(2n) & = & \mathfrak{usp}(2n) & + & [\mathfrak{su}(2n)-\mathfrak{usp}(2n)] & & \\ & & \downarrow & & \downarrow & & \\ & & \mathfrak{usp}(2n) & + & i\,[\mathfrak{su}(2n)-\mathfrak{usp}(2n)] & = & \mathfrak{su}^*(2n) \end{array}$$

These real forms are summarized in Table 3.

The root spaces A_1 $[SU(2)]$, B_1 $[SO(3)]$, and C_1 $[U(1;Q)\simeq USp(2;C)]$ are equivalent. As a result, the different real forms of their complex extensions are related to each other. Similar remarks hold for the real forms of $B_2=C_2$, $D_2=A_1+A_1$, and $D_3=A_3$. The relations among these real forms are summarized in Table 4. This table is useful in inferring "spinor representations" among classical groups. Thus, $SO(3)$ has spinor representations based on $SU(2)$ and $Sp(1)$; $SO(4)$ has spinor representations based on $SU(2)\times SU(2)$; $SO(5)$ has spinor representations based on $USp(4)$; and $SO(6)$ has spinor representations based on $SU(4)$.

For completeness, the real forms for the exceptional Lie algebras are collected in Table 5.

Real forms of the complex extension of a simple Lie algebra are almost uniquely distinguished by an index. This is the trace of the Cartan-Killing form (2), once the appropriate factors of i have been absorbed into it. If n_c is the dimension of the maximal compact subgroup, $\chi=\mathrm{tr}\ (g)=+1(n-n_c)-1(n_c)=n-2n_c$. The index ranges from $-n$ for the compact real form (for which $n_c=n$) to $+l$ for the least compact real form.

Table 4: Equivalence among real forms of the simple classical Lie algebras.

A_1	$=$	B_1	$=$	C_1	χ
$\mathfrak{su}(2)$	$=$	$\mathfrak{so}(3)$	$=$	$\mathfrak{sp}(1) = \mathfrak{usp}(2)$	-3
$\mathfrak{su}(1,1) = \mathfrak{sl}(2;R)$	$=$	$\mathfrak{so}(2,1)$	$=$	$\mathfrak{sp}(2;R)$	$+1$

D_2	$=$	A_1	$+$	A_1	χ
$\mathfrak{so}(4)$	$=$	$\mathfrak{so}(3)$	$+$	$\mathfrak{so}(3)$	-6
$\mathfrak{so}^*(4)$	$=$	$\mathfrak{so}(3)$	$+$	$\mathfrak{so}(2,1)$	-2
$\mathfrak{so}(3,1)$	$=$	$\mathfrak{sl}(2;C)$			0
$\mathfrak{so}(2,2)$	$=$	$\mathfrak{so}(2,1)$	$+$	$\mathfrak{so}(2,1)$	$+2$

B_2	$=$	C_2	χ
$\mathfrak{so}(5)$	$=$	$\mathfrak{sp}(2) = \mathfrak{usp}(4)$	-10
$\mathfrak{so}(4,1)$	$=$	$\mathfrak{sp}(1,1) = \mathfrak{usp}(2,2)$	-2
$\mathfrak{so}(3,2)$	$=$	$\mathfrak{sp}(4;R)$	$+2$

D_3	$=$	A_3	χ
$\mathfrak{so}(6)$	$=$	$\mathfrak{su}(4)$	-15
$\mathfrak{so}(5,1)$	$=$	$\mathfrak{su}^*(4)$	-5
$\mathfrak{so}^*(6)$	$=$	$\mathfrak{su}(3,1)$	-3
$\mathfrak{so}(4,2)$	$=$	$\mathfrak{su}(2,2)$	$+1$
$\mathfrak{so}(3,3)$	$=$	$\mathfrak{sl}(4;R)$	$+3$

Table 5: Real forms of the exceptional Lie algebras.

Root Space	$\text{Class}_{\text{Rank(Character)}}$	Maximal Compact Subgroup Root Space	Dimension
G_2	$G_{2(-14)}$	G_2	14
	$G_{2(+2)}$	$A_1 + A_1$	6
F_4	$F_{4(-52)}$	F_4	52
	$F_{4(-20)}$	B_4	36
	$F_{4(+4)}$	$C_3 + A_1$	24
E_6	$E_{6(-78)}$	E_6	78
	$E_{6(-26)}$	F_4	52
	$E_{6(-14)}$	$D_5 + D_1$	46
	$E_{6(+2)}$	$A_5 + A_1$	38
	$E_{6(+6)}$	C_4	36
E_7	$E_{7(-133)}$	E_7	133
	$E_{7(-25)}$	$E_6 + D_1$	79
	$E_{7(-5)}$	$D_6 + A_1$	69
	$E_{7(+7)}$	A_7	63
E_8	$E_{8(-248)}$	E_8	248
	$E_{8(-24)}$	$E_7 + A_1$	136
	$E_{8(+8)}$	D_8	120

14 Riemannian Symmetric Spaces

Exponentiation lifts Lie algebras to Lie groups and subspaces in Lie algebras into submanifolds in Lie groups. In particular, exponentiation of a Cartan decomposition

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \downarrow & & \downarrow & & \downarrow \\ G & = & K & \times & (P = G/K) \end{array}$$

lifts the subspace $\mathfrak{p}$ to the quotient $(P = G/K)$.

A metric may be defined on the Lie group G as follows. Define the distance between the identity and some nearby point $g(\epsilon) = EXP(\epsilon X) = EXP(\delta x^i X_i)$ by

$$ds^2(0) = G_{rs}\delta x^r \delta x^s$$

Move I and $g(\epsilon)$ to the neighborhood of any point $g(x) \in G$ by left multiplication: $g(x)I \to g(x)$, $g(x)g(\delta x^i X_i) \to g\left((x+dx)^i X_i\right)$. The infinitesimals $dx^i(x)$ at x (defined by $g(x)$) and $\delta x^i = dx^i(0)$ at I are linearly related

$$\delta x^i = M^i{}_j(x) dx^j(x)$$

By requiring that the distance ds between I and $g(\delta x^i X_i)$ at the identity be the same as the distance between $g(x^i X_i)I$ and $g(x^i X_i)g(\delta x^i X_i) = g\left((x+dx)^i X_i\right)$ at $g(x^i X_i)$ leads to the condition

$$ds^2 = G_{rs}(0)\delta x^r \delta x^s = G_{rs}(0) M^r{}_i(x) M^s{}_j(x) dx^i(x) dx^j(x) = G_{ij}(x) dx^i(x) dx^j(x)$$

An invariant metric $G(x)$ over the Lie group G is defined by

$$\begin{array}{rcl} G_{ij}(x) & = & G_{rs}(0) M^r{}_i(x) M^s{}_j(x) \\ G(x) & = & M^t(x) G(0) M(x) \end{array}$$

It is useful to identify $G(0)$ with the Cartan-Killing inner product on $\mathfrak{g}$. Since $M(x)$ is nonsingular, the signature of $G(x)$ is invariant over the group.

The invariant metric on G can be restricted to subspaces $K \subset G$ and $P = G/K \subset G$. The signature on these subspaces is the same as the signature on the subspaces $\mathfrak{k}$ and $\mathfrak{p}$ in $\mathfrak{g}$. Thus, if G is compact, the invariant metric is negative definite on K and on $P = G/K$ and positive definite on the analytically continued space $P' = G'/K$. In short, it is definite (negative, positive) on P, P'. These spaces are Riemannian spaces. They are globally symmetric. They have been investigated by studying the properties of the secular equation of the Lie algebra $\mathfrak{g}$, restricted to the subspace $\mathfrak{p}$:

$$\det\ \left[R(p^i P_i) - \lambda I\right] = \sum_j (-\lambda)^{n-j} \hat{\phi}_j(p) = 0 \tag{4}$$

where the P_i are basis vectors that span $\mathfrak{p}$. The coefficients $\hat{\phi}_j(p)$ in the secular equation (4) for Riemannian symmetric spaces are related to the coefficients $\phi_j(x)$ in the secular equation (1) for Lie algebras. A rank for the Riemannian symmetric space $P = EXP(\mathfrak{p})$ can be defined from the secular equation following exactly the prescription followed for the Lie algebra $\mathfrak{g}$. The rank of the Riemannian symmetric space $P = EXP(\mathfrak{p})$ is

1. The number of functionally independent coefficients $\hat{\phi}_j(p)$ in the secular equation.

Table 6: All classical Riemannian symmetric spaces.

Root Space	Quotient	Dimension	Rank	χ
A_{p+q-1}	$SU(p,q)/S[U(p)\otimes U(q)]$	$2pq$	$\min(p,q)$	$1-(p-q)^2$
A_{n-1}	$SL(n;R)/SO(n)$	$\frac{1}{2}(n+2)(n-1)$	$n-1$	$n-1$
A_{2n-1}	$SU^*(2n)/USp(2n)$	$(2n+1)(n-1)$	$n-1$	$-2n-1$
B_{p+q}	$SO(p,q)/SO(p)\otimes SO(q)$	pq	$\min(p,q)$	$pq-\frac{1}{2}p(p-1)-\frac{1}{2}q(q-1)$
D_{p+q}	$SO(p,q)/SO(p)\otimes SO(q)$	pq	$\min(p,q)$	$pq-\frac{1}{2}p(p-1)-\frac{1}{2}q(q-1)$
D_n	$SO^*(2n)/U(n)$	$n(n-1)$	n	$-n$
C_{p+q}	$USp(2p,2q)/USp(2p)\otimes USp(2q)$	$4pq$	$\min(p,q)$	$-2(p-q)^2-(p+q)$
C_n	$Sp(2n;R)/U(n)$	$n(n+1)$	n	$+n$

2. The number of independent roots of the secular equation.

3. The dimension of the maximal Euclidean subspace in P.

4. The number of independent (Laplace-Beltrami) operators that commute with all displacement operators P_i: $\Delta_j(P)=\hat{\phi}_j(p^i\to P_i)$.

Rank 1 Riemannian symmetric spaces are isotropic as well as homogeneous.

Tables 3 and 5 contain all the information required to enumerate all the classical and exceptional Riemannian symmetric spaces. All the classical Riemannian symmetric spaces are tabulated in Table 6. The exceptional Riemannian symmetric spaces can be constructed from the information in Table 5 following the procedure used to construct Table 6 from Table 3.

As particular examples of Riemannian symmetric spaces we consider the compact spaces $SO(p+q)/[SO(p)\times SO(q)]$ and their noncompact counterparts $SO(p,q)/[SO(p)\times SO(q)]$. These spaces have rank $\min(p,q)$, dimension pq, and can be represented explicitly in matrix form as

$$\left[\begin{array}{c|c} 0 & X \\ \hline \sigma X^t & 0 \end{array}\right] \to EXP\left[\begin{array}{c|c} 0 & X \\ \hline \sigma X^t & 0 \end{array}\right] = \left[\begin{array}{c|c} D_p & Y \\ \hline \sigma Y^t & D_q \end{array}\right]$$

Here X is a $p\times q$ matrix and $\sigma=+1$ for the noncompact case and -1 for the compact case. The block diagonal matrices D_p and D_q are defined from the metric-preserving conditions ($M^tI_{p+q}M=I_{p+q}$, $M^tI_{p,q}M=I_{p,q}$)

$$D_p^2=I_p+\sigma YY^t \qquad D_q^2=I_q+\sigma Y^tY$$

The pq coordinates in the Riemannian symmetric spaces can be taken as the pq elements of the submatrix Y.

These Riemannian symmetric spaces can be treated as algebraic submanifolds in R^K, $K=pq+\frac{1}{2}q(q+1)$. The K coordinates on R^K can be identified with the pq matrix elements of Y and the $\frac{1}{2}q(q+1)$ matrix elements of the real symmetric matrix D_q. These coordinates obey the $\frac{1}{2}q(q+1)$ algebraic constraints defined by

$$D_q^2-\sigma Y^tY=I_q$$

For $SO(3)/SO(2)$ and $SO(2,1)/SO(2)$ this condition is determined from the matrix

$$\left[\begin{array}{c|c} \left[I_2 + \left(\begin{array}{cc}\sigma x & \sigma y\end{array}\right)\left[\begin{array}{c} x \\ y \end{array}\right]\right]^{1/2} & \begin{array}{c} x \\ y \end{array} \\ \hline \begin{array}{cc}\sigma x & \sigma y\end{array} & z \end{array}\right] \qquad \text{to be} \qquad z^2 - \sigma(x^2 + y^2) = 1$$

For $\sigma = -1$ the space is the sphere S^2 defined by $z^2 + (x^2 + y^2) = 1$. For $\sigma = +1$ the space is the two-sheeted hyperboloid H_2^2 defined by $z^2 - (x^2 + y^2) = 1$. More specifically it is the upper sheet containing $(0, 0, 1)$ of the two-sheeted hyperboloid. The second sheet occurs in the coset $O(2, 1)/SO(2)$. The symmetric spaces $SO(n + 1)/SO(n)$ and $SO(n, 1)/SO(n)$ are the sphere S^n and the upper sheet of the two-sheeted hyperboloid H_{2+}^n. Both have dimension n and rank 1. The spaces are simply connected, homogeneous, and isotropic.

For $SO(4, 2)/SO(4) \times SO(2)$ the eight dimensional algebraic manifold is defined by the three constraints in R^{11}

$$\left[\begin{array}{cc} y_9 & y_{10} \\ y_{10} & y_{11} \end{array}\right]^2 - \sigma \left[\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ y_5 & y_6 & y_7 & y_8 \end{array}\right] \left[\begin{array}{cc} y_1 & y_5 \\ y_2 & y_6 \\ y_3 & y_7 \\ y_4 & y_8 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

The compact analytically continued space $SO(6)/SO(4) \times SO(2)$ is obtained by setting $\sigma = -1$. These spaces have dimension eight and rank two. They are homogeneous but not isotropic. For each, there are "two inequivalent directions." There are two independent Laplace-Beltrami operators on these spaces, one quadratic and one quartic.

The complete list of globally symmetric pseudo Riemannian symmetric spaces can be constructed almost as easily. Two linear operators, T_1 and T_2 are introduced that obey $T_1^2 = I, T_2^2 = I, T_1 T_2 = T_2 T_1 \neq I$. The two are used to split $\mathfrak{g}$ into subspaces

$$T_1 \mathfrak{g}_{\sigma\tau} = \sigma \mathfrak{g}_{\sigma\tau} \qquad T_2 \mathfrak{g}_{\sigma\tau} = \tau \mathfrak{g}_{\sigma\tau}$$

where $\sigma = \pm 1$, $\tau = \pm 1$. The decomposition and double rotation

$$\begin{array}{ccccccccc} \mathfrak{g} & = & \mathfrak{g}_{++} & + & \mathfrak{g}_{+-} & + & \mathfrak{g}_{-+} & + & \mathfrak{g}_{--} \\ & \downarrow T_1 & & & & & & & \\ \mathfrak{g}' & = & \mathfrak{g}_{++} & + & \mathfrak{g}_{+-} & + & i(\mathfrak{g}_{-+} & + & \mathfrak{g}_{--}) \\ & \downarrow T_2 & & & & & & & \\ \mathfrak{g}'' & = & \mathfrak{g}_{++} & + & i\mathfrak{g}_{+-} & + & i(\mathfrak{g}_{-+} & + & i\mathfrak{g}_{--}) \end{array}$$

generates a noncompact subgroup K'' as well as a pseudo-Riemannian symmetric space P''

$$K'' = EXP\left(\mathfrak{g}_{++} + i\mathfrak{g}_{+-}\right) \qquad P'' = EXP\left(i\mathfrak{g}_{-+} + \mathfrak{g}_{--}\right)$$

These have also been classified.

The simplest example of a Riemannian pseudo symmetric space is $SO(2, 1)/SO(1, 1)$:

$$\mathfrak{so}(2,1) \rightarrow \left[\begin{array}{cc|c} 0 & \theta_3 & \theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \hline \theta_2 & \theta_1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & \theta_1 \\ \hline 0 & \theta_1 & 0 \end{array}\right] + \left[\begin{array}{cc|c} 0 & \theta_3 & \theta_2 \\ -\theta_3 & 0 & 0 \\ \hline \theta_2 & 0 & 0 \end{array}\right]$$

$$\downarrow$$

$$M = \left[\begin{array}{cc|c} z & x & y \\ -x & * & * \\ \hline y & * & * \end{array}\right]$$

The metric-preserving condition $M^t I_{2,1} M = I_{2,1}$ leads to the constraint equation $z^2 + x^2 - y^2 = 1$. This space is the single-sheeted hyprboloid H_1^2. It is two dimensional and has rank 1, but it is not isotropic. Intersections with the plane $x = 0$ are hyperbolas and with the planes $y =$ const. are circles. This space is not simply connected.

15 Summary

Lie groups are among the most powerful mathematical tools available to Physicists. They play a major role in Physics because they occur as transformation groups from coordinate system to coordinate system in real space (rotation group $SO(3)$, Lorentz group $O(3,1)$, Galilei group, Poincaré group $ISO(3,1)$) or in spaces describing internal degrees of freedom ($SU(2)$ for spin or isospin, $SU(3)$ for quarks and color, $SU(4)$ for spin-isospin, ...).

It is remarkable that a beautiful classification theory for simple (the building blocks) Lie groups exists, because of the rather amorphous nature of the definitions of a Lie group. In a search for structure, the first step in the analysis of Lie groups is linearization of the group multiplication law in the neighborhood of the identity to a linear vector space on which there is a Lie algebra structure. This in itself is sufficient to create a strong connection to Quantum Mechanics. Although there is not a 1:1 correspondence between Lie groups and their Lie algebras, there is a very beautiful connection between them. This relates algebra (discrete invariant subgroups) and topology (homotopy groups) in an elegant way.

The structure of Lie algebras is described using tools from linear algebra: secular equations and inner products. Together, these tools are used to reduce Lie algebras to their basic units: nilpotent and solvable invariant subalgebras and semisimple and simple Lie algebras. The commutation relations for simple Lie algebras can be put into a canonical form using another miracle of this theory: a positive definite root space that summarizes the properties of the secular equation and the Cartan-Killing inner product. As the secular equation can only be solved exactly over an algebraically closed field, the classification of simple Lie algebras covers complex Lie algebras. Each complex extension has several real forms. These are easily classified.

Even more remarkable is the connection between simple Lie groups and Riemannian spaces that "look the same everywhere." All Riemannian symmetric spaces are quotients of a simple Lie group by a subgroup that is maximal in some precise sense (Cartan decomposition sense). Cartan was able to classify all Riemannian symmetric spaces as a consequence of his classification of all the real forms of all the simple Lie groups. The algebraic tools used to classify Lie algebras (secular equations, Dynkin diagrams) were used again to classify these spaces (Dynkin diagrams $\rightarrow$ Araki-Satake diagrams). These spaces are classified by a root space, group-subgroup pair, dimension, rank, and character. Construction of invariant operators (Casimir invariants, Laplace-Beltrami operators) is algorithmic.

Nonsemisimple Lie groups/algebras can be constructed from simple Lie algebras by carefully introducing singular change of basis transformations. This leads to "group contraction," not discussed above. In this way the Poincaré group can be constructed systematically from the groups $SO(3,2)$ or $SO(4,1)$: $SO(3,2) \rightarrow ISO(3,1)$, $SO(4,1) \rightarrow ISO(3,1)$ in the limit of "large R." Here R is the "radius" of some universe of hyperbolic nature, with signature $(3,2)$ or $(4,1)$. The Galilei group can be constructed by contraction from the Poincaré group in the limit $c = 3 \times 10^{10}$ cm/sec $\rightarrow \infty$.

We have not discussed here the theory of the representations of Lie groups. A beautiful theorem by Wigner and Stone guarantees that the tensor representations of a compact group are complete. Gel'fand has given expressions for the complete set of tensor representations of the classical compact Lie groups. They are expressed by "dressing" the appropriate Dynkin diagrams or else in terms of irreducible representations of the symmetric group S_n. Gel'fand has also given explicit, analytic, closed form expressions for the matrix elements of any of the shift operators in any of these representations. For the noncompact real forms most of the unitary irreducible representations can be obtained from these expressions for matrix elements ("master analytic representation") by appropriate analytic continuation.

Since Lie groups exist at the interface of algebra and topology, it is to be expected that there is a very close relation with the theory of special functions. In fact, the theory of special functions forms an important chapter in the theory of Lie groups. On the topological side, the shift operators $E_{\boldsymbol{\alpha}}$ (think $J_{\pm}$) have coordinate representations $\langle x'|E_{\boldsymbol{\alpha}}|x\rangle$ involving first order differential operators. On the algebraic side the matrix elements $\langle \mathbf{n}'|E_{\boldsymbol{\alpha}}|\mathbf{n}\rangle$ are square roots of products of integers (divided by products of integers). These topological and algebraic expressions are related to each other in a myriad of ways. All of the standard properties of special functions (Rodriguez formulas, recursion relations in coordinates and indices, differential equations, generating functions, ...) occur in a systematic way in a Lie theoretic formulation of this subject.

Finally, no review or even book could do justice to the applications that Lie group theory finds in Physics.

The rich interplay that exists between freedom and rigidity of structure found in Lie group theory can be found in only the purest works of art — for example, the fugues of Bach.

16 Further Reading

R. Gilmore (1974) *Lie Groups, Lie Algebras, and Some of Their Applications*, NY: Wiley (republished (2005), NY: Dover).

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